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Exchange graphs for mutation-finite non-integer quivers

Anna Felikson [a,*], Philipp Lampe [b]

[a] Department of Mathematical Sciences, Durham University, Science Laboratories, South Road, Durham, DH1 3LE, United Kingdom
[b] Abteilung für Mathematik und ihre Didaktik, Europa-Universität Flensburg, Auf dem Campus 1, 24943 Flensburg, Germany

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Skew-symmetric non-integer matrices with real entries can be viewed as quivers with non-integer arrow weights. Such quivers can be mutated following the usual rules of quiver mutation. Felikson and Tumarkin show that mutation-finite non-integer quivers admit geometric realisations by partial reflections. This allows us to define a geometric notion of seeds and thus to define the exchange graphs for mutation classes. In this paper we study exchange graphs of mutation-finite quivers. The concept of finite type generalises naturally to mutation-finite non-integer quivers. We show that for all non-integer quivers of finite type there is a well-defined notion of an exchange graph, and this notion is consistent with the classical notion of exchange graph of integer mutation types coming from cluster algebras. In particular, exchange graphs of finite type quivers are finite. We also show that exchange graphs of rank 3 affine quivers are finite modulo the action of a finite-dimensional lattice (but unlike the integer case, the rank of the lattice is higher than 1 for non-integer quivers).

1. Introduction and summary

Cluster algebras were introduced and studied in a series of four articles by Fomin–Zelevinsky, see [12,13,15], and Berenstein–Fomin–Zelevinsky, see [2]. Many beautiful properties of cluster algebras are visualised by the exchange graph. For example, cluster algebras attached to Dynkin diagrams have finite exchange graphs, see [13], and cluster algebras attached to affine Dynkin diagrams have exchange graphs of linear growth, see Felikson–Shapiro–Tumarkin [7, Theorem 10.8] and Felikson–Shapiro–Thomas–Tumarkin [6, Theorem 1.1].

In Fomin–Zelevinsky's setting, all quivers are integer-valued, that is, they are given by exchange matrices with integer entries. More generally, one can consider quivers whose multiplicities of arrows are not necessarily integers. Such quivers correspond to exchange matrices with real entries. Those objects appeared naturally in the context of Painlevé-VI differential equations, see Dubrovin–Mazzocco [4], and have been studied in the context of root systems and associahedra, see Fomin–Reading [10], almost periodicity, see Lampe [17], dynamical systems, see Machacek–Ovenhouse [19], and representation theory, see Duffield–Tumarkin [5]. However, a concept of cluster variables has not yet been developed for quivers with real weights, making it challenging to introduce a notion of seeds and exchange graphs in the non-integer case.

Fortunately, there are also geometrically flavoured constructions originating from Barot–Geiß–Zelevinsky's concept of a quasi-Cartan companion, see [1], that were developed, for example by Speyer–Thomas [25], Seven [23,24], and Reading [21]. In particular, Felikson–Tumarkin [8,9] used these ideas to investigate mutation-finite real quivers. More precisely, [9] clas-

* Corresponding author.
E-mail addresses: anna.felikson@durham.ac.uk (A. Felikson), philipp.lampe@uni-flensburg.de (P. Lampe).

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sifies mutation-finite real quivers and shows that all non-integer finite mutation type quivers have a geometric realisation where a seed is given by a set of vectors in some quadratic space and a mutation is represented by partial reflection, changing some vectors in the set. These realisations are acting either on a positive definite quadratic vector space (finite type), or on positive semi-definite spaces of corank 1 and 2 (affine and extended affine types respectively). Note that [8] and [9] provide geometric realisations only for individual seeds (with a guarantee that after mutation one gets a geometric realisation of the mutated seed, but without paying attention to the entirety of all seeds in the mutation class and in particular to observing necessary periodicities that are expected when considering integer subquivers).

In this paper we extend the construction to show that it is possible to define seed mutations and exchange graphs for any non-integer quiver of finite type. It turns out that some compatibility checks are required in order to define seed mutations and exchange graphs consistently (i.e. to match with what we have in the classical case of integer quivers). We show that for any real quiver of finite type there exists a way to satisfy these compatibility checks, i.e. to define the realisation of all seeds together and to define the exchange graph. We describe the whole family of compatible ways to define realisations of the mutation class and show that all of those originating from quivers in the same mutation class lead to the same exchange graph.

The case of affine type is more involved. We focus on rank 3 affine mutation classes. A geometric realisation of an affine seed of rank 3 is given by a triangle on Euclidean plane with π-rational angles. We show that for every rank 3 mutation-finite affine quiver there exists a compatible realisation of the entire mutation class, which allows to define an exchange graph. The exchange graph in this case turned out to be a finite graph modulo the action of a lattice of some finite dimension. The crucial difference with the integer case is that the corresponding lattice is no longer one-dimensional, as was the case in all classical affine types.

For higher rank affine quivers and extended-affine quivers, we were not able to verify if there is a compatible way to define the geometric realisation of the entire exchange graph, so we cannot yet define exchange graphs in these settings.

In the geometric realisations considered here, a seed is represented by a set of vectors $v_1, \dots, v_n$ in some quadratic space V. A mutation μ_k in direction v_k then acts by changing the sign of v_k and changing some of the vectors into their reflection images with respect to the plane orthogonal to v_k. Given the corresponding quiver Q, there are two ways to define this action: either by reflecting the vector v_i when there is an arrow from v_i to v_k in Q, or by reflecting v_i when there is an arrow from v_k to v_i. These two ways are equivalent in the context of [8] and [9] (they lead to seeds with equivalent properties), but for defining the exchange graph covering all the seeds it is important to distinguish the two ways to apply. In particular, since we want the mutation to be an involution, we need to use one method or the other depending on the properties of the seed (i.e. both the set of vectors and Q).

A natural way to make this choice in finite type is to use the positivity/negativity of roots in the corresponding (possibly non-crystallographic) root systems. However, there is a freedom in choosing the initial seed with respect to the set of primitive roots. Furthermore, it is not *a priori* clear whether a given choice will be compatible with some expected properties of the exchange graph. Fortunately, we can always find a compatible choice and all compatible choices will lead to defining the same exchange graph:

Theorem A *(Theorem 5.4). Given a seed of finite type (i.e. realised in $\mathbb{S}^n$), there exists a compatible choice of a positive vector, and all compatible choices of positive vectors yield the same exchange graph.*

The crucial step in reconciling our theory with the cluster algebra classical theory is the formulation of a compatibility constraint, in particular in rank 2 (Definition 3.10 and Remark 3.11, based on the results of Lemma 3.7).

For the affine type, it turned out that there is not much freedom in setting a compatible definition of the mutation. We examine rank 3 affine mutation classes and find two possibilities for the compatible definition. One of these definitions turned out to be difficult to handle, but the other one allowed to check that it is indeed a compatible choice, and hence a way to define the exchange graph.

Theorem B *(Theorem 9.4, Corollary 9.10, Theorem 10.1). Given a rank 3 seed of affine type (i.e. realised in $\mathbb{E}^2$ by a triangle with angles $r_1\pi$, $r_2\pi$ and $r_3\pi$ with $r_1, r_2, r_3 \in \mathbb{Q}$), there exists a lattice L such that the following conditions hold:*

(1) The lattice acts on the exchange graph and the quotient of the group action is finite.
(2) The rank of the lattice is equal to $\mathrm{rk}_{\mathbb{Z}}(L) = \varphi(d)/2$ or $\varphi(d)$ where φ is Euler's totient function and d the least common denominator of r_1, r_2, r_3.

To prove Theorem B we construct a line $b \subseteq \mathbb{E}^2$ (called the belt line) that intersects every triangle associated to an acyclic seed. The line is related to a billiard problem in Euclidean geometry. The lattice L encodes the number theory of lengths of vectors parallel to b that translate one seed into another seed. After constructing a number of translation vectors geometrically, we use number theory to prove that these vectors are linearly independent over the integers. More precisely, we work in the cyclotomic number field and apply a Dedekind determinant and a Verlinde formula. The proof of Theorem B (1) also uses the Schwarz–Milnor Lemma, see [22] and [20]. We observe that the geometry of the belt line b also holds for real rank 3 quivers of finite type (realised in $\mathbb{S}^2$) and even for mutation-infinite cases represented by partial reflections in $\mathbb{S}^2$, $\mathbb{E}^2$, or the hyperbolic plane $\mathbb{H}^2$.

As an application of the above results, we compute the growth of the exchange graphs of mutation-finite real quivers. For a natural number n we denote by $gr(n)$ the number of seeds that can be reached from the initial seed by at most n mutations. We are interested in the growth of the function.

Corollary *(Corollary 9.13).* The exchange graph Γ of a mutation-finite real rank 3 quiver distinct from the Markov quiver has polynomial growth. In case when the quiver is affine, the polynomial growth rate is equal to the rank of the lattice L from Theorem B.

