Branching random walk in a random time-independent environment

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Abstract

In a lattice population model, particles move randomly from one site to another as independent random walks, split into two offspring, or die. If duplication and

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mortality rates are equal and take the same value over all lattice sites, the resulting model is a critical branching random walk (characterized by a mean total number of offspring equal to 1). There exists an asymptotical statistical equilibrium, also called steady state. In contrast, when duplication and mortality rates take independent random values drawn from a common nondegenerate distribution (so that the difference between duplication and mortality rates has nonzero variance), then the steady state no longer exists. Simultaneously at all lattice sites, if the difference between duplication and mortality rates takes strictly positive values with strictly positive probability, the total number of particles grows exponentially. The lattice $\mathbb{Z}^d$ includes large connected sets where the duplication rate exceeds the death rate by a positive constant amount, and these connected sets provide the growth of the total population. This is the supercritical regime of branching processes. On the other hand, if the difference between duplication and mortality rates is almost surely negative or null except when it is almost surely zero, then the total number of particles vanishes asymptotically. The steady state can be reached only if the difference between duplication and mortality rates is almost surely zero.

keywords: branching random walk; contact population model; random environment; steady state

1 Introduction

In a contact process on the d-dimensional lattice $\mathbb{Z}^d$, $d \geq 1$, the particles move independently of one another on $\mathbb{Z}^d$ as random walks, split into two offspring, or die. The birth-and-death mechanism is controlled at each lattice site $x \in \mathbb{Z}^d$ by the duplication rate $\lambda(x)$ and the mortality rate $\mu(x)$.

When rates $\lambda(x) \equiv \mu(x) \equiv \lambda_0 > 0$ are constant for all $x \in \mathbb{Z}^d$, the walk is a critical branching random walk, characterized by a mean total number of offspring equal to 1 (Sewastjanow, 1974). Molchanov and Whitmeyer (2017) proved that if the underlying random walk is transient —with strictly positive probability, particles never return to the initial lattice site after a finite random time (Durrett, 2010: p. 190)—, then the distribution of the particle field asymptotically approaches a statistical equilibrium, also called steady state.

However, if $\lambda(x)$ and $\mu(x)$ are random fields on the lattice, does the population process converge to a steady state? One might speculate that having the expectations of $\lambda(x)$ and $\mu(x)$, which characterize the random environment, equal to the same constants for all $x \in \mathbb{Z}^d$, allows the convergence of the particle field distribution to a stochastic equilibrium. We show that this is never the case if $\lambda(\cdot) - \mu(\cdot) \not\equiv 0$.

With Theorems 1 and 2 below, we show that if the random vectors $(\lambda(x), \mu(x))_{x\in\mathbb{Z}^d}$, are drawn independently from the same non-degenerate distribution, then simultaneously at all $x \in \mathbb{Z}^d$, either the population vanishes asymptotically or the population grows exponentially. The latter case is due to the existence of arbitrarily large connected sets of sites $x \in \mathbb{Z}^d$, where the random vectors $(\lambda(x), \mu(x))_{x\in\mathbb{Z}^d}$ satisfy $\lambda(x) - \mu(x) \geq \delta_0$ for some constant $\delta_0 > 0$. The exponential growth of the population is heterogeneous over the lattice, that is, "intermittent": almost all particles are concentrated near large enough sets where the potential is positive (Molchanov, 2012; König, 2016).

2 Model

2.1 The random environment

$N(t, y)$ is the total number of particles at the lattice site $y \in \mathbb{Z}^d$ at time $t \geq 0$. Initially, there is a single particle at each site, $N(0, y) = 1$ for all $y \in \mathbb{Z}^d$. Particles can independently of one another move as continuous-time random walks on $\mathbb{Z}^d$, die, or split into two offspring, where duplication and mortality rates (reflecting the environment) are random:

$$\big(\lambda(x, \omega_m), \mu(x, \omega_m)\big)_{x\in\mathbb{Z}^d,\, \omega_m\in\Omega_m} \tag{1}$$

defined on some fixed probability space $(\Omega_m, \mathcal{F}_m, \mathbb{P}_m)$. $\lambda(x) = \lambda(x, \omega_m)$ is the random splitting or duplication rate and $\mu(x) = \mu(x, \omega_m)$ is the random mortality rate at site $x \in \mathbb{Z}^d$ (but constant over time). The vectors $\big(\lambda(x), \mu(x)\big) = \big(\lambda(x, \omega_m), \mu(x, \omega_m)\big)$, $x \in \mathbb{Z}^d$, are independent of one another with common non-degenerate distribution in $(\Omega_m, \mathcal{F}_m, \mathbb{P}_m)$

(König, 2016, 2021: and references therein about surveys of random potentials). For the sake of simplicity we also assume that $\lambda(x)$ and $\mu(x)$ are uniformly bounded, that is, for some $c_0 > 0$,

$$\mathbb{P}_m\big(0 \leq \lambda(x) \leq c_0, 0 \leq \mu(x) \leq c_0\big) = 1 \qquad \text{for all } x \in \mathbb{Z}^d. \tag{2}$$

2.2 The process

Given a realization of the random environment from Eq. (1), particles either:

split: at every $x \in \mathbb{Z}^d$, particles split into two offspring particles independently of one another at rate $\lambda(x) \geq 0$;

die: at every $x \in \mathbb{Z}^d$, particles die independently of one another at rate $\mu(x) \geq 0$;

or move: the particles jump independently from one another with generator $\kappa\mathcal{L}_a$, where $\kappa > 0$ is the diffusive coefficient and $\mathcal{L}_a$ is defined by

$$(\mathcal{L}_a\psi)(x) = \sum_{y\in\mathbb{Z}^d\setminus\{0\}} \big(\psi(x+y) - \psi(x)\big)a(y) \equiv \sum_{y\in\mathbb{Z}^d} \big(\psi(x+y) - \psi(x)\big)a(y), \tag{3}$$

where ψ is any bounded function defined on the lattice, $a(y)$, $y \in \mathbb{Z}^d$, is a symmetric probability kernel, defined by:

$$a(y) \geq a(0) = 0\,, \qquad a(y) \equiv a(-y)\,, \qquad \sum_{y\in\mathbb{Z}^d} a(y) = 1. \tag{4}$$

We assume that the jump kernel $a(y)$ decreases sufficiently fast as $|y| \to \infty$, so that all its exponential moments are finite. Equivalently,

$$\sum_{y\in\mathbb{Z}^d} a(y)\,\cosh\big((\theta, y)\big) < \infty \tag{5}$$

for all $\theta \in \mathbb{R}^d$, where (θ, y) is the inner product in $\mathbb{R}^d$. The corresponding continuous-time random walk is also assumed irreducible, that is, it is supported on the full lattice $\mathbb{Z}^d$. A sufficient irreducibility condition is that $a(y) > 0$ for all $|y| = 1$.

For a given realization of the random environment, that is, for fixed $\omega_m \in \Omega_m$, the random dynamics governed by Eq. (3) and (4) generates the so-called *quenched* (Sznitman, 1998) expectation denoted by $\mathbb{E}$ and probability denoted by $\mathbb{P}$. The averages over $\omega_m \in \Omega_m$ are the *annealed* expectation $\mathcal{E}$ and probability $\mathcal{P}$. With $\langle \cdot \rangle$ denoting averaging over $\omega_m \in \Omega_m$, $\mathcal{E}(\cdot) = \langle \mathbb{E}(\cdot) \rangle$.