For cluster algebras with integer exchange matrices we have a gap between polynomial growth of small degree and exponential growth. More precisely, unless a mutation-finite cluster algebra with an integer exchange matrix has exponential growth, it either has linear growth (in the case of affine cluster algebras) or quadratic or cubic growth (for 3 sporadic families of cluster algebras), see Fomin–Shapiro–Thurston [11, Proposition 11.1] and Felikson–Shapiro–Thomas–Tumarkin [6, Theorem 1.1]. In contrast to the integer case, the polynomial growth rate, i.e. the rank of the lattice L, is larger than 1 for all non-integer cases.

The article is organised as follows. In Section 2 we recall from [8] the notion of geometric realisations. In particular, we use geometry to define a notion of a seed for real exchange matrices. Then, in Section 3, we examine rank 2 mutations, which as a substructure form the core of the compatibility checks we consider in Section 4. In Section 5 we prove Theorem A. The rest of the paper concerns rank 3 affine quivers. In Section 6 we construct initial seeds for affine quivers. In this section we focus on the case where all angles are rational multiples of π with an odd least common denominators; we explore the case of even least common denominators in Section 10. In Section 7 we recall the definition of the cyclotomic field and study number-theoretic properties of sines and cosines of angles in triangles associated to the seeds. Section 8 is devoted to the proof of the linear independence of a set of translation vectors over the integers. In Section 9 we determine the rank of L and present our main result about the structure of the exchange graph, using quasi-isometries. In this section we also introduce growth rates and prove that the exchange graph of an affine rank 3 quiver has polynomial growth. Section 10 contains an analysis of changes for even common denominators.

2. Geometric realisations and their mutations

2.1. Real exchange matrices and their mutations

A crucial notion in the theory of cluster algebras is the mutation of a skew-symmetric matrix. The notion was introduced by Fomin–Zelevinsky [12] for integer matrices, but the same definition makes sense for real matrices. We fix an integer $r \geq 1$ called the *rank*.

Definition 2.1 *(Matrix mutation [12]).* The *mutation* μ_k, $k \in [1, r]$, of a real skew-symmetric $r \times r$ matrix $B = (b_{ij})$ is the skew-symmetric $r \times r$ matrix $\mu_k(B) = B' = (b'_{ij})$ with entries

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\}; \\ b_{ij} + (b_{ik}|b_{kj}| + |b_{ik}|b_{kj})/2 & \text{otherwise.} \end{cases}$$

Such a matrix B is called *exchange matrix*. Note that real skew-symmetric $r \times r$ matrices are in correspondence with weighted quivers on r vertices without loops and 2-cycles (where we introduce an arrow $i \to j$ with weight b_{ij} whenever $b_{ij} > 0$). We say that a skew-symmetric matrix B is *acyclic* if its quiver does not contain oriented cycles.

Notice that matrix mutation is an involution, that is, $(\mu_k \circ \mu_k)(B) = B$ for all B and all k. The following definition is a straightforward generalisation of usual terminology for exchange matrices with integer entries to real entries.

Definition 2.2 *(Matrix mutation classes).* Let $B \in \mathrm{Mat}_{r\times r}(\mathbb{R})$ be an exchange matrix.

(1) A skew-symmetric matrix $B' \in \mathrm{Mat}_{r\times r}(\mathbb{R})$ is called *mutation-equivalent* to B if there exists a sequence $(k_1, \ldots, k_s)$ of indices such that $B' = (\mu_{k_s} \circ \ldots \circ \mu_{k_1})(B)$.
(2) The *mutation class* of B is the set of all B' that are mutation-equivalent to B.
(3) We say that B is *mutation-finite* if the mutation class of B is finite.
(4) We say that B is *mutation-acyclic* if it is mutation-equivalent to an acyclic exchange matrix. Otherwise we say it is *mutation-cyclic*.

2.2. Geometric realisation of seeds and their mutations

For a skew-symmetric matrix with integer entries we can construct a cluster algebra by introducing clusters and cluster variables. For a skew-symmetric matrix with non-integer entries it is not easy to define notions of cluster variables and clusters. Building on work of Seven [23] and Barot–Geiß–Zelevinsky [1], Felikson–Tumarkin [8] consider a notion of geometric seeds and realise their mutations by partial reflections. In the rest of the section we give a short overview of the construction.

Remark 2.3. We point out that in different contexts one can construct different geometric realisations. In this article we restrict ourselves to the construction below.

We fix a *quadratic space* V of dimension r, that is, an r-dimensional real vector space V together with a symmetric bilinear form $(\cdot,\cdot): V \times V \to \mathbb{R}$. The following definitions are motivated by results of Barot–Geiß–Zelevinsky [1, Proposition 3.2] and Seven [23, Theorem 1.4] (with integer entries replaced by real entries).

Definition 2.4 *(Geometric realisations).* Suppose that B is a real skew-symmetric $r \times r$ matrix. An ordered basis $\mathbf{v} = (v_1, \ldots, v_n)$ of V is called a *geometric realisation* of B if the following conditions hold:

(1) For every $i \in \{1, \ldots, n\}$ we have $(v_i, v_i) = 2$.
(2) For two distinct $i, j \in \{1, \ldots, n\}$ we have $|(v_i, v_j)| = |b_{ij}|$.

We identify two geometric realisations that are obtained from each other by a permutation of their entries.

Definition 2.5 *(Admissibility).* A geometric realisation of a matrix B by vectors $\mathbf{v} = (v_1, \ldots, v_n)$ is called *admissible* if for every chordless oriented cycle $(i_1, \ldots, i_s)$ in the quiver of B the number of positive scalar products $(v_{i_j}, v_{i_{j+1}})$ is odd (here indices are to be read mod s), whereas in every chordless non-oriented cycle such a number is even. A geometric realisation of a mutation class is called *admissible* if the realisation of every quiver is admissible.

The terminology of admissible geometric realisations comes from the article by Seven [23]. In the following we always mean admissible geometric realisation when we speak about geometric realisations.

Remark 2.6. Let $B = (b_{ij})$ be a skew-symmetric matrix. A *quasi-Cartan companion* of B is a matrix $A = (a_{ij})$ such that $a_{ii} = 2$ for all i and $|a_{ij}| = |b_{ij}|$. The *Gram matrix* of $\mathbf{v} \in V^r$ is the matrix $((v_i, v_j)_{ij})$. The conditions (1) and (2) in Definition 2.4 imply that the Gram matrix of a geometric realisation of B is a quasi-Cartan companion of B, see [1].

Definition 2.7 *(Seeds).* A *seed* of rank r is a pair $(\mathbf{v}, B)$ where B is a real skew-symmetric $r \times r$ matrix and $\mathbf{v}$ a geometric realisation of B.

Remark 2.8. A realisation of a 2×2 exchange matrix is a linearly independent pair (v_1, v_2) inside a 2-dimensional vector space V satisfying conditions (1) and (2) in Definition 2.4 for all $i, j \in \{1, 2\}$. Given a seed of rank $r > 2$, we can construct a seed of rank 2 by removing $r - 2$ vectors and deleting the corresponding rows and columns of the matrix. This seed is called a *subseed* of rank 2.

As in Lie theory, where the roots of a root system are divided into positive and negative roots, we declare certain vectors $v \in V\backslash\{0\}$ to be *positive*, and certain vectors $v \in V\backslash\{0\}$ to be *negative*. We denote the set of positive elements by $V^+ \subseteq V$ and the set of negative elements by $V^- \subseteq V$. We will give a precise definition of positive and negative vectors later in the text, see Definition 4.2 in Section 4.1. As a rough idea, we demand that the definition be made to recover the classical mutation of seeds when restricting to the case of integer exchange matrices. Moreover, the condition implies that $v \in V^+$ if and only if $-v \in V^-$.

Definition 2.9 *(Mutations of seeds).* Suppose that $(\mathbf{v}, B)$ is a seed of rank r. The *mutation* μ_k, $k \in [1, r]$, of $(\mathbf{v}, B)$ is $\mu_k(\mathbf{v}, B) = (\mu_k(\mathbf{v}), \mu_k(B))$ where $\mu_k(\mathbf{v}) = (v'_1, \ldots, v'_r)$ is defined in the following way. If $v_k \in V^+$, then we put

$$v'_i = \begin{cases} -v_i & \text{if } i = k; \\ v_i & \text{if } (v_k \in V^+ \text{and } i \neq k \text{ and } b_{ik} > 0) \text{ or } (v_k \in V^- \text{and } i \neq k \text{ and } b_{ik} < 0); \\ v_i - (v_i, v_k)v_k & \text{if } (v_k \in V^+ \text{and } i \neq k \text{ and } b_{ik} < 0) \text{ or } (v_k \in V^- \text{and } i \neq k \text{ and } b_{ik} > 0); \end{cases}$$

It is easy to check that the mutation μ_k is well-defined, i.e. $\mu_k(\mathbf{v}, B)$ is again a seed. A related statement for exchange matrices with integer entries is due to Barot–Geiß–Zelevinsky [1, Proposition 3.2].

If v_k is positive, then $-v_k$ is negative. We can conclude that μ_k is an involution, that is, $(\mu_k \circ \mu_k)(\mathbf{v}, B) = (\mathbf{v}, B)$ for all k and $(\mathbf{v}, B)$. The mutation μ_k is called *positive* if $v_k \in V^+$ and it is called *negative* otherwise.