2.3 The probability generating function

Consider the total number $n(t, x, y)$ of particles at the lattice site $y \in \mathbb{Z}^d$ at time $t \geq 0$, generated by a single particle at $x \in \mathbb{Z}^d$ at time 0. Then $n(0, x, y) = 1$ if $x = y$ and $n(0, x, y) = 0$ otherwise. The sizes of the sub-populations $\big(n(t, x, y)\big)_{y \in \mathbb{Z}^d}$ are mutually independent, in the sense that, for any positive integer M and any distinct sites $x_1, \ldots x_M$, any finite set $\Gamma \subset \mathbb{Z}^d$ and any random vectors $\big(n(t, x_1, y)\big)_{y \in \Gamma}, \ldots, \big(n(t, x_M, y)\big)_{y \in \Gamma}$ are independent of one another. For each $y \in \mathbb{Z}^d$ and $t \geq 0$, the population size $N(t, y)$ is a sum of the independent sub-population sizes:

$$N(t, y) = \sum_{x \in \mathbb{Z}^d} n(t, x, y). \tag{6}$$

Fix $\omega_m \in \Omega_m$ and finite $\Gamma \subseteq \mathbb{Z}^d$. Then the population size

$$n(t, x, \Gamma, \omega_m) := \sum_{y \in \Gamma} n(t, x, y, \omega_m) \tag{7}$$

in Γ has the quenched probability generating function

$$u_z(t, x, \Gamma, \omega_m) := \mathbb{E} z^{n(t, x, \Gamma, \omega_m)}. \tag{8}$$

It satisfies

$$\begin{aligned} \frac{\partial u_z}{\partial t}(t, x, \Gamma, \omega_m) &= \kappa \left(\mathcal{L}_a u_z\right)(t, x, \Gamma, \omega_m) + \lambda(x, \omega_m) u_z^2(t, x, \Gamma, \omega_m) \\ &\quad - \left(\lambda(x, \omega_m) + \mu(x, \omega_m)\right) u_z(t, x, \Gamma, \omega_m) + \mu(x, \omega_m), \end{aligned} \tag{9}$$

(Kolmogorov, Petrovskii, and Piskunov, 1937), with initial condition

$$u_z(0, x, \Gamma, \omega_m) = \begin{cases} z, & x \in \Gamma, \\ 1, & x \notin \Gamma. \end{cases} \tag{10}$$

Differentiating Eq. (9) k times with respect to z at $z = 1_-$ yields quenched factorial moments:

$$m_k(t, x, \Gamma, \omega_m) := \mathbb{E}\big(n(n-1)\cdots(n-k+1)\big), \tag{11}$$

where $n := n(t, x, \Gamma, \omega_m)$. In particular, the first quenched moment is solution to

$$\begin{aligned} \frac{\partial m_1}{\partial t}(t, x, \Gamma, \omega_m) &= \kappa\left(\mathcal{L}_a m_1\right)(t, x, \Gamma, \omega_m) + V(x, \omega_m) m_1(t, x, \Gamma, \omega_m) \\ &= \left(\mathcal{H} m_1\right)(t, x, \Gamma, \omega_m), \end{aligned} \tag{12}$$

where the Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H} \equiv \mathcal{H}(\omega_m) := \kappa \mathcal{L}_a + V(x, \omega_m), \tag{13}$$

with the random potential

$$V(x) \equiv V(x, \omega_m) := \lambda(x, \omega_m) - \mu(x, \omega_m). \tag{14}$$

The initial condition for Eq. (12) is

$$m_1(0, x, \Gamma) \equiv \mathbb{1}_\Gamma(x) := \begin{cases} 1, & \text{if } x \in \Gamma, \\ 0, & \text{if } x \notin \Gamma. \end{cases} \tag{15}$$

Equations for higher moments $m_k(t, x, \Gamma, \omega_m)$, $k \geq 2$, use the same Hamiltonian as in Eq. (13).

Do these population models provide particle field solutions converging to a statistical equilibrium? When rates are constant and equal on the lattice ($\lambda(x) \equiv \mu(x) \equiv \lambda_0 > 0$ for all $x \in \mathbb{Z}^d$), and the underlying random walk with generator $\kappa\mathcal{L}_a$ is transient, Molchanov and Whitmeyer (2017) proved that the factorial moments $m_k(t,x,\Gamma)$ in Eq. (11) converge asymptotically. This implies that, in the critical contact model with transient walk, the distribution of the population field $N(t,\Gamma) = \sum_{x\in\mathbb{Z}^d} n(t,x,\Gamma)$ with finite sets $\Gamma \subset \mathbb{Z}^d$ converges asymptotically in law to a steady state, which is a stationary ergodic field. Chernousova and Molchanov (2019) also proved the existence of a steady state when, in a critical case of branching process, each particle produces an arbitrary total number of offspring with distribution of jumps that is symmetric around the parent particle. Yarovaya (2013) and Bulinskaya (2021) and references therein have analyzed other aspects of branched random walks with heterogeneous (but non-random) birth-death processes.

Balashova, Molchanov, and Yarovaya (2021) proved that if the underlying random walk is recurrent —it returns an infinite number of times to the initial lattice site almost surely, (Durrett, 2010: p. 190)—, then, asymptotically, the population size $N(t,\Gamma)$ solution in the critical contact model is intermittent, as clusters emerge and almost all particles are concentrated near large enough sets where the potential is positive. In particular, $N(t,\Gamma) \xrightarrow[t\to\infty]{\mathbb{P}-\text{a.s.}} 0$ for any finite set $\Gamma \subset \mathbb{Z}^d$.

The contact models of Molchanov and Whitmeyer (2017) and Balashova, Molchanov, and Yarovaya (2021) in homogeneous deterministic environment and satisfying the criticality condition $\lambda(x) \equiv \mu(x) \equiv \lambda_0 > 0$ for all $x \in \mathbb{Z}^d$ lack realism. Moreover, they are unstable to small perturbations of the parameters.

Our model introduced in section 2 extends the contact population model to the case of random duplication and mortality rates, as in Eq. (1). If the distribution of the random potential in Eq. (14) has bounded positive density on some interval (v_-, v_+), then, according to the theory of random Schrödinger operators (Aizenman and Warzel, 2015), the spectrum Sp of $\mathcal{H}(\omega_m)$ satisfies

$$\mathrm{Sp}\left(\mathcal{H}(\omega_m)\right) = \mathrm{Sp}\left(\kappa\mathcal{L}_a\right) + \mathrm{range}(V(\cdot,\omega_m)) = [-\kappa\alpha + v_-, v_+], \tag{16}$$

where range$V(\cdot,\omega_m)$ denotes the range of $V(\cdot,\omega_m)$. Here

$$\mathrm{Sp}\,(\kappa\mathcal{L}_a) \equiv \mathrm{range}\big(\kappa\hat{\mathcal{L}}_a\big) = \Big(\kappa\Big(\min_{k\in[0,2\pi)^d}\hat{a}(k)-1\Big),0\Big) \equiv [-\kappa\alpha,0], \tag{17}$$

where $\hat{a}(k)=\sum_{x\in\mathbb{Z}^d}e^{i(k,x)}a(x)$ and $\hat{\mathcal{L}}_a(k)=\hat{a}(k)-1\le 0$, $k\in[0,2\pi)^d$, are the associated Fourier transforms.

In the spectral theory of random operators $\mathcal{H}(\omega_m)$ (Molchanov, 2012; Aizenman and Warzel, 2015), at least near the edges $-\kappa\alpha+v_-$ and v_+ of the interval expressed in Eq. (16), the point spectrum of the random operator $\mathcal{H}(\omega_m)$ has exponentially decreasing eigenfunctions. These eigenfunctions are associated with the extreme values of the random potential $V(x)$. These spectral properties of $\mathcal{H}(\omega_m)$ are the basis of Theorems 1 and 2 below.

Albeverio et al. (2000) showed that quenched and annealed moments of all orders grow in a non-regular and intermittent manner. However, the $\mathbb{P}_m$-almost sure (for all realizations of the random environment expressed in Eq. (1)) behavior of the field $N(t,y)$ at $t\to\infty$ cannot always be characterized by its moments.

Here, we prove that the branching random walk model in non-degenerate stationary random environment (that is, with time-independent potential $V(x)=\lambda(x)-\mu(x)$ with strictly positive variance) has no steady state.

2.4 Results

The values of the potential $V\equiv V(x):=\lambda(x)-\mu(x)$ are independent and identically distributed at sites $x\in\mathbb{Z}^d$. Our first result is that, if the distribution of the potential V allows strictly positive values, the contact model displays an exponential growth.

Theorem 1. *If*

$$\mathbb{P}_m\big(V>0\big)>0, \tag{18}$$

then, for a $\mathbb{P}_m$-almost-sure realization of the random environment in Eq. (1), the particle *field* $(N(t,y))_{y\in\mathbb{Z}^d}$*, grows exponentially: there is a* $\gamma > 0$ *such that, for each* $y \in \mathbb{Z}^d$*,*

$$\mathbb{P} - a.s., \qquad \liminf_{t\to\infty} \frac{\ln N(t,y)}{t} \geq \gamma. \tag{19}$$

In subsection 3.1 below, we show that the property in Eq. (19) holds for any γ such that $\mathbb{P}_m\big(V > \gamma\big) > 0$.