Definition 2.10 *(Exchange graphs).* The *exchange graph* of a seed $(\mathbf{v}, B)$ contains as vertices all seeds $(\mathbf{v}', B')$ obtained from $(\mathbf{v}, B)$ by sequences of mutations. Two vertices $(\mathbf{v}', B')$ and $(\mathbf{v}'', B'')$ are connected by an edge if and only if they are related by a single mutation.

Two seeds $(\mathbf{v}, B)$ and $(\mathbf{v}', B')$ are called *mutation-equivalent* if there exists a sequence of indices $(k_1, \dots, k_s)$ such that $(\mathbf{v}', B') = (\mu_{k_s} \circ \dots \circ \mu_{k_1})(\mathbf{v}, B)$. By definition, if $(\mathbf{v}, B)$ and $(\mathbf{v}', B')$ are mutation-equivalent, then their exchange graphs coincide.

Remark 2.11. (a) The definitions in Section 2.2 are set in such a way that for the case of integer exchange matrices admitting geometric realisations, this newly defined notion of the exchange graph would give the same as the exchange graph constructed from corresponding cluster algebras. We will not give the definition of cluster algebra here (see [12]) as we will not use it later, but we will mention that by the rank of a cluster algebra we mean the size of the corresponding exchange matrix, and by cluster algebras of Dynkin types A, B, C, D, E, F, G we mean the finite cluster algebras associated to the orientations of the corresponding Dynkin diagrams.

(b) The proof of Theorem 5.4 below shows that for the Dynkin types the new definition of the exchange graph (together with the standard Lie theoretic notion of positive and negative vectors) coincides indeed with the one coming from cluster algebras.

3. Seed mutations in rank 2

3.1. The description of mutations of rank 2 seeds

This section is devoted to the mutation of seeds of rank 2. The aim of the subsection is to study the adequacy of distinct notions of positive and negative vectors.

Here a seed is given by a skew-symmetric 2×2 matrix B together with a realisation $\mathbf{v} = (v_1, v_2)$ in a 2-dimensional real vector space V, which we may identify with the vector space $V = \mathbb{R}^2$. After projectivisation, we may identify the space $V/\mathbb{R}^+$ with the circle $\mathbb{S}^1$.

The matrix B is determined by the entry b_{12} thanks to skew-symmetry. In what follows, in regard to [9, Lemma 6.5], we suppose that $b_{12} = 2\cos(\alpha)$ where α is a rational multiple of π, because otherwise we will produce a mutation-infinite seed. To be concrete, let us fix the angle as follows.

Definition 3.1 *(Fundamental angle).* For a rank 2 seed $(\mathbf{v}, B)$ with $b_{12} = 2\cos(\alpha)$, we put $\alpha = \frac{a}{b}\pi$ where $a, b \geq 1$ are natural coprime numbers, and call α the *fundamental angle*.

Notice that $|b_{12}| \leq 2$. We combine this inequality with condition (2) in Definition 2.4. Sylvester's criterion implies that the bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is positive definite. Without loss of generality we may assume that $V = \mathbb{E}^2$ is the Euclidean plane equipped with the standard scalar product $\langle \cdot, \cdot \rangle$.

For $v \in V \backslash \{0\}$ we put $\Pi_v = \{w \in \mathbb{E}^2 \mid \langle w, v \rangle < 0\}$ and $l_v = \{w \in \mathbb{E}^2 \mid \langle w, v \rangle = 0\}$. Notice that Π_v is an (open) half-plane bounded by the line l_v.

The map $v \mapsto \Pi_v$ induces a bijection between the set of vectors $v \in \mathbb{E}^2$ with $\langle v, v \rangle = 2$ and the set of (open) half-planes in $\mathbb{E}^2$ whose boundary line passes through the origin. Because of that we identify the pair $\mathbf{v} = (v_1, v_2)$ with the intersection of the half-planes $\Pi_{v_1} \cap \Pi_{v_2}$. This intersection is a sector given by an angle of size α centered at the origin. The size α determines $|b_{12}|$. To obtain a geometric model of the seed $(\mathbf{v}, B)$ we have to encode the sign of b_{12}, which we do by an arrow. More precisely, we orient (the sides of the sector corresponding to) the vectors $v_1 \to v_2$ if $b_{12} > 0$, $v_2 \to v_1$ if $b_{12} < 0$, and do not connect them if $b_{12} = 0$.

The mutation process is now realised by partial reflections, which means the following. Suppose that the seed $(\mathbf{v}, B)$ is realised by a sector with an angle whose sides s_1 and s_2 correspond to v_1 and v_2 (together with an arrow between v_1 and v_2). Then the mutation of the seed at v_2, according to Definition 2.9, either replaces s_1 with the reflection s_1 across s_2, or leaves it unchanged (depending on the location of v_2 and orientation of the arrow).

The following definition is an obvious attempt to define positive and negative vectors. Under some assumptions it recovers the classical notion of positivity in Lie theory when we think of the vectors v_1 and v_2 as positive simple roots in a finite type root system.

Definition 3.2 *(Positivity via reference point).* Fix $u \in \mathbb{E}^2 \backslash \{0\}$ (we will call u a *reference point*). We declare v to belong to the set V^+ if $\langle u, v \rangle > 0$, and declare v to belong to the set V^- if $\langle u, v \rangle < 0$.

3.2. Long and short periods

Example 3.3 *(Short period).* We consider a geometric realisation $\mathbf{v} = (v_1, v_2)$ of the exchange matrix

$$B = \begin{pmatrix} 0 & 2\cos(\pi/3) \\ -2\cos(\pi/3) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

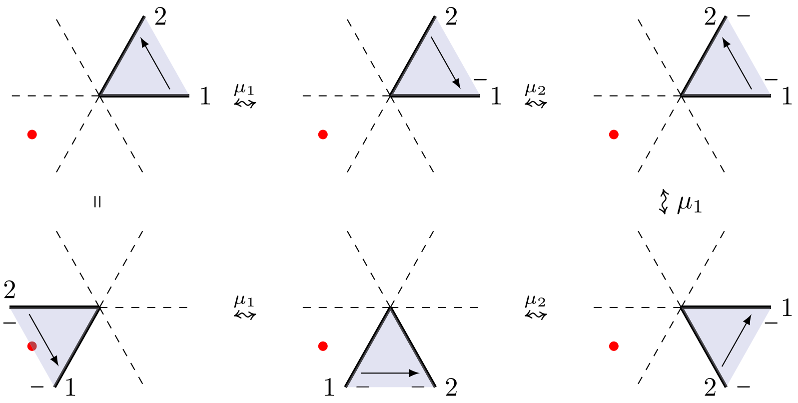

Fig. 1. Geometric description of seeds and mutations in rank 2.

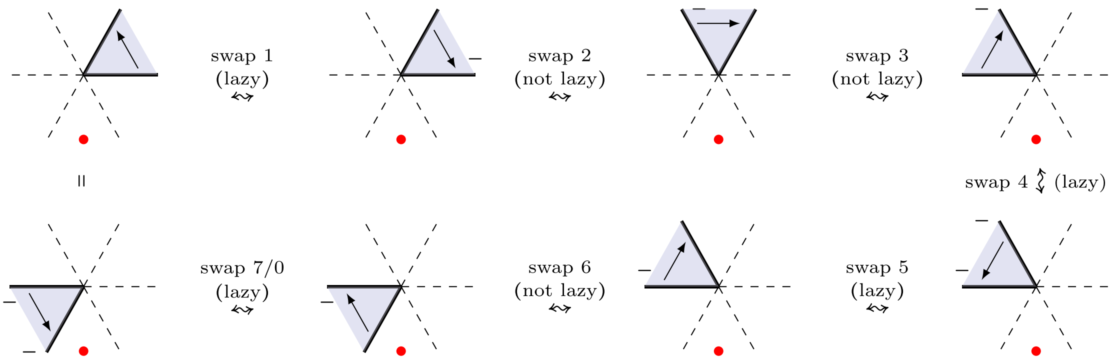

Fig. 2. Seed mutation with period 7.

We draw the seed $(\mathbf{v}, B)$ in the top left of corner of Figs. 1 and 2. Here we visualise the sector $\Pi_{v_1} \cap \Pi_{v_2}$ as shaded regions. The vectors v_1 and v_2 themselves are not drawn explicitly, but we can recover them by considering the two outward normal vectors of length $\sqrt{2}$ to the sides of the sector. The reference point u is shown as a bold dot. Definition 3.2 asserts that v_i, $i \in \{1, 2\}$, is positive if and only if the line l_{v_i} divides the plane into two half-planes such that u lies in the same half-plane as the sector.