If V now is almost surely negative or null except when it is almost surely zero, then the particle field $N(t,y)$, $y \in \mathbb{Z}^d$, goes extinct:

Theorem 2. *If*

$$\mathbb{P}_m\,(V \leq 0) = 1 \qquad but \qquad \mathbb{P}_m\,(V < 0) > 0, \tag{20}$$

then there exists a constant $c > 0$ *and, for all* $x \in \mathbb{Z}^d$*, there is a* $t_*(x) \in [0,\infty)$*, such that*

$$\mathcal{E}N(t,x) \leq e^{-ct} \qquad \text{for all } t \geq t_*(x). \tag{21}$$

In particular, $\mathcal{P}\,(N(t,x) > 0)$ *decreases exponentially at rate* c *as* $t \to \infty$*.*

The large-time behavior of the particle field $(N(t,x))_{x\in\mathbb{Z}^d}$ is closely related to that of its first quenched moment $M_1(t,x) = \mathbb{E}N(t,x)$, with $M_1(0,x) = 1$ for all $x \in \mathbb{Z}^d$. By the Feynman-Kac representation (Gärtner and Molchanov, 1990: Th. 2.1),

$$M_1(t,x) = \mathbb{E}_x \exp\left(\int_0^t V(X_s)\,ds\right), \tag{22}$$

where $(X_s)_{s\in[0,t]}$ is a random walk with generator $\kappa\mathcal{L}_a$ from Eq. (3) and initial condition $X_0 = x$. Heuristically, under Eq. (18), the main contribution to $M_1(t,x)$ and $N(t,x)$ is expected to result from trajectories that spend enough time in the regions of the lattice where the potential V is uniformly positive. On the other hand, under Eq. (20), the integral in Eq. (22) is always negative or null while most trajectories eventually hit a large enough region of the lattice where the potential V is uniformly negative (subcritical case, Athreya and Ney (1972)), thus forcing the particle field to decrease asymptotically.

By the upper bound expressed in Eq. (21) and Borel-Cantelli lemma (Feller, 1968),

$N(t_k, x) = 0$, $\mathcal{P}$-almost surely, for each fixed $x \in \mathbb{Z}^d$ and each sequence $0 \le t_0 < t_1 < t_2 < \ldots$ with $\sum_{k\ge 0} e^{-ct_k} < \infty$. The same holds for $N(t, B) = \sum_{x\in B} N(t, x)$, where $B \subset \mathbb{Z}^d$ is finite. The overall vanishing of the particle field at all times is more subtle.[1]

3 Proofs

The condition in Eq. (18) and the independence of the environment at sites $x \in \mathbb{Z}^d$ $\mathbb{P}_m$-almost-surely generate large clusters where the branching process is supercritical (section 3.1). This causes the population field located in these clusters to increase exponentially, irrespectively of the environment elsewhere. The result is exponential growth of the population everywhere in $\mathbb{Z}^d$, as stated in Theorem 1.

On the other hand, governed by Eq. (20), for each $x \in \mathbb{Z}^d$, the annealed moment $\mathcal{E}N(t, x)$ of the particle field $N(t, x)$ vanishes asymptotically (section 3.2).

3.1 Proof of Theorem 1

Our proof is based on stochastically lower bounding the process $N(t, x)$ by another population process for which the exponential growth in Eq. (19) is easier to prove. A key ingredient for the latter is the fact that under the condition in Eq. (18), the analogue of the operator $\mathcal{H}$ from Eq. (13) has a principal (also called dominant) eigenvalue, which is both positive and strictly above the rest of the spectrum of $\mathcal{H}$.

3.1.1 Comparison

For fixed $x_0 \in \mathbb{Z}^d$, $\big(n(t, y)\big)_{t\ge 0, y\in\mathbb{Z}^d}$ is the sub-population branching process with the same rates as $N(t, .)$ in section 2 but starting from a single particle at x_0:

$$n(0, y) = \mathbb{1}_{x_0}(y) = \begin{cases} 1, & y = x_0, \\ 0, & y \ne x_0. \end{cases} \tag{23}$$

[1] Combining a suitable discrete version (Antal, Peter, 1995) of Sznitman (1998)'s method of enlargement of obstacles with the population survival analysis from Engländer and Peres (2017), yields the $\mathbb{P}_m$-almost-sure asymptotic behavior of the field $(N(t, x))_{x\in\mathbb{Z}^d}$ in the subcritical regime in Eq. (20). This extension is beyond the scope of this article.

Branching random walks $n(t, y)$ possess the stochastic monotonicity property, which is the existence of a stochastic lower bound as in Eq. (51) below, the proof of which we present in section A.1:

Lemma 3. *For arbitrary $x_0 \in \mathbb{Z}^d$ and duplication and mortality rates $\big(\lambda^\epsilon(x), \mu^\epsilon(x)\big)_{x\in\mathbb{Z}^d}$, $\epsilon \in \{+, -\}$, satisfying*

$$\lambda^+(x) \geq \lambda^-(x) \quad \text{and} \quad \mu^+(x) \leq \mu^-(x) \tag{24}$$

for all $x \in \mathbb{Z}^d$, the associated branching random walks $(n^\epsilon(t, y))_{y\in\mathbb{Z}^d}$, $\epsilon = +, -$ are the total numbers of particles located at site y at time t, with initial condition expressed in Eq. (23), can be defined on a common probability space in such a way that the inequality

$$n^+(t, y) \geq n^-(t, y) \tag{25}$$

holds for all $t \geq 0$ and all $y \in \mathbb{Z}^d$.

Lemma 3 allows for stochastically lower-bounding the population process $N(t, y)$ using a simpler branching random walk for which the exponential growth of Theorem 1 is easier to prove.

Given rates $\big(\lambda(x), \mu(x)\big)_{x\in\mathbb{Z}^d}$, $\big(N(t, y)\big)_{t\geq 0, y\in\mathbb{Z}^d}$ and $\big(n(t, x, y)\big)_{y\in\mathbb{Z}^d}$ are the associated branching random walk and sub-populations, as in Eq. (6). By continuity of probability, the condition in Eq. (18) implies that there exist positive constants δ_1, $\widehat{\lambda}$, and $\widehat{\mu}$ such that

$$\mathbb{P}_m\big(\lambda(x) > \widehat{\lambda}, \widehat{\mu} > \mu(x)\big) > 0, \quad \text{where} \quad \widehat{\lambda} - \widehat{\mu} > \delta_1 > 0. \tag{26}$$

By independence of $\big(\lambda(x), \mu(x)\big)$ from one lattice site $x \in \mathbb{Z}^d$ to another, for each finite $Q \subset \mathbb{Z}^d$,

$$\mathbb{P}_m\big(\lambda(x) > \widehat{\lambda}, \widehat{\mu} > \mu(x) \text{ for all } x \in Q\big) > 0. \tag{27}$$

For simplicity, we choose Q in Eq. (27) as a lattice cube whose appropriate size we will evaluate in section 3.1.3. Given such Q, define

$$\lambda_Q(x) = \begin{cases} \widehat{\lambda}, & \text{if } x \in Q, \\ 0, & \text{if } x \notin Q, \end{cases} \qquad \mu_Q(x) = \begin{cases} \widehat{\mu}, & \text{if } x \in Q, \\ c_0, & \text{if } x \notin Q, \end{cases} \tag{28}$$

where $c_0 > 0$ is the common finite upper bound expressed in Eq. (2), and $\big(N_Q(t, y)\big)_{t\geq 0, y\in\mathbb{Z}^d}$ and $\big(n_Q(t, x, y)\big)_{t\geq 0, x, y\in\mathbb{Z}^d}$ are the associated branching random walk and sub-populations:

$$N_Q(t, y) := \sum_{x\in Q} n_Q(t, x, y). \tag{29}$$

As the inequalities

$$\lambda_Q(x) \leq \lambda(x) \quad \text{and} \quad \mu_Q(x) \geq \mu(x) \tag{30}$$

hold for all $x \in \mathbb{Z}^d$, that is, the coupling condition in Eq. (24) is satisfied, Lemma 3 allows for coupling the original branching random walk and the Q-modified process (with rates from Eq. (28)) to provide the inequalities

$$N_Q(t, y) \leq N(t, y) \qquad \text{and} \qquad n_Q(t, x, y) \leq n(t, x, y) \tag{31}$$

for all $t \geq 0$ and all x, $y \in \mathbb{Z}^d$. This is consistent with the intuition that the harsher environment $\big(\lambda_Q(x), \mu_Q(x)\big)_{x\in\mathbb{Z}^d}$ suppresses the population growth faster.

The potential $V_Q(x) := \lambda_Q(x) - \mu_Q(x)$ of the Q-modified process satisfies

$$V_Q(x) = \begin{cases} -c_0, & \text{if } x \notin Q, \\ \widehat{\lambda} - \widehat{\mu}, & \text{if } x \in Q. \end{cases} \tag{32}$$

In particular, $V_Q(x) > \delta_1 > 0$ in Q and $V_Q(x) < 0$ outside Q.