Fig. 1 shows the mutation process. Here we mutate at sources when passing through the picture in clockwise direction. During the mutation process sectors with angles $\pi/3$ and $2\pi/3$ occur. Sometimes it is convenient not to draw sectors with obtuse angles, instead we label the sides of the sector: here a minus sign attached to a half-plane Π_{v_i}, $i \in \{1, 2\}$, indicates that we draw $\{w \in \mathbb{E}^2 \mid \langle w, v_i\rangle > 0\}$ instead of $\Pi_{v_i} = \{w \in \mathbb{E}^2 \mid \langle w, v_i\rangle < 0\}$. We obtain a periodic sequence of seeds of period 5. Fig. 3 shows the 5-periodic mutation process for the same seed and the same reference point as Fig. 1, but here sectors are drawn entirely without signs. The figure also shows the exchange graph of the seed, a pentagon, which is an associahedron of type A_2, compare Fomin–Reading [10, Figure 4.4].

Note that when we intersect the half-planes Π_v (with $v \in V\setminus\{0\}$) with the circle $\mathbb{S}^1 \subseteq \mathbb{E}^2$, they become half-circles; similarly, when we intersect the sectors $\Pi_{v_1} \cap \Pi_{v_2}$ (with $v_1, v_2 \in V\setminus\{0\}$) with the circle $\mathbb{S}^1 \subseteq \mathbb{E}^2$, they become arcs and their arc lengths reflect the angles at which the planes intersect. We therefore also call the projectivisation $\mathbb{S}^1 \subset \mathbb{E}^2$ a geometric realisation of $(\mathbf{v}, B)$.

Example 3.4 *(Long period).* Fig. 2 shows the mutation process for the same seed as in Example 3.3 with a different reference point. We obtain a periodic sequence of seeds of period 7. Notice that the matrix B defines a cluster algebra of type A_2, which admits 5 seeds. Hence, the short periods correspond to the periods that occur in cluster algebras (attached to integer exchange matrices), whereas larger periods yield larger exchange graphs. We draw the locus of the reference points u which yield 5-periodic sequences of seeds in Fig. 4.

Our aim is to describe which choices result in short periods, and which ones result in long periods. To do this, we define the *swap* of a seed $((v_1, v_2), B)$ to be the seed $((v_2, v_1), \tau(B))$ where $\tau(B)$ is obtained from B by a simultaneous reordering of the rows and columns given by the permutation (12). The swap is denoted by τ.

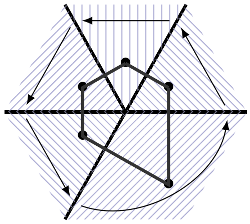

Fig. 3. The construction of the associahedron of type A_2 via geometric mutations.

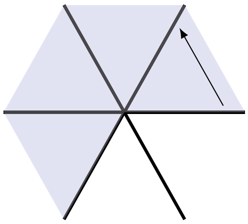

Fig. 4. The locus of reference points (shaded) for which mutation has a short period.

Let us fix an initial seed $(\mathbf{w}, B)$, given by a sector with an angle of size α as in Definition 3.1. For technical reasons it will be convenient to write the pair of vectors as $\mathbf{w} = (w_0, -w_1)$. Without loss of generality we may assume that $b_{12} > 0$. We consider the map $\tau \circ \mu_2$. Note that

$$(\tau \circ \mu_2)((w_0, -w_1), B) = ((w_1, -w_2), B')$$

for some $w_2 \in V$ and a matrix $B' = (b'_{ij})$. Notice that $B' = (\tau \circ \mu_2)(B) = \tau(-B) = B$. In particular, $b'_{12} > 0$. Iteration yields seeds $(\tau \circ \mu_2)^n(\mathbf{w}, B) = ((w_n, -w_{n+1}), B)$ parametrised by natural numbers $n \in \mathbb{N}$ all of which have the same exchange matrix. Explicitly, we have

$$w_{n+1} = \begin{cases} -w_{n-1} + \langle w_{n-1}, w_n \rangle w_n & \text{if } w_n \text{ is positive;} \\ -w_{n-1} & \text{if } w_n \text{ is negative.} \end{cases}$$

Definition 3.5 *(Lazy mutations).* We say that $n \in \mathbb{N}$ is *lazy* if w_n is negative. If this is the case, then we also call the mutation $(w_{n-1}, -w_n) \leftrightsquigarrow (w_n, -w_{n+1})$ lazy.

Remark 3.6.

(1) If n is lazy, then $\langle w_{n+1}, w_n \rangle = -\langle w_n, w_{n-1} \rangle$, else $\langle w_{n+1}, w_n \rangle = \langle w_n, w_{n-1} \rangle$.
(2) Lazy numbers come in pairs. Suppose that $n \in \mathbb{Z}$ is lazy. Then $w_{n+1} = -w_{n-1}$ so that exactly one of the numbers $n-1$ and $n+1$ is lazy.

For every n the angle between w_n and w_0 is an integer multiple of π/b. It follows that the sequence $(\tau \circ \mu_2)^n(\mathbf{w}, B)$ must be periodic.

Lemma 3.7. The period p of $\tau \circ \mu_2$ is equal to $b + 2a$ or to $3b - 2a$. Furthermore, the following holds:

(a) If the angle between the initial vectors w_0 and $-w_1$ is acute (i.e. $2a < b$), then $3b - 2a > b + 2a$. In this case $p = b + 2a$ is the smaller of the possible periods unless w_0 is negative and w_1 is positive. In the latter case $p = 3b - 2a$.
(b) On the other hand, if the angle between the initial vectors w_0 and $-w_1$ is obtuse (i.e. $2a > b$), then $3b - 2a < b + 2a$. In this case $p = 3b - 2a$ is the smaller of the two possible periods if w_0 is negative and w_1 is positive.

Proof. For every $n \in \mathbb{Z}$ we consider the linear map $f_n : V \to V$ defined by $w_{n-1} \mapsto w_n$ and $w_n \mapsto w_{n+1}$. If n is not lazy, then f_n is a rotation around the origin by the angle $\psi \in [0, \pi]$ for which $\langle w_{n-1}, w_n \rangle = 2\cos(\psi)$. Since lazy numbers come in pairs, the value of $\langle w_{n-1}, w_n \rangle$ is the same for all non-lazy numbers n by Remark 3.6 (1). In other words, for every non-lazy number n the map f_n is a rotation by the same angle ψ in the same direction. We have $\psi = \alpha$ or $\psi = \pi - \alpha$, depending on the sign of w_0 and w_1.

To find the period p, we consider several cases. First, suppose that w_1 is negative. In this case, the number 1 is not lazy, and hence f_1 is a rotation so that $\psi = \alpha$. Second, suppose that w_0 is positive. In this case 0 is not lazy, and hence f_0 is a rotation so that $\psi = \alpha$. Now suppose that w_1 is positive and w_0 negative. In this case, 0 and 1 are both lazy. It follows that 2 is not lazy and hence f_2 is a rotation so that $\psi = \pi - \alpha$.

Let us write $\psi = c\pi/b$ with $c \in \{a, b-a\}$. In the case $c = a$, we have to show that $p = b + 2a$; in the case $c = b - a$, we have to show that $p = 3b - 2a$. Therefore, we have to show $p = b + 2c$ in both cases. To this end, we show that there are exactly b non-lazy mutations and exactly c pairs of lazy mutations along a fundamental period of seeds. We denote the number of non-lazy mutations along a period by b', and the number of pairs of lazy mutations by c'. We claim $b' = b$ and $c' = c$.

The composition $f_{n+1} \circ f_n$ is a rotation by π when $(n, n+1)$ is a pair of lazy numbers. Since f_n is a rotation by ψ for every non-lazy number n, we must have $c'\pi + b'\psi = 2\pi m$ for some integer m. The equation implies that $c' + b' \cdot \frac{c}{b} = 2m$ is an integer. Because of the coprimality of b and c, we conclude that b' must be a multiple of b. It cannot be zero because there must be non-lazy mutations. Hence $b' \geq b$.

There are b lines through the origin intersecting the sides of the initial sector under an angle that is a multiple of π/b. These lines divide the plane into $2b$ sectors with angles π/b, which we can label consecutively by elements in $\mathbb{Z}/b\mathbb{Z}$ when we identify opposite sectors. Now suppose we perform b non-lazy mutations. The occurring seeds are given by sectors made of c smaller sectors whose labels are given by

$$[k+1, k+c], [k+c+1, k+2c], \ldots, [k+(b-1)c+1, k+bc], [k+1, k+c]$$

for some $k \in \mathbb{Z}/b\mathbb{Z}$. (Here, we denote $[r, s] = \{r, r+1, \ldots, r+s\}$.) The non-lazy mutations are interrupted by pairs of lazy mutations. Lazy mutations happen between intervals $[k+(l-1)c+1, k+lc]$ and $[k+lc+1, k+l(c+1)]$ if $[k+(l-1)c+1, k+lc]$ contains a fixed sector $t \in \mathbb{Z}/b\mathbb{Z}$ (determined by the reference point u). Hence the number of pairs of lazy mutations is equal to the number of elements in the sequence $k+1, k+2, \ldots, k+bc$ that are equal to t. This number is equal to c, and hence $p = b + 2c$. □

There are three types of cluster algebras of rank 2 with finitely many cluster variables, namely A_2, B_2, and G_2. The entries of the exchange matrix $B = (b_{ij})$ fulfil the relation $|b_{12}b_{21}| = 1 = 4\cos^2(\pi/3)$ in type A_2, $|b_{12}b_{21}| = 2 = 4\cos^2(\pi/4)$ in type B_2, and $|b_{12}b_{21}| = 3 = 4\cos^2(\pi/6)$ in type G_2. Since the angles $\pi/3$, $\pi/4$ and $\pi/6$ are acute, we model these classical cluster algebras geometrically through a situation like in case (a) of Lemma 3.7. The orbits of the classical cluster transformation have lengths 5, 6, and 8, respectively. This corresponds to the shorter of the two possible periods in the lemma, e.g. for A_3, where $(a, b) = (1, 3)$, we have $5 = 2a + b \neq 3b - 2a$. This means that we should choose the reference point in such a way that only small periods occur as rank 2 substructures in order for our model to mimic classical cluster algebra behaviour.

Example 3.8. Let us illustrate Lemma 3.7 in the case of the 7-periodic seed mutation from Fig. 2. The twisted mutation map $\tau \circ \mu_2$ yields seven seeds $(w_0, -w_1), (w_1, -w_2), \ldots, (w_6, -w_0)$. We have two pairs of adjacent lazy indices, namely 0, 1 and 4, 5. For example, for $i = 4$, the lazy step $(w_3, -w_4) \leftrightsquigarrow (w_4, -w_5)$ exchanges w_3 with its negative $w_5 = -w_3$ (and changes the sign of w_4). The numbers 2, 3, 6 are not lazy. For example, for $i = 3$, the non-lazy step $(w_2, -w_3) \leftrightsquigarrow (w_3, -w_4)$ exchanges w_2 with w_4, which is obtained from w_2 by a rotation by $2\pi/3$ in counterclockwise direction. The other two non-lazy steps are also given by rotations by the same angle, so that the three non-lazy mutations have the effect of a full rotation.

Definition 3.9 *(Short and long periods).* We refer to the smaller of the two periods in Lemma 3.7 as a *short* period and to the larger of the two periods in Lemma 3.7 as a *long* period.

In other words, $\min(b+2a, 3b-2a)$ is the short period and $\max(b+2a, 3b-2a)$ is the long period. In particular, if the fundamental angle is acute, then $b + 2a$ is short and $3b - 2a$ is long.

Definition 3.10 *(Compatibility of the reference point).* We say that the choice $u \in V$ is *compatible* with the seed $((w_0, -w_1), B)$ if mutation of this seed (with respect to the reference point u) has a short period.

This definition of compatibility ensures that we get periods 5, 6, or 8, respectively, for an initial seed of type A_2, B_2 and G_2, respectively, in accordance with classical cluster algebra theory.

Remark 3.11. Let us describe the locus of compatible reference points geometrically. Let $(\mathbf{v}, B)$ be a seed of rank 2 such that $\Pi_{v_1} \cap \Pi_{v_2}$ is a sector with angle $\alpha = \angle(A, O, B)$ and its sides are oriented $OA \to OB$.

(a) If α is an acute angle, then the reference point u is compatible with $(\mathbf{v}, B)$ unless u lies inside the supplementary angle $\angle(-B, O, A)$ (the red region in the left side of Fig. 5).
(b) If α is an obtuse angle, then the reference point u is compatible with $(\mathbf{v}, B)$ if and only if u lies inside the supplementary angle $\angle(-B, O, A)$ (the green region in the right side of Fig. 5).

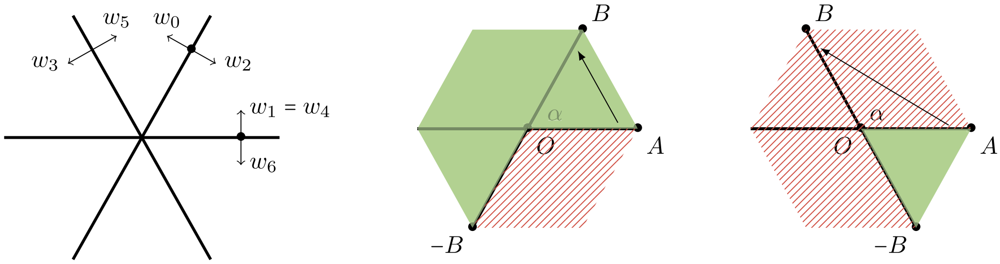

Fig. 5. Left: the vectors w_i. Middle and Right: the locus of compatible (smoothly shaded, green when in colour) and incompatible (hatched, red when in colour) reference points.

4. Compatibility in higher rank

4.1. Compatible reference points

The Definition 2.9 of a seed mutation requires a choice of positive and negative vectors. Like in the situation in rank 2, as discussed in Section 3, we define positive and negative vectors through a reference point, compare also Macdonald [18, Section 1.2]. To construct the set $V^+ \subseteq V$, we fix a reference point $u \in V$.

Every vector $v \in V$ defines a half-space Π_v by taking $\Pi_v = \{w \in V \mid \langle v, w\rangle < 0\}$.

Definition 4.1 *(Positive vectors).* Given a *reference point* u, we say that $v \in V$ is *positive* if u belongs to Π_v. In this case we write $v \in V^+$. We say v belongs to V^- if $-v \in V^+$.

Definition 4.2 *(Compatibility in rank $r > 2$).* We say that the reference point u is *compatible* with the seed $(\mathbf{v}, B)$ if u is compatible with every rank 2 subseed of $(\mathbf{v}, B)$ in the sense of Definition 3.10. We say that u is *compatible* if it is compatible with every seed in the mutation class of $(\mathbf{v}, B)$.

As we have seen in Section 3, not every notion of positivity will recover the exchange graphs for classical cluster algebras. Moreover, Definition 4.2 is clearly a necessary condition for exchange graphs to agree with classical exchange graphs. Hence, it is natural to impose this condition in general. We will see later that it is also a sufficient condition to recover classical exchange graphs. We fix an initial seed $(\mathbf{v}_0, B)$ and a reference point u, and denote the resulting exchange graph by Γ.

Definition 4.3 *(Finite and affine seeds).* Suppose that $(\mathbf{v}, B)$ is a seed of rank n.

(1) We say that $(\mathbf{v}, B)$ is of *finite type*, if its geometric realisation lies on the sphere $\mathbb{S}^{n-1}$. We say it is *of infinite type* otherwise.
(2) We say that $(\mathbf{v}, B)$ is of *affine* if its geometric realisation lies in the Euclidean space $\mathbb{E}^{n-1}$.

Remark 4.4 *(Finite and affine quivers).* As the exchange matrix is the essential part from which we can determine the seed $(\mathbf{v}, B)$, we also use the terminology from Definition 4.3 when considering exchange matrices only. In particular,

(1) an exchange matrix B is of *finite type* if its geometric realisation lies on the sphere $\mathbb{S}^{n-1}$;
(2) an exchange matrix B is *affine* if its geometric realisation lies in the Euclidean space $\mathbb{E}^{n-1}$.

Remark 4.5. In Definition 4.4 the term *affine* is justified because the condition is satisfied for all affine quivers in the usual setting with integer exchange matrices. In particular, quivers of type $\tilde{A}_2$, $\tilde{B}_2$, etc. satisfy the condition.

5. Exchange graphs for finite types

The classification of (non-integer) mutation-finite matrices is obtained in [9, Theorem B]. It states that such matrices are either coming from orbifolds or are in one of four infinite series, or in the finite list of matrices listed in of [9, Table 1.1]. In particular, it states that non-integer mutation matrices not coming from an orbifold are all having geometric realisations in a space with positively definite quadratic form, or semi-positive quadratic space of corank 1 or at least 2, respectively. In these cases one says that the matrix is of finite, affine or extended affine type respectively.

Remark 5.1. As we will use the classification of finite type and of rank 3 affine type, we reproduce these classifications in Fig. 6 left and right respectively. The mutation classes in these figures are represented by the corresponding quivers with real weights according to the following notation:

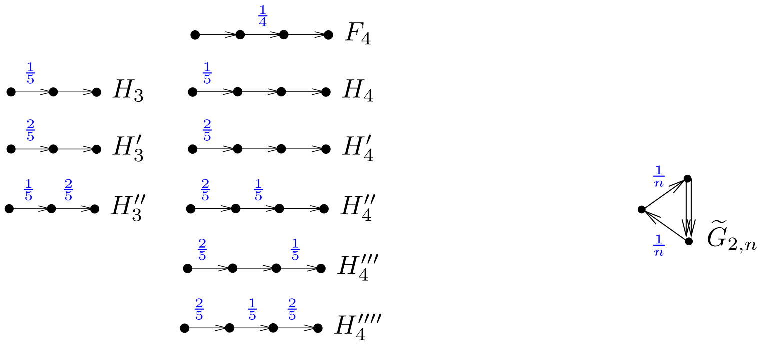

Fig. 6. Non-integer mutation-finite quivers not coming from a surface or orbifold: finite types (left) and affine rank 3 types (right).

- label p/q on an arrow from vertex v_i to v_j means that in the exchange matrix one has $b_{ij} = 2\cos\frac{\pi p}{q}$.