As in section 2.3, the time variation of the associated factorial moments is expressed in terms of the Schrödinger operator

$$\left(\mathcal{H}_Q\phi\right)(x) := \big(\kappa\mathcal{L}_a + V_Q(x)\big)\phi(x), \tag{33}$$

as in Eq. (13). Moreover,

$$V_Q(x) = -c_0 + \Lambda \sum_{z \in Q} \delta(x - z) \qquad \text{with} \qquad \Lambda := \widehat{\lambda} - \widehat{\mu} + c_0, \tag{34}$$

so that the operator $\mathcal{H}_Q$ is analogous to the operator in the branching random walk with finitely many centers of generation (Molchanov and Yarovaya, 2012a,b), though in our case its spectrum is shifted by $-c_0 < 0$.

We now prove the existence of the strictly positive principal eigenvalue of the generator of the Q-modified process.

3.1.2 An auxiliary model of supercritical branching random walk in finite domain

To show that the auxiliary model with branching rates $\big(\lambda_Q(x), \mu_Q(x)\big)_{x \in \mathbb{Z}^d}$ and the jump generator $\kappa\mathcal{L}_a$ display pointwise exponential growth of the population field $(N(t,y))_{y \in \mathbb{Z}^d}$, we again use Lemma 3 to stochastically compare the model of Eq. (3) and (28) to its version restricted to a finite domain in $\mathbb{Z}^d$. The latter is a continuous-time multi-type branching process (Athreya and Ney, 1972: V): define a type of a particle by its location site. Then the size of the population of particles of type y is equal to the size of the population of particles located at site y. Under the conditions of irreducibility of the jump kernel $a(.)$, the $\mathbb{P}$-almost-surely exponential growth of population (restricted to a finite domain) results from a continuous-time version of Kesten and Stigum (1966)'s theorem.

For an integer $\ell \geq 0$, Q_ℓ is the lattice cube $[-\ell, \ell]^d \cap \mathbb{Z}^d$ of side length $2\ell+1$. The cube Q_ℓ is centered at the origin. We write $Q_\ell(z)$ for the image $z + Q_\ell$ of Q_ℓ by translation, with center at $z \in \mathbb{Z}^d$. Given positive integers ℓ_1, ℓ_2, and ℓ_3 to be fixed below, define

$$L := \ell_1 + \ell_2, \qquad \widetilde{L} := \ell_1 + \ell_2 + \ell_3 \equiv L + \ell_3, \tag{35}$$

$$Q := Q_L, \qquad \text{and} \qquad \widetilde{Q} := Q_{\widetilde{L}}. \tag{36}$$

The restricted version of the model of Eq. (3) and (28) is defined using the restricted jump rates $\big(\kappa a(y-x)\big)_{x,y\in\widetilde{Q}}$ and the branching rates $\big(\lambda_{\widetilde{Q}}(x),\mu_{\widetilde{Q}}(x)\big)_{x\in\mathbb{Z}^d}$, with

$$\lambda_{\widetilde{Q}}(x) := \lambda_Q(x) \quad \text{and} \quad \mu_{\widetilde{Q}}(x) := \mu_Q(x) + q_{\widetilde{Q}}(x) \qquad \text{for all } x \in \widetilde{Q}, \tag{37}$$

where

$$q_{\widetilde{Q}}(x) := \kappa \sum_{z \text{ such that } x+z\notin\widetilde{Q}} a(z) \tag{38}$$

is the combined rate for a particle at site $x\in\widetilde{Q}$ to jump outside $\widetilde{Q}$ in the model of Eq. (3) and (28). The restricted model is a version of the model of Eq. {(3), (28)} with mortality rate $\mu_Q(x)$ set to infinity at all lattice sites $x\in\mathbb{Z}^d\setminus\widetilde{Q}$.

By Lemma 3, given the initial condition in Eq. (23) with arbitrary $x_0\in\widetilde{Q}$, the population size of the restricted model satisfies $\tilde{n}(t,x_0,y)=0$ for all $y\in\mathbb{Z}^d\setminus\widetilde{Q}$. By Lemma 3, we couple it stochastically to the population size of the model in Eq. (3) and (28), so that the bounds

$$n_Q(t,x_0,y) \geq \tilde{n}(t,x_0,y) \tag{39}$$

hold for all $y\in\mathbb{Z}^d$ and $t\geq 0$. Indeed, a particle located at $x\in\widetilde{Q}$ lives for an exponentially-distributed time with parameter

$$\Lambda_Q(x) := \lambda_Q(x) + \mu_Q(x) + \kappa > 0. \tag{40}$$

Then, it either

- splits into two particles located at $x\in\widetilde{Q}$ with probability $\lambda_Q(x)/\Lambda_Q(x)$,
- becomes a single particle located at $y\in\widetilde{Q}$ with probability $\kappa a(y-x)/\Lambda_Q(x)$,
- dies out (producing no particles) with probability $(\mu_Q(x)+q_{\widetilde{Q}}(x))/\Lambda_Q(x)$.

Following Georgii and Baake (2003), $N_{x,y}$, $x,y\in\widetilde{Q}$, is the population size at $y\in\widetilde{Q}$ generated after a single step by a particle located at $x\in\widetilde{Q}$. Its expectation satisfies

$$\mathsf{E}N_{x,y} = \frac{2\lambda_Q(x)}{\Lambda_Q(x)}\,\delta_{xy} + \frac{\kappa a(y-x)}{\Lambda_Q(x)}\,(1-\delta_{xy}), \tag{41}$$

where δ_{xy} is the Kronecker function δ. The long-term behavior of this multitype branching process is controlled by the Perron eigenvalue $\tilde{\gamma}$ of the generator matrix $\mathcal{G} = (g_{x,y})_{x,y\in\widetilde{Q}}$ (Georgii and Baake, 2003: Eq. (2.3); Athreya and Ney, 1972: 202), where

$$g_{x,y} := \Lambda_Q(x)\big(\mathsf{E}N_{x,y} - \delta_{xy}\big) = \begin{cases} \lambda_{\widetilde{Q}}(x) - \mu_{\widetilde{Q}}(x) - \kappa \sum_{y\in\widetilde{Q}} a(y-x), & \text{if } x = y, \\ \kappa a(y-x), & \text{if } x \neq y. \end{cases} \tag{42}$$

Moreover,

$$V_{\widetilde{Q}}(x) := \lambda_{\widetilde{Q}}(x) - \mu_{\widetilde{Q}}(x) \equiv V_Q(x) - q_{\widetilde{Q}}(x) \qquad \text{for all } x \in \widetilde{Q}. \tag{43}$$

The following lemma is key to our analysis of the supercritical regime.

Lemma 4. *Consider $a(\,\cdot\,)$ the jump kernel from Eq.* (4), *$\kappa > 0$ an arbitrary jump rate, and finite cubes Q and $\widetilde{Q} \supset Q$ in $\mathbb{Z}^d$ defined as in Eq.* (35) *and* (36). *Define the deterministic matrix $\widehat{G}_{\widetilde{Q}} = (\hat{g}_{x,y})_{x,y\in\widetilde{Q}}$ as*

$$\hat{g}_{x,y} = \begin{cases} \hat{v}_x - \kappa, & \textit{if } x = y, \\ \kappa a(y-x), & \textit{if } x \neq y, \end{cases} \tag{44}$$

where $\hat{v}_x$, $x \in \widetilde{Q}$, are real numbers. $\delta > 0$ and $\eta > 0$ are arbitrary constants.

There exist positive integers ℓ_1, ℓ_2, and ℓ_3 such that, for the generated cubes Q and $\widetilde{Q} \supset Q$, if $\hat{v}_x \geq \delta + \eta$ for all $x \in Q$, then the principal eigenvalue, which is real, of $\widehat{G}_{\widetilde{Q}}$ is strictly greater than $\delta > 0$. Furthermore, given such Q and $\widetilde{Q}$, the same result holds for each extended matrix $\widehat{G}_W = (\hat{g}_{x,y})_{x,y\in W}$ with entries as in Eq. (44), *where the cube $\widetilde{Q}$ is replaced by any finite lattice domain $W \supset \widetilde{Q}$.*

The proof of Lemma 4 is based upon the min-max theorem (Appendix A.2). We use the uniform deterministic bound expressed in Lemma 4 to prove Theorem 1.

3.1.3 Almost-sure population growth

Recall that the positive potential condition in Eq. (18) implies that there is a $\delta > 0$ such that $\mathbb{P}_m\big(V > \delta\big) > 0$. By continuity of probability, there exists a $\eta > 0$ such that this bound can be improved to

$$\mathbb{P}_m\big(V > \delta + \eta\big) > 0. \tag{45}$$

As in Eq. (26), this implies the existence of positive $\widehat{\lambda}$ and $\widehat{\mu}$ such that

$$\mathbb{P}_m\big(\lambda(x) > \widehat{\lambda}, \widehat{\mu} > \mu(x)\big) > 0, \quad \text{where} \quad \widehat{\lambda} - \widehat{\mu} > \delta + \eta > 0. \tag{46}$$

For the rest of this section, fix such δ, η, $\widehat{\lambda}$, and $\widehat{\mu}$.

ℓ_1, ℓ_2, and ℓ_3 are the lengths defined in Lemma 4. $L := \ell_1 + \ell_2$ and $\widetilde{L} := L + \ell_3$ are the scales expressed in Eq. (35). Given such L and $\widetilde{L}$, $\widetilde{\mathbb{Z}}^d := (2\widetilde{L} + 1)\mathbb{Z}^d$ is the integer sub-lattice of step $2\widetilde{L}+1$ in each direction. For each shifted cube $Q_L(z) \equiv z + Q_L$, $z \in \widetilde{\mathbb{Z}}^d$, as in Eq. (36), consider the event

$$A_L(z) := \big\{\lambda(x) > \widehat{\lambda}, \widehat{\mu} > \mu(x) \quad \text{for all} \quad x \in Q_L(z)\big\}. \tag{47}$$

Because the $\lambda(x)$ are independent from one lattice site $x \in \mathbb{Z}^d$ to another, likewise for $\mu(x)$, the probability of the event $A_L(z)$ is

$$\mathbb{P}_m\big(A_L(z)\big) \equiv \Big(\mathbb{P}_m\big(\lambda(x) > \widehat{\lambda}, \widehat{\mu} > \mu(x)\big)\Big)^{|Q_L|} > 0, \tag{48}$$

where $|Q_L| = (2L+1)^d$ is the total number of lattice sites in the lattice cube Q_L. By Borel-Cantelli's lemma, the cardinality of

$$\widetilde{\mathcal{A}} := \big\{z \in \widetilde{\mathbb{Z}}^d : A_L(z) \text{ holds } \big\} \tag{49}$$

is $\mathbb{P}_m$-almost-surely infinite.

Given a realization of the environment, fix $z \in \widetilde{\mathcal{A}}$ and consider the corresponding integer-lattice branching random walk

$$N_Q^{(z)}(t,y) := \sum_{x\in\mathbb{Z}^d} n_Q^{(z)}(t,x,y)\; x \in \mathbb{Z}^d, \tag{50}$$

with rates from Eq. (28), where (z) indicates that $Q \equiv Q(z)$. As in section 3.1.1, Lemma 3 implies the stochastic bounds

$$N_Q^{(z)}(t,y) \le N(t,y) \qquad \text{and} \qquad n_Q^{(z)}(t,x,y) \le n(t,x,y) \tag{51}$$

for all $t \ge 0$ and all $x,\, y \in \mathbb{Z}^d$.

On the other hand, given $z \in \widetilde{\mathcal{A}}$ fixed in Eq. (49), consider branching random walks $\widetilde{N}_Q^{(z)}(t,y)$ and their associated sub-populations $\tilde{n}_Q^{(z)}(t,x,y)$ as in section 3.1.2. It represents the branching random walk in Eq. (50) restricted to $\widetilde{Q}(z)$, namely where each particle dies when jumping from $\widetilde{Q}(z)$. By Lemma 3, both processes $N_Q^{(z)}(t,y)$ and $\widetilde{N}_Q^{(z)}(t,y)$ can be coupled such that sub-population sizes are ordered as

$$\tilde{n}_Q^{(z)}(t,x,y) \le n_Q^{(z)}(t,x,y) \tag{52}$$

for all $t \ge 0$, $x \in \widetilde{Q}(z)$, and $y \in \mathbb{Z}^d$. $\tilde{n}_Q^{(z)}(t,x,y) \equiv 0$ for all $y \notin \widetilde{Q}(z)$.

The generator $\widehat{G}_{\widetilde{Q}(z)}$ of the process restricted to $\widetilde{Q}(z)$ satisfies the conditions of Lemma 4 and so its principal eigenvalue $\tilde{\gamma}(z)$ is strictly positive with $\tilde{\gamma}(z) > \delta > 0$. By irreducibility of the jump kernel $a(\,\cdot\,)$, the generator $\widehat{G}_{\widetilde{Q}(z)}$ is *positive regular* in the sense of Athreya and Ney (1972: 202) and Georgii and Baake (2003: 1093). By the continuous-time version of Kesten-Stigum theorem (Georgii and Baake, 2003: Theorem 2.1), there is a random variable $W = W(z) \ge 0$ such that, $\mathbb{P}$-almost surely,

$$\tilde{n}_Q^{(z)}(t,x,y)\, e^{-\tilde{\gamma}(z)t} \to W\pi_y \tag{53}$$

as $t \to \infty$, where $\pi = (\pi_y, y \in \widetilde{Q}(z))$ is the (strictly positive) left eigenvector of the generator $\widehat{G}_{\widetilde{Q}(z)}$ associated with the eigenvalue $\tilde{\gamma}(z) > 0$. For different $z \in \widetilde{\mathcal{A}}$, the random variables $W(z)$ are independent and identically distributed, with

$$q \equiv q_W^{(z)} := \mathbb{P}\big(W(z) > 0\big) > 0, \tag{54}$$

due to the branching regularity condition (Georgii and Baake, 2003: Eq. (2.4)):

$$\mathbb{E}\big(N_{xy} \log N_{xy}\big) < \infty \qquad \text{for all} \quad x, y \in \widetilde{Q}(z), \tag{55}$$

where N_{xy} is the total number of particles located at y resulting from a single split of one particle located at site x. In addition, $\mathbb{P}($ the process survives$|W(z) > 0) = 1$. Being solely determined by the dynamics of the process in $\widetilde{Q}(z)$, the survival events $\{W(z) > 0\}$ are independent from one site $z \in \widetilde{\mathcal{A}}$ to another. As a result, the cardinality of

$$\big\{z \in \widetilde{\mathcal{A}} : W(z) > 0\big\} \tag{56}$$

is $\mathbb{P}_m \times \mathbb{P}$-almost surely infinite. Thanks to the last item of Lemma 4, the key result of Eq. (19) of Theorem 1 follows.

3.2 Proof of Theorem 2

Our argument is that, under Eq. (20), $\mathcal{P}$-almost surely, the integral $\int_0^t V(X_s)\, ds$ in Eq. (22) tends to $-\infty$ linearly fast. Therefore, the first annealed moment $\mathcal{E}N(t,x)$ vanishes exponentially when $t \to \infty$.

Given the condition in Eq. (20), there exists $\delta_2 > 0$ with

$$\varepsilon := \mathbb{P}_m\big(V \leq -\delta_2\big) > 0. \tag{57}$$

We show that the upper bound expressed in Eq. (21) holds for each $c \in (0, \frac{\varepsilon}{4}\delta_2)$. By Eq. (57), the independent random variables

$$\xi_y := \mathbb{1}_{V(y) \leq -\delta_2}(\omega_m) \tag{58}$$

have common Bernoulli distribution with success probability ε.

Fix $x \in \mathbb{Z}^d$ and consider the random sojourn times of the random walk X_s starting from $X_0 = x$:

$$\tau(t,x,y) := \int_0^t \mathbb{1}_y(X_s)\,ds, \qquad y \in \mathbb{Z}^d. \tag{59}$$

For each fixed $t \geq 0$, the positive or null random variables $\tau(t,x,y)$ satisfy

$$\sum_{y\in\mathbb{Z}^d} \tau(t,x,y) \equiv t \tag{60}$$

and the integral in Eq. (22) is upper bounded:

$$\int_0^t V(X_s)\,ds \leq -\delta_2 S_t(x) \quad \text{where} \quad S_t \equiv S_t(x) := \sum_{y\in\mathbb{Z}^d} \tau(t,x,y)\xi_y. \tag{61}$$

By linearity of the expectation, the first $\mathbb{P}_m$-averaged-over-$\omega_m \in \Omega_m$ moment of S_t is

$$\langle S_t \rangle = \sum_{y\in\mathbb{Z}^d} \tau(t,x,y)\langle \xi_y \rangle = \sum_{y\in\mathbb{Z}^d} \tau(t,x,y)\varepsilon = \varepsilon t. \tag{62}$$

The associated variance satisfies

$$\begin{aligned}\big\langle (S_t - \varepsilon t)^2 \big\rangle &= \sum_{y_1,y_2\in\mathbb{Z}^d} \tau(t,x,y_1)\tau(t,x,y_2)\big\langle (\xi(y_1)-\varepsilon)(\xi(y_2)-\varepsilon)\big\rangle \\ &\equiv \sum_{y\in\mathbb{Z}^d} \tau^2(t,x,y)\big\langle (\xi(y)-\varepsilon)^2\big\rangle = \sum_{y\in\mathbb{Z}^d} \tau^2(t,x,y)\varepsilon(1-\varepsilon),\end{aligned} \tag{63}$$

because, by independence of ξ_y from one $y \in \mathbb{Z}^d$ to another, the off-diagonal terms (with $y_1 \neq y_2$) in the sum have zero expectation.