Remark 5.2 *(Domain associated to a seed).* Given a seed $(\mathbf{v}, B)$ represented by a tuple of vectors $\mathbf{v} = (v_1, \dots, v_n)$, by the *domain* associated to the seed we mean $\bigcap_{i=1}^n \Pi_{v_i}$.

Remark 5.3. The Definition 2.5 of admissibility of a realisation immediately implies that if the domain belonging to a seed is acute-angled, then the seed is acyclic.

Theorem 5.4. Let B be an exchange matrix of finite type. Then

(1) there are geometric realisations $(\mathbf{v}, B)$ with the reference point in a compatible position;
(2) in every such realisation, the compatible reference point belongs to some acute-angled domain;
(3) every choice of a reference point in an acute-angled domain gives such a realisation;
(4) all such realisations result in the same exchange graph.

Proof. The classification of exchange matrices of finite type is given in [9, Theorem B] (see also Table 1.1 there): it says that except for surface and orbifold cases and cases with integer matrices, there are finitely many types to consider, all coming from non-crystallographic root system H_3 (3 mutation classes), H_4 (5 mutation classes) and F_4. The mutation classes coming from surfaces or orbifolds of finite type are exhausted by a disc, a punctured disc, and a disc with a unique orbifold point, i.e. by types A_n, D_n and B_n. So, taking into account also mutation classes with integer exchange matrices, we need to consider the following types: infinite series A_n, B_n, D_n, and finite list E_6, E_7, E_8, F_4, three mutation classes coming from type H_3 and five mutation classes coming from type H_4. Notice that each of them is associated to a (possibly non-crystallographic) root system.

We will consider the non-crystallographic cases separately from the crystallographic ones.

Non-crystallographic mutation classes: the existence of a geometric realisation is shown in [9, Lemma 6.3]. For every geometric realisation (i.e. an appropriate choice of vectors associated to the initial seed from the finite root system H_3 or H_4 respectively) we consider all possible choices of the reference point (placing it consequently inside each of the fundamental chambers of the group action) and show the statements of the theorem explicitly, by computer-assisted computation: we consider all these cases, check which of them result in a reference point in a compatible position and explicitly compare the exchange graphs produced from the cases where the reference point is compatible.

Crystallographic mutation classes: Let Δ be the root system of B, let $w_1, \dots, w_n$ be the simple roots of Δ, and let $F = \bigcap \Pi_{w_i}$ be the fundamental simplex defined by the simple roots. Our reasoning here will be based on the following two observations:

- the root system Δ has only the following angles between the roots: $\pi/2$, $\pi/3$, $2\pi/3$, $\pi/4$, $3\pi/4$. In particular, this implies that any acute angle has a form of π/n;
- the simple roots of Δ together with any orientation of the acyclic quiver from the corresponding mutation class give a geometric realisation for any acyclic seed.

(1) To show existence of a compatible realisation, consider the set of simple roots of Δ, this will be the set of vectors associated to the initial seed (we take any acyclic seed as the initial one). We place the reference point inside F. Then the notion of positive/negative roots of the root system coincides with the notion of positivity given by Definition 2.9. It is shown in [23,24] that the mutation of Y-seeds (i.e. pairs $(\mathbf{c}, B)$ with a c-vector $\mathbf{c}$) is using the same formulae as given in Definition 4.1. Since the standard mutation of seeds from root systems has only rank 2 subseeds with a short period, the reference point must be compatible.

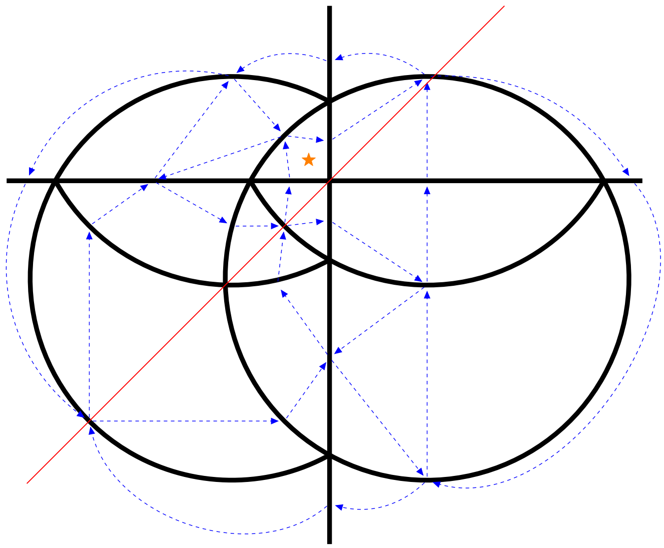

Fig. 7. Realising the exchange graph of type A_3 on the sphere.

(2) If the reference point were in an obtuse-angled domain, there would be a subseed of rank 2 with a long period, compare the rightmost picture in Fig. 5, in contradiction to the compatibility of the reference point.

(3) Suppose that a reference point lies in an acute-angled domain P. Then the angles of P are $\pi/2$, $\pi/3$, $\pi/4$, i.e. P is a Coxeter polytope. The reflection group G_P generated by the reflections with respect to the sides of P should coincide with the reflection group W obtained from the initial seed. This implies that P should be one of the fundamental chambers of W (otherwise G_P is a subgroup of W of index higher than 1). Hence, the situation coincides with the one constructed in (1), and the reference point is compatible again.

(4) Based on [25], Seven in [23, Corollary 1.7] and [23, Theorem 1.3] gives an explicit formula for mutations of Y-seeds in the case of integer acyclic exchange matrices (i.e. for a mutation of the pair $(\mathbf{c}, B)$ where $\mathbf{c}$ is a c-vector). The formula given in [23,24] coincides with the mutation rule of geometric realisations of seeds given in Definition 2.9 (given that the initial seed was the set of simple roots and the reference point was inside the initial fundamental domain F). As Y-seeds are in bijection with the seeds of cluster algebras, see Fomin–Zelevinsky [14, Theorem 1.10], this implies that we obtain always the same exchange graphs as one has in cluster algebra settings. It is known from [13, Theorem 1.7] that the exchange matrix for a cluster algebra does not depend on the choice of the initial seed in it. □

Remark 5.5. The exchange graph originating from the mutation class H_3 coincides with the one given by Fomin–Reading in [10], the exchange graphs for the classes H_3' and H_3'' are provided in [8]. For the rank 4 mutation classes, the exchange matrices are too big to be drawn here: the computer calculations show that the number of vertices of the exchange graphs are given by 105 (for F_4), 280 (for H_4), 495 (for H_4'), 352 (for H_4''), 420 (for H_4''') and 570 (for H_4'''').

Remark 5.6. In the setting of non-integer matrices, c-vectors are not mutated in the same as the geometric realisations considered in this paper. Moreover, mutation of Y-seeds cannot be defined in the same way as in [23], since the sign coherence breaks after some sequences of mutations.

Example 5.7. An example of a geometric picture of the exchange graph of type A_3 is shown in Fig. 7. To obtain the picture on the plane, we project the sphere to the plane using the stereographic projection from a right-angled vertex. The bold lines bound the domains associated to seeds (here, projections of spherical triangles). The reference point is represented by a star. For each seed, inside the domain we draw the corresponding quiver (in dashed lines). The exchange graph is the dual graph to the decomposition of the plane to the domains. The cluster complex is the dual complex to the decomposition.

Remark 5.8 *(Belt line b).* In Fig. 7, a spherical line b of the sphere is drawn as a straight oblique line. The line b has formidable properties. First, it is a line which intersects precisely the acyclic seeds. Second, the spherical triangle of every cyclic seed admits two vertices such that the orthogonal projection of the vertex to the opposite side lies on the spherical line b. One can check that a spherical line with the same properties exists for every compatible realisation in Theorem 5.4. We will see later in Section 6.3 that a line with the same properties exists for exchange graphs in the Euclidean plane.

6. Realisations of rank 3 affine quivers

6.1. Realisations in the Euclidean plane

Consider a mutation-finite exchange matrix B of rank 3. The rest of the paper concerns the rank 3 affine type. We choose a geometric realisation $\mathbf{v} = (v_1, v_2, v_3)$ of B inside a quadratic space $V = \mathbb{R}^3$. Now we describe how to realise the seed $(\mathbf{v}, B)$ in the Euclidean plane $\mathbb{E}^2$ after projectivisation.

There are two bilinear forms $V \times V \to \mathbb{R}$. First, we denote the standard Euclidean form by angular brackets $\langle\cdot,\cdot\rangle\colon V \times V \to \mathbb{R}$. Moreover, as in Definition 2.4, we denote the bilinear form defining the quadratic space V by round brackets $(\cdot,\cdot)\colon V \times V \to \mathbb{R}$. The form $(\cdot,\cdot)$ is positive semi-definite of corank 1. I.e. there exists a unique up to proportionality vector $e_1 \in V\backslash\{0\}$ such that $(e_1, w) = 0$ for all $w \in V$. Extend e_1 to an ordered basis $\mathbf{e} = (e_1, e_2, e_3)$ of V by choosing vectors e_2 and e_3. Then we pick a hyperplane $P \subseteq \mathbb{R}^3$ that is parallel to e_2 and e_3 and does not contain the origin. To be concrete, let us put $P = \{(w_1, w_2, w_3) \in \mathbb{R}^3 \mid w_1 = 1\}$. Here coordinates are written with respect to the basis $\mathbf{e}$.

Let $v = (v_1, v_2, v_3) \in V$, and suppose that v is not parallel to e_1, that is, $(v_2, v_3) \neq (0, 0)$. Then the orthogonal complement $v^\perp = \{w \in V \mid \langle v, w\rangle = 0\}$ is not parallel to P. Hence the intersection of $l_v = v^\perp \cap P$ defines a line in P. The line bounds the half-plane $\Pi_v = \{w \in V \mid \langle v, w\rangle < 0\} \cap P$, and the intersection $\Pi_{v_1} \cap \Pi_{v_2} \cap \Pi_{v_3}$ defines a planar triangle (or an infinite region bounded by three lines). The concept of *geometric realisation* now means that if $|b_{ij}| = |(v_i, v_j)| = 2\cos(\pi t_{ij})$ for two indices i and j, then πt_{ij} is the angle between the sides of the triangle associated to v_i and v_j. The quiver encodes the placements of the signs inside the skew-symmetric matrix B.

Remark 6.1. Another geometric description can be constructed as follows. Consider the Euclidean plane $\mathbb{E}^2 = \mathbb{R}^2$. Recall that a map $f\colon \mathbb{E}^2 \to \mathbb{R}$ is *affine linear* if there exists a row vector $a \in \mathbb{R}^2$ and a real number b such that $f(x) = ax + b$ for all x. We denote by F the set of all affine linear maps $\mathbb{E}^2 \to \mathbb{R}$. Clearly, F is a 3-dimensional vector space. We define a symmetric bilinear form on F by setting $(f, g) = (a, c)$ when $f(x) = ax + b$ and $g(x) = cx + d$ for all x. Note that the bilinear form is degenerate and its radical consists of all constant maps.

For every $f \in F$ define $l_f = f^{-1}(0)$ and $\Pi_f = f^{-1}([0,\infty))$. If f is non-constant, then $l_f \subseteq \mathbb{E}^2$ is a line and $\Pi_f \subseteq \mathbb{E}^2$ is a half-plane bounded by l_f. Notice that l_f and Π_f do not change when we multiply f by a positive scalar. In this way, a seed $(\mathbf{v}, B)$ defines a triangle Δ in the Euclidean plane whose angles encode the entries of B.

Both constructions are dual to each other. Vectors $v = (v_1, v_2, v_3) \in V$ are in bijection with linear maps $f_v\colon V \to \mathbb{R}$ defined by $f_v(w_1, w_2, w_3) = v_1 w_1 + v_2 w_2 + v_3 w_3$. The hyperplane $P = \mathbb{R}^2$ embeds in $\mathbb{R}^3$ via $(w_2, w_3) \to (1, w_2, w_3)$. The restriction of f_v to P is an affine linear map $P \to \mathbb{R}$. Conversely, every affine linear map $P \to \mathbb{R}$ extends uniquely to a linear map f_v from V to $\mathbb{R}$. The zero set of the restriction of f_v to P defines the same line l_v as above. The viewpoint of affine linear maps is used in the context of affine root systems, see Macdonald [18, Section 1.2]. Angles measured in $P = \mathbb{R}^2$ using the Euclidean bilinear form on $\mathbb{R}^2$ agree with angles measured in F using the bilinear form above.

6.2. The initial configuration

The bulk of the paper is devoted to geometric mutations of rank 3 affine quivers as Euclidean triangles with rational angles. According to [8] an initial seed can be realised by two parallel lines and a third line intersecting the parallel lines under the angle π/d for some positive integer d, see Fig. 8.

Remark 6.2. We can construct other matrices in the same mutation class using considerations as in the proof of [8, Theorem 6.11]. In particular, the mutation class has a matrix of the form

$$\begin{pmatrix} 0 & 2\cos(\pi t_3) & 2\cos(\pi t_2) \\ -2\cos(\pi t_3) & 0 & 2\cos(\pi t_1) \\ -2\cos(\pi t_2) & -2\cos(\pi t_1) & 0 \end{pmatrix},$$

where (t_1, t_2, t_3) is a permutation of $(\frac{1}{d}, \frac{k}{d}, \frac{d-k-1}{d})$ for some natural number $k \in [0, \frac{d}{2}]$. Note that the sum of the angles πt_i is equal to π, so that they form a Euclidean triangle. We plug in $k = \frac{d-1}{2}$ if d is odd, and $k = \frac{d}{2}$ if d is even. After writing $d = 2n + 1$ or $d = 2n$ for some $n \geq 1$, we can say that B is mutation-finite if and only if it is mutation-equivalent to

$$\begin{pmatrix} 0 & 2\cos(\pi t_3) & 2\cos(\pi t_2) \\ -2\cos(\pi t_3) & 0 & 2\cos(\pi t_1) \\ -2\cos(\pi t_2) & -2\cos(\pi t_1) & 0 \end{pmatrix},$$

where (t_1, t_2, t_3) is one of the triples

$$\left(\frac{n}{2n+1}, \frac{n}{2n+1}, \frac{1}{2n+1}\right), \quad \left(\frac{n-1}{2n}, \frac{1}{2}, \frac{1}{2n}\right) \qquad (n \geq 1).$$

In fact, a stronger statement is true.

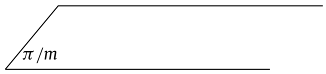

Fig. 8. A seed with parallel sides.

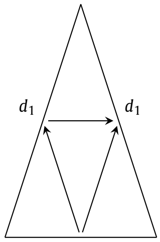

Fig. 9. The initial seed $(\mathbf{v}_0, B_0)$.

Proposition 6.3. Let $n_1, n_2, n_3 \in [1, d]$ with $n_1 + n_2 + n_3 = d$. The mutation class of $(\mathbf{v}, B)$ contains a triangle with angles $\frac{n_1}{d}\pi$, $\frac{n_2}{d}\pi$ and $\frac{n_3}{d}\pi$ if and only if $\gcd(n_1, n_2, n_3) = 1$.

Proof. Suppose that $\frac{n_1}{d}\pi$, $\frac{n_2}{d}\pi$ and $\frac{n_3}{d}\pi$ are the angles of a triangle Δ and n_1, n_2, n_3 have a non-trivial common divisor e. Then the angles of every triangle in the mutation class of Δ are multiples of $e\pi/(d)$. This mutation class cannot contain Δ.

For the converse direction, notice that the matrix B is mutation-finite by the π-rationality of the angles. Hence, we can apply Felikson–Tumarkin's classification of mutation classes for mutation-finite exchange matrices in rank 3, see [8, Section 6]. In particular, a seed whose triangle has angles $\frac{n_1}{d}\pi$, $\frac{n_2}{d}\pi$ and $\frac{n_3}{d}\pi$ is mutation-equivalent to $(\mathbf{v}, B)$. □

Remark 6.4. Note that the number d is the least common denominator of the three rational multiples of π in any geometric realisation obtained from $(\mathbf{v}, B)$ by sequences of mutations. We consider separately the cases where d is even or odd. In this Section 6.2 and in Sections 7, 8, 9 we study the case where d is odd. Section 10 is devoted to the case where d is even.

We fix an integer $n \geq 1$, and put $d = 2n + 1$. We construct an initial seed as follows.

Definition 6.5 *(Initial triangle).*

(a) We define the *fundamental angle* to be $\alpha = \pi/(2n + 1)$.
(b) The *initial triangle* $\Delta_0 \subseteq \mathbb{E}^2$ is an isosceles triangle with angles α, $n\alpha$ and $n\alpha$.
(c) The *initial seed* $(\mathbf{v}_0, B_0)$ is given by the initial triangle Δ_0 together with an acyclic quiver as shown in Fig. 9.

Remark 6.6. By virtue of Remark 6.2 the choice of an initial seed in Definition 6.5 is not a loss of generality.

6.3. The initial acyclic belt

Definition 6.7 *(Acyclic belts).*

(1) An *acyclic belt* is a maximal connected full subgraph of the exchange graph containing only vertices $(\mathbf{v}, B)$ with acyclic exchange matrices B.
(2) The acyclic belt obtained from $(\mathbf{v}_0, B_0)$ by sequences of mutations at sinks or sources is called the *initial acyclic belt*. It is denoted by I and we label the vertices of I so that $I_0 = (\mathbf{v}_0, B_0)$ and that $I_{n+1} = (\mathbf{v}_{n+1}, B_{n+1})$ is obtained from $I_n = (\mathbf{v}_n, B_n)$ by a mutation at a source for all $n \in \mathbb{Z}$.

Example 6.8 *(Type $\widetilde{A}_2$).* Let us look at the case $n = 1$. In this case, the matrix B has integer entries and it defines a cluster algebra of type $\tilde{A}_2$. Its exchange graph is called the *brick wall* and it is shown in Fig. 12, left. Moreover, it is known how to visualise the seeds by equilateral triangles in the Euclidean plane, see Fig. 12, right. Note that there is exactly one acyclic belt, and this acyclic belt consists of equilateral triangles.