By Eq. (62), the first annealed moment of the sum in Eq. (61) satisfies

$$\mathcal{E} S_t \equiv \mathbb{E}\big\langle S_t \big\rangle = \varepsilon t, \tag{64}$$

so that to upper bound the annealed variance of the sum in Eq. (61):

$$\mathcal{E}(S_t - \varepsilon t)^2 \equiv \mathbb{E}\big\langle (S_t - \varepsilon t)^2 \big\rangle = \varepsilon(1-\varepsilon) \sum_{y \in \mathbb{Z}^d} \mathbb{E}\tau^2(t,x,y), \tag{65}$$

where

$$\begin{aligned} \mathbb{E}\tau^2(t,x,y) &\equiv \int_0^t \int_0^t \mathbb{E}_x\big(\mathbb{1}_y(X_{s_1})\mathbb{1}_y(X_{s_2})\big)\, ds_1\, ds_2 \\ &= 2 \iint_{0 \le s_1 \le s_2 \le t} p(s_1,x,y)p(s_2 - s_1, y, y)\, ds_1\, ds_2, \end{aligned} \tag{66}$$

where the last term results from the symmetry of the random walk X_s, $s \ge 0$, as in Eq. (4), and from the property $\mathbb{E}_x\big(\mathbb{1}_y(X_s)\big) = p(s,x,y)$, which is valid for all $s \ge 0$ and lattice sites x and y. By homogeneity of the random walk X_s, $s \ge 0$, (its transition probabilities satisfy $p(s,y,y) \equiv p(s,0,0)$ for all $s \ge 0$ and $y \in \mathbb{Z}^d$):

$$\begin{aligned} \mathbb{E}\tau^2(t,x,y) &\equiv 2\int_0^t p(s_1,x,y) \left(\int_{s_1}^t p(s_2 - s_1, 0, 0)\, ds_2 \right) ds_1 \\ &\le 2 \int_0^t p(u,0,0)\, du \int_0^t p(s_1,x,y)\, ds_1. \end{aligned} \tag{67}$$

Based on Eq. (65) and on the identity $\sum_{y\in\mathbb{Z}^d} p(s_1,x,y) \equiv 1$, the last inequality yields

$$\mathcal{E}(S_t - \varepsilon t)^2 \le 2\varepsilon(1-\varepsilon)t \int_0^t p(u,0,0)\, du, \tag{68}$$

where the integral is upper bounded due to the following lemma.

Lemma 5. *For all $t > 0$, the transition probability $p(t,x,y)$ of a homogeneous symmetric irreducible random walk on $\mathbb{Z}^d$ satisfies the inequality*

$$p(t,0,0) \le \frac{c}{t^{\frac{d}{2}}}, \tag{69}$$

where $c = c(d) > 0$ is a finite constant.

Proof in section A.3.

As the upper bound expressed in Eq. (69) of the probability $p(t,0,0)$ of return, Inequality (68) becomes

$$\mathcal{E}(S_t-\varepsilon t)^2 \equiv \mathbb{E}\big\langle (S_t-\varepsilon t)^2\big\rangle \le \begin{cases} C_1(\varepsilon)t^{\frac{3}{2}} \text{ if } d=1,\\ C_2(\varepsilon)t\ln t \text{ if } d=2,\\ C_d(\varepsilon)t \text{ if } d\ge 3,\end{cases} \tag{70}$$

where C_d are finite positive constants, written in terms of ε. By Chebyshev's inequality (Feller, 1968: IX, 6; Durrett, 2010: Eq. (1.6.1)), for every $\alpha > 0$,

$$\mathcal{P}\Big(\Big|\frac{S_t}{t}-\varepsilon\Big| > \alpha\Big) \le \alpha^{-2}\mathcal{E}\Big(\frac{S_t}{t}-\varepsilon\Big)^2, \tag{71}$$

where, by Eq. (70), the last expression decreases at least as fast as $t^{-\frac{1}{2}}$ when $t \to \infty$.

For $\alpha = \frac{\varepsilon}{2}$ and $t = t_n := 2^n, n \in \mathbb{N}$,

$$\sum_{n\in\mathbb{N}} \mathcal{P}\Big(\Big|\frac{S_{t_n}}{t_n}-\varepsilon\Big| > \frac{\varepsilon}{2}\Big) < \infty. \tag{72}$$

By the first Borel-Cantelli lemma (Feller, 1968: Lemma VIII.3.2), there exists a random $n^* = n^*(x)$ with $\mathcal{P}\{n^* < \infty\} = 1$ such that

$$|S_{t_n} - \varepsilon t_n| > \frac{\varepsilon}{2} t_n \tag{73}$$

for all $n \ge n^*$. As S_t in Eq. (61) is a non-decreasing function of $t \ge 0$,

$$S_t \ge \frac{\varepsilon}{4} t \qquad \text{for all } t \ge t_{n^*} = 2^{n^*}, \tag{74}$$

which, from Feynman-Kac formula in Eq. (22) and the uniform point-wise upper bound in Eq. (61), yields Eq. (21):

$$\mathcal{E}N(t,x) \equiv \mathcal{E}\exp\Big(\int_0^t V(X_s)\,ds\Big) \le \mathcal{E}\exp\big(-\delta_2 S_t(x)\big) \le \exp\Big(-\frac{\varepsilon\delta_2}{4}t\Big) \tag{75}$$

for all $t \ge t_{n^*}$.

By Markov's inequality (Durrett, 2010: Th. 1.6.4)), if ξ is a random variable such that $\xi \geq 0$ almost surely and $a > 0$, then:

$$\mathbb{P}\big(\xi \geq a\big) \leq \frac{\mathbb{E}\xi}{a}. \tag{76}$$

Then

$$\mathcal{P}\big(N(t,x) > 0\big) \equiv \mathcal{P}\big(N(t,x) \geq 1\big) \leq \mathcal{E}N(t,x), \tag{77}$$

which implies that the probability on the left-hand side decreases exponentially fast. □

Conclusion

The distribution of population sizes governed by a critical branching random walk (also called "contact process") on the d-dimensional lattice $\mathbb{Z}^d$, $d \geq 1$, with constant duplication rate (λ) and mortality rate (μ), if the underlying random walk is transient, converges to a statistical equilibrium (Molchanov and Whitmeyer, 2017).

If, instead of being constant, these rates are random such that the vectors $(\lambda(x), \mu(x))_{x\in\mathbb{Z}^d}$ are independent from one lattice site to another and identically distributed, under the condition that the potential $V(x) = \lambda(x) - \mu(x)$ has a non-degenerate distribution (with nonzero variance), we showed that a steady state no longer exists: If the event $\big\{V(x) > 0\big\}$ has a strictly positive probability, then the contact process is supercritical with population size growing exponentially fast. Alternatively, if $V(x) \leq 0$ with probability one, while the event $\big\{V(x) < 0\big\}$ has strictly positive probability, the population size $N(t,x)$ vanishes asymptotically, for each $x \in \mathbb{Z}^d$. In particular, the annealed —average over events $\omega_m \in \Omega_m$— probability of the event $\big\{N(t,x) > 0\big\}$ decreases exponentially fast.

A Proofs

A.1 Proof of Lemma 3

Our argument uses the coupling technique (Lindvall, 1992) and is similar to that in Chernousova, Hryniv, and Molchanov (2020: Theorem 3).

We proceeded by induction, constructing branching random walks $(n^+(t,y))_{y\in\mathbb{Z}^d}$ and $(n^-(t,y))_{y\in\mathbb{Z}^d}$ on a common probability space, one change at a time, while making sure that the partial order $n^+(t,y) \geq n^-(t,y)$ in Eq. (25) always holds. The latter condition holds for $t = 0$ because, from Eq. (23),

$$n^+(0,y) = n^-(0,y) = \mathbb{1}_{x_0}(y), \qquad \text{for all } y \in \mathbb{Z}^d. \tag{78}$$

If $(n^+(t,y))_{y\in\mathbb{Z}^d}$ and $(n^-(t,y))_{y\in\mathbb{Z}^d}$ have been successfully constructed up to time $t \geq 0$, while preserving the partial order in Eq. (25), consider the almost-surely finite sets

$$Y_0 := \big\{y \in \mathbb{Z}^d : n^-(t,y) > 0\big\} \quad \text{and} \quad Y_1 := \big\{y \in \mathbb{Z}^d : n^+(t,y) > n^-(t,y)\big\}, \tag{79}$$

and denote $k_y^0 := n^-(t,y) \geq 1$ for $y \in Y_0$ and $k_y^1 := n^+(t,y) - n^-(t,y) \geq 1$ for $y \in Y_1$. Y_0 is the support of n^-, and $Y_0 \cup Y_1$ the support of n^+.

For each $y \in Y_0$, consider the independent exponential random variables

$$\begin{gathered}\zeta_{1,y}^0 \sim \mathrm{Exp}(\kappa\, k_y^0), \qquad \zeta_{2,y}^0 \sim \mathrm{Exp}(\lambda_y^-\, k_y^0), \qquad \zeta_{3,y}^0 \sim \mathrm{Exp}(\mu_y^+\, k_y^0), \\ \zeta_{4,y}^0 \sim \mathrm{Exp}\big((\lambda_y^+ - \lambda_y^-)\, k_y^0\big), \qquad \zeta_{5,y}^0 \sim \mathrm{Exp}\big((\mu_y^- - \mu_y^+)\, k_y^0\big),\end{gathered} \tag{80}$$

where Exp denotes the exponential law and $\zeta \sim \mathrm{Exp}(\nu)$ with $\nu \geq 0$ if $P(\zeta > s) = e^{-\nu s}$ for all $s \geq 0$. In particular, $\zeta \sim \mathrm{Exp}(0)$ is almost surely infinite, $P(\zeta = +\infty) = 1$. Likewise, for each $y \in Y_1$, consider the independent exponential random variables

$$\zeta_{1,y}^1 \sim \mathrm{Exp}(\kappa\, k_y^1), \qquad \zeta_{2,y}^1 \sim \mathrm{Exp}(\lambda_y^+\, k_y^1), \qquad \zeta_{3,y}^1 \sim \mathrm{Exp}(\mu_y^+\, k_y^1). \tag{81}$$

Denote

$$\bar{\zeta} := \min\Big(\min\{\zeta^0_{\ell,y} : \ell = 1, \cdots, 5, y \in Y_0\}, \min\{\zeta^1_{\ell,y} : \ell = 1, 2, 3, y \in Y_1\}\Big). \tag{82}$$

Two cases are possible. First, if

$$\bar{\zeta} \equiv \zeta^1_{\ell,y} \qquad \text{for some } \ell \in \{1, 2, 3\} \text{ and } y \in Y_1, \tag{83}$$

then

$$n^\epsilon(s, y) := n^\epsilon(t, y) \qquad \text{for all } t \le s < t + \bar{\zeta} \text{ and } y \in \mathbb{Z}^d, \quad \epsilon = +, -. \tag{84}$$

Set $n^-(t + \bar{\zeta}, y) \equiv n^-(t, y)$ and define the single-particle change in n^+ is as:

- if $\bar{\zeta} \equiv \zeta^1_{1,y}$, then a single particle at y jumps to $y + z \in \mathbb{Z}^d$ with probability $a(z)$, that is, $n^+(t + \bar{\zeta}, x) = n^+(t, x) + \mathbb{1}_{y+z}(x) - \mathbb{1}_y(x)$ for all $x \in \mathbb{Z}^d$;
- if $\bar{\zeta} \equiv \zeta^1_{2,y}$, then a single particle is born at y, that is, $n^+(t+\bar{\zeta}, x) = n^+(t, x)+\mathbb{1}_y(x)$ for all $x \in \mathbb{Z}^d$;
- if $\bar{\zeta} \equiv \zeta^1_{3,y}$, then a single particle at y dies, that is, $n^+(t + \bar{\zeta}, x) = n^+(t, x) - \mathbb{1}_y(x)$ for all $x \in \mathbb{Z}^d$.

Otherwise, necessarily,

$$\bar{\zeta} \equiv \zeta^0_{\ell,y} \qquad \text{for some } \ell \in \{1, 2, 3, 4, 5\} \text{ and } y \in Y_0. \tag{85}$$

Then we let

$$n^\epsilon(s, y) := n^\epsilon(t, y) \qquad \text{for all } t \le s < t + \bar{\zeta} \text{ and } y \in \mathbb{Z}^d,\ \epsilon = +, -, \tag{86}$$

and define the single-particle changes in n^ϵ, $\epsilon = +, -$, at time $t + \bar{\zeta}$ as:

- if $\bar{\zeta} \equiv \zeta^0_{1,y}$, then, in each process, a single particle at y jumps to $y + z \in \mathbb{Z}^d$ with probability $a(z)$, that is, $n^\epsilon(t + \bar{\zeta}, x) = n^\epsilon(t, x) + \mathbb{1}_{y+z}(x) - \mathbb{1}_y(x)$, $\epsilon = +, -$, for all $x \in \mathbb{Z}^d$;

- if $\bar{\zeta} \equiv \zeta^0_{2,y}$, then, in each process, a single particle is born at site y, that is, $n^\epsilon(t + \bar{\zeta}, x) = n^\epsilon(t, x) + \mathbb{1}_y(x)$, $\epsilon = +, -$, for all $x \in \mathbb{Z}^d$;

- if $\bar{\zeta} \equiv \zeta^0_{3,y}$, then a single particle at site y dies in each process, that is, $n^\epsilon(t + \bar{\zeta}, x) = n^\epsilon(t, x) - \mathbb{1}_y(x)$, $\epsilon = +, -$, for all $x \in \mathbb{Z}^d$;

- if $\bar{\zeta} \equiv \zeta^0_{4,y}$, then a single particle is born at site y in n^+ only, that is, $n^+(t + \bar{\zeta}, x) = n^+(t, x) + \mathbb{1}_y(x)$, with $n^-(t + \bar{\zeta}, x) \equiv n^-(t, x)$ for all $x \in \mathbb{Z}^d$;

- if $\bar{\zeta} \equiv \zeta^0_{5,y}$, then a single particle at site y dies in n^- only, that is, $n^-(t + \bar{\zeta}, x) = n^-(t, x) - \mathbb{1}_y(x)$, with $n^+(t + \bar{\zeta}, x) \equiv n^+(t, x)$ for all $x \in \mathbb{Z}^d$.

The random fields $(n^\epsilon(s, \cdot))_{0 \le s \le t+\bar{\zeta}}$, $\epsilon = +, -$, have the correct distributions, while the partial order condition Eq. (25) extends to the whole time interval $[0, t + \bar{\zeta}]$.

Lemma 3 follows by induction. □

A.2 Proof of lemma 4

For an arbitrary vector $f \in \mathbb{R}^{\widetilde{Q}}$, denote

$$(f, f) := \sum_{x \in \widetilde{Q}} (f_x)^2 \qquad \text{and} \qquad (\widehat{G}_{\widetilde{Q}} f, f) := \sum_{x,y \in \widetilde{Q}} \hat{g}_{x,y} f_x f_y. \tag{87}$$

From Eq. (44), we apply the min-max theorem by constructing the cubes Q and $\widetilde{Q} \supset Q$ in $\mathbb{Z}^d$ and a vector $f \in \mathbb{R}^{\widetilde{Q}}$ for which the Rayleigh–Ritz (Horn and Johnson, 1985) quotient $(\widehat{G}_{\widetilde{Q}} f, f)/(f, f)$ is strictly greater than $\delta > 0$.

First, define the first absolute moment M_1 of the kernel $a(z)$, $z \in \mathbb{Z}^d$:

$$M_1 := \sum_{z \in \mathbb{Z}^d} a(z)\, |z|, \tag{88}$$

where $|z|$ is the Euclidean distance to the origin in $\mathbb{Z}^d$. By Eq. (5), $M_1 < \infty$. Given the lattice cube Q_ℓ of side length $2\ell + 1$ with center at the origin $0_{\mathbb{Z}^d}$ of $\mathbb{Z}^d$, $\bar{q}_\ell$ is the single-jump escape rate from $0_{\mathbb{Z}^d}$ to the complement of Q_ℓ,

$$\bar{q}_\ell := \kappa \sum_{z \notin Q_\ell} a(z). \tag{89}$$

Given arbitrary integers $\ell_1 > 0$, $\ell_2 > 0$, and L as in Eq. (35), define

$$f_z := \begin{cases} 1 - \frac{1}{\ell_2}\mathrm{dist}(z, Q_{\ell_1}), & \text{if } \mathrm{dist}(z, Q_{\ell_1}) \leq \ell_2, \\ 0, & \text{otherwise}, \end{cases} \tag{90}$$

where $\mathrm{dist}(z, Q_{\ell_1}) := \min_{w \in Q_{\ell_1}} |z - w|$ is the distance from $z \in \mathbb{Z}^d$ to the cube $Q_{\ell_1} \subset \mathbb{Z}^d$. f_z is a Lipschitz function of z, as, for all $z, w \in \mathbb{Z}^d$:

$$|f_z - f_w| \leq \frac{|z - w|}{\ell_2}. \tag{91}$$

It vanishes outside Q_L, and $0 \leq f_z \leq 1$ for all $z \in \mathbb{Z}^d$. In particular,

$$(f, f) = \sum_{x \in \widetilde{Q}} (f_x)^2 \equiv \sum_{x \in Q_L} (f_x)^2 \leq |Q_L| = (2L + 1)^d. \tag{92}$$

It remains to lower bound the quadratic form $(\widehat{G}_{\widetilde{Q}} f, f)$. First, decompose

$$\begin{aligned} (\widehat{G}_{\widetilde{Q}} f, f) &= \sum_{x \in Q} f_x \hat{g}_{x,x} f_x + \sum_{x \in Q} f_x \sum_{y \in Q \setminus \{x\}} \hat{g}_{x,y} f_y \\ &= \sum_{x \in Q} \big(\hat{g}_{x,x} + \sum_{y \in Q \setminus \{x\}} \hat{g}_{x,y}\big)(f_x)^2 + \sum_{x \in Q} f_x \sum_{y \in Q \setminus \{x\}} \hat{g}_{x,y}(f_y - f_x), \end{aligned} \tag{93}$$

and from Eq. (44), for each $x \in Q := Q_L \equiv Q_{\ell_1 + \ell_2}$,

$$\hat{g}_{x,x} + \sum_{y \in Q \setminus \{x\}} \hat{g}_{x,y} = \hat{v}_x - q_{\widetilde{Q}}(x) \geq \hat{v}_x - \bar{q}_{\ell_3}, \tag{94}$$

where we used the fact that $\bigcup_{x \in Q} Q_{\ell_3}(x) \subset \widetilde{Q}$, which implies that each escape rate $q_{\widetilde{Q}}(x)$, $x \in Q$, from Eq. (38), is upper bounded by $\bar{q}_{\ell_3}$. Also, $\bar{q}_{\ell_3} \to 0$ as $\ell_3 \to \infty$. As a result, for ℓ_3 such that $3\bar{q}_{\ell_3} < \eta$, we get

$$\hat{v}_x - \bar{q}_{\ell_3} > \delta + \frac{2}{3}\eta > 0 \tag{95}$$

for all $x \in Q$. The first sum on the right-hand side of Eq. (93) is lower-bounded by

$$\sum_{x\in Q_{\ell_1}} \big(\hat{g}_{x,x} + \sum_{y\in Q\setminus\{x\}} \hat{g}_{x,y}\big)(f_x)^2 \geq \big(\delta + \frac{2}{3}\eta\big)|Q_{\ell_1}| = \big(\delta + \frac{2}{3}\eta\big)(2\ell_1+1)^d. \tag{96}$$

On the other hand, because of the Lipschitz bound in Eq. (91), the last sum in Eq. (93) satisfies

$$\begin{aligned}\Big|\sum_{x\in Q} f_x \sum_{y\in Q\setminus\{x\}} \hat{g}_{x,y}(f_y - f_x)\Big| &\leq \frac{\kappa}{\ell_2}\sum_{x\in Q}\sum_{y\in Q\setminus\{x\}} a(y-x)|y-x| \\ &\leq \frac{M_1\kappa}{\ell_2}|Q| = \frac{M_1\kappa}{\ell_2}(2L+1)^d,\end{aligned} \tag{97}$$

where M_1 defined in Eq. (88). Assume that $\ell_1 > \ell_2$, so that $2L+1 = 2(\ell_1+\ell_2)+1 < 2(2\ell_1+1)$ and

$$\Big|\sum_{x\in Q} f_x \sum_{y\in Q\setminus\{x\}} \hat{g}_{x,y}(f_y - f_x)\Big| \leq \frac{2^d M_1\kappa}{\ell_2}(2\ell_1+1)^d. \tag{98}$$

For an integer $\ell_2 > 0$ satisfying $2^d M_1 \kappa < \frac{\eta}{3}\ell_2$,

$$(\widehat{G}_{\widetilde{Q}} f, f) \geq \big(\delta + \frac{2}{3}\eta\big)(2\ell_1+1)^d - \frac{\eta}{3}(2\ell_1+1)^d = \big(\delta + \frac{\eta}{3}\big)(2\ell_1+1)^d. \tag{99}$$

Hence the Rayleigh-Ritz quotient satisfies

$$\frac{(\widehat{G}_{\widetilde{Q}} f, f)}{(f,f)} \geq \frac{\big(\delta + \frac{\eta}{3}\big)(2\ell_1+1)^d}{(2(\ell_1+\ell_2)+1)^d} > \delta, \tag{100}$$

provided ℓ_1 is sufficiently large and satisfies $\ell_1 > \ell_2 > 3 \times 2^d M_1 \frac{\kappa}{\eta} > 0$. By the min-max principle, the principal eigenvalue (which is real) $\hat{\gamma}$ of $\widehat{G}_{\widetilde{Q}}$ is

$$\hat{\gamma} := \sup_{f\neq 0_{\mathbb{Z}^d}} \frac{(\widehat{G}_{\widetilde{Q}} f, f)}{(f,f)} > \delta. \tag{101}$$

The first item of Lemma 4 follows.

Alternatively, given a lattice domain $W \supset \widetilde{Q}$, consider the extended matrix $\widehat{G}_W \equiv (\hat{g}_{x,y})_{x,y\in W}$ with entries as in Eq. (44). With the same argument for the vector $f = (f_x)_{x\in W}$ with components from Eq. (90), the principal eigenvalue (which is real) of $\widehat{G}_W$ remains strictly greater than $\delta > 0$ for all such $W \subset \mathbb{Z}^d$. □

For $\overline{G}_{\widetilde{Q}}$ as in Eq. (44) with $\hat{g}_{x,x} \equiv \delta + \eta - \kappa$ for all $x \in Q$, by uniformity of the bound expressed in Eq. (95), there exists $\bar{\gamma} > \delta$ such that the principal eigenvalue $\hat{\gamma}$ of $\widehat{G}_{\widetilde{Q}}$ satisfies $\hat{\gamma} \geq \bar{\gamma} > \delta$ uniformly in the values of other diagonal entries $\bar{g}_{x,x}$, $x \in \widetilde{Q} \setminus Q$.

A.3 Proof of Lemma 5

In terms of the inverse Fourier transform for the random walk with generator $\mathcal{L}_a$ (as in Eq. (3)), the probability $p(t,0,0)$ of return satisfies

$$p(t,0,0) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} e^{t\kappa\hat{\mathcal{L}}_a(k)}\, dk, \tag{102}$$

where, by symmetry postulated in Eq. (4), the Fourier transform $\hat{\mathcal{L}}_a$ is real-valued:

$$\hat{\mathcal{L}}_a(k) \equiv \sum_{z\in\mathbb{Z}^d} \big(\cos(k,z) - 1\big) a(z), \qquad k \in [-\pi,\pi)^d. \tag{103}$$

Following the argument in Spitzer (2001: §7) and using the irreducibility of the random walk, $\gamma > 0$ is such that the characteristic function of jumps satisfies

$$\sum_{z\in\mathbb{Z}^d\setminus\{0\}} \cos(k,z) a(z) \leq 1 - \gamma|k|^2 \tag{104}$$

for all $k \in [-\pi,\pi)^d$. Then

$$\begin{aligned} p(t,0,0) =& \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} e^{t\kappa\hat{\mathcal{L}}_a(k)}\, dk \leq \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} e^{-t\kappa\gamma|k|^2}\, dk \\ \leq& \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t\kappa\gamma|k|^2}\, dk, \end{aligned} \tag{105}$$

from which the bound in Eq. (69) follows. □

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