Definition 6.9 *(Belt line b).* Define $b \subseteq \mathbb{E}^2$ to be the line connecting the two feet of the altitudes in the initial triangle corresponding to the sink and the source in the initial quiver Q_0. The line is shown in Fig. 10.

Remark 6.10 *(Billiard).* The line b is well-studied in Euclidean geometry. It plays an important role in the solution of Fagnano's problem, which is related to triangular billiards. The problem asks to find three points on the three sides of a given

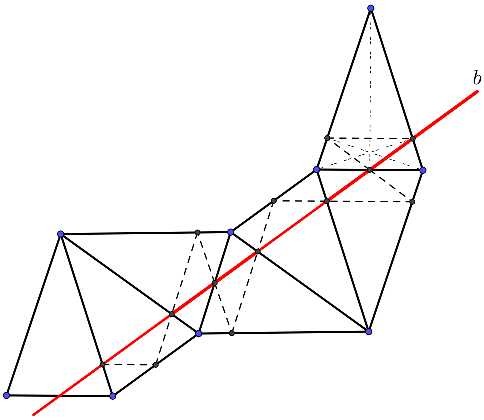

Fig. 10. The belt line b. The feet of altitudes and the lines connecting them are highlighted by dotted lines.

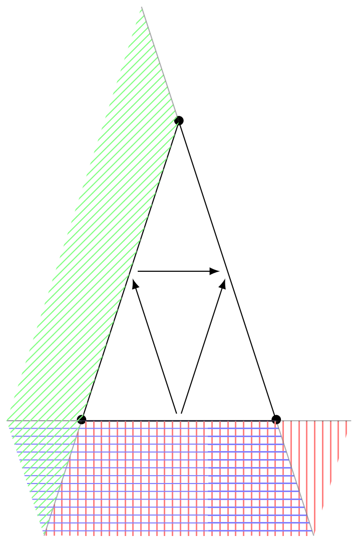

Fig. 11. Locus of reference points not compatible with the initial seed.

triangle such that the perimeter of the triangle formed by the three points is as small as possible. For an acute-angled triangle, the minimum is attained when the points are the feet of the altitudes of the triangle, and the perimeter of the minimal triangle forms a 3-periodic billiard orbit, see Tabachnikov [26, Chapter 7]. The name b stands both for acyclic *belt* and *billiard*.

Now, we need to find a compatible choice of a reference point.

Proposition 6.11. Let u be a compatible reference point. Then u lies either

(a) infinitely far away on line b, in the direction of the arrow from the source to the sink in the initial triangle, or
(b) inside an acute triangle that represents the seed $I_{-k} = (\mathbf{v}_{-k}, B_{-k})$ for some $k \geq 0$ (which is obtained from the initial seed $I_0 = (\mathbf{v}_0, B_0)$ by a series of sink mutations).

Proof. Starting with the initial seed we perform a sequence of mutations at sinks, which produce the seeds $I_0, I_{-1}, I_{-2}, \ldots$ according to the labelling from Definition 6.7. For $k \in \mathbb{N}$ we denote the triangle corresponding to the seed I_k by Δ_k. We claim that if u does not lie inside the triangle Δ_{-k}, then Δ_{k+1} is obtained by reflecting Δ_k across the side of it which corresponds to the sink in I_k, compare Fig. 13.

For a proof of the claim let us fix $k \geq 0$ and assume that u does not lie inside Δ_{-k}. Without loss of generality we assume that $\mathbf{v}_k = (w_1, w_2, w_3)$ is labelled so that the index 1 corresponds to the sink in B_k. By definition, u must be compatible with the seed I_k at every index, that is, at all angles of the triangle Δ_k. Remark 3.11 allows us to determine the loci of compatibility for each angle explicitly. The loci are given by sectors around the 3 angles; the sectors of non-compatible points for the three angles are shaded differently in Fig. 11. The line l_{w_1}, the side line of Δ_k corresponding to the sink,

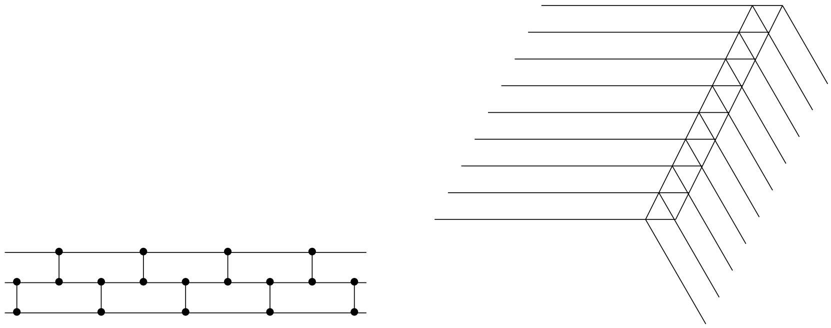

Fig. 12. The exchange graph (left) and the geometric description of the seeds (right) of a cluster algebra of type $A_2^{(1)}$.

divides the plane into two half-planes. Since u does neither lie inside Δ_k nor inside the forbidden regions, it must lie the half-plane that does not contain the triangle Δ_k. Hence the mutation of I_k at 1 is not lazy and Δ_{k+1} is obtained by reflecting Δ_k across l_{w_1}.

The discussion in the previous paragraph about the location of u implies that either case (b) occurs or Δ_{k+1} is obtained from Δ_k by a reflection across a side for every $k \geq 0$. For the rest of the proof we assume that (b) does not occur.

The composition of two reflections across two lines intersecting each other at an angle φ is a rotation by angle 2φ. The composition of three rotations by angles 2α, $2n\alpha$ and $2n\alpha$ is a translation due to $2\alpha + 2n\alpha + 2n\alpha = 2\pi$. Hence, after 6 steps we obtain a triangle Δ_{-6} which can be obtained from Δ_0 by a translation by a vector w. We obtain regions of compatibility for Δ_{-6} similar to those Δ_0 in Fig. 11 but shifted by vector w. Analogously, we can continue to mutate at sources, and obtain similar regions of compatibility for every sixth triangle.

It is well-known in the literature that the line b intersects the occurring triangles in the feet of two altitudes, see for example Section 4.5 about Fagnano's theorem in Coxeter–Greitzer's book [3]. From this we can conclude that the translation vector w is parallel to b.

A tracking of angles shows that the base sides of the triangles Δ_{-1} and Δ_{-4} in seeds -1 and -4 are parallel to b. The compatibility of the reference point of the seeds Δ_{-1-6k} and Δ_{-4-6k} for all $k \geq 0$ implies that u lies between the two lines defined by the base sides of the triangles Δ_{-1} and Δ_{-4}. This implies that u lies infinitely far away on the line b. □

Condition 6.12. In this article we study case (a) of Proposition 6.11, that is, we assume that the reference point u lies infinitely far away on the line b, in the direction of the arrow from the source to the sink in the initial triangle.

Remark 6.13. The choice of case (a) seems less natural than case (b), but it allows us to prove Theorem B. We conjecture that the choices (a) and (b) yield the same exchange graph, so that Theorem B is true also in case (b). However, our proof is not directly transferable to case (b).

The initial acyclic belt is shown in Fig. 13 for the example $n = 2$ (that is, $\alpha = \frac{\pi}{5}$).

Remark 6.14. We can describe the notion of positivity given by the reference point u from Proposition 6.11 in geometric terms. Suppose that a seed $(\mathbf{v}, B)$ is given by a triangle Δ. Let s be a side of Δ, and without loss of generality let us assume that s corresponds to the component v_1 of $\mathbf{v} = (v_1, v_2, v_3)$. The extension of s divides $\mathbb{E}^2$ into two half-planes. Let $z \in \mathbb{E}^2$ be a vector perpendicular to s such that z lies in a different half-plane than Δ. Then it is easy to see that there exists a vector $b^\perp \in \mathbb{E}^2$ having the property that v_1 (and hence the mutation of $(\mathbf{v}, B)$ at v_1) is positive if and only if $(b^\perp, z) > 0$. Here $(\cdot,\cdot) : \mathbb{E}^2 \times \mathbb{E}^2 \to \mathbb{R}$ denotes the standard scalar product. By construction, the line $b^\perp$ is orthogonal to the line b.

The discussion in the proof of Proposition 6.11 shows the following.

Proposition 6.15. The initial acyclic belt has the following geometric structure.

(1) The domain of every geometric realisation I_n with $n \in \mathbb{Z}$ inside the initial acyclic belt is an acute-angled triangle. The line b intersects every triangle I_n in two points. These points are the feet of two altitudes in I_n.
(2) Suppose that $n \in \mathbb{Z}$. The domains of the geometric realisation I_n and I_{n+6} are obtained from each other by a translation by a vector parallel to b.

Remark 6.16. Proposition 6.15 is true for every other choice of an acyclic seed (given by an acute-angled initial triangle and an acyclic quiver) when we choose a reference point infinitely far away on the line through the feet of the altitudes corresponding to the sink and the source. Hence it is true for every acyclic belt. *A priori*, two different acyclic belts could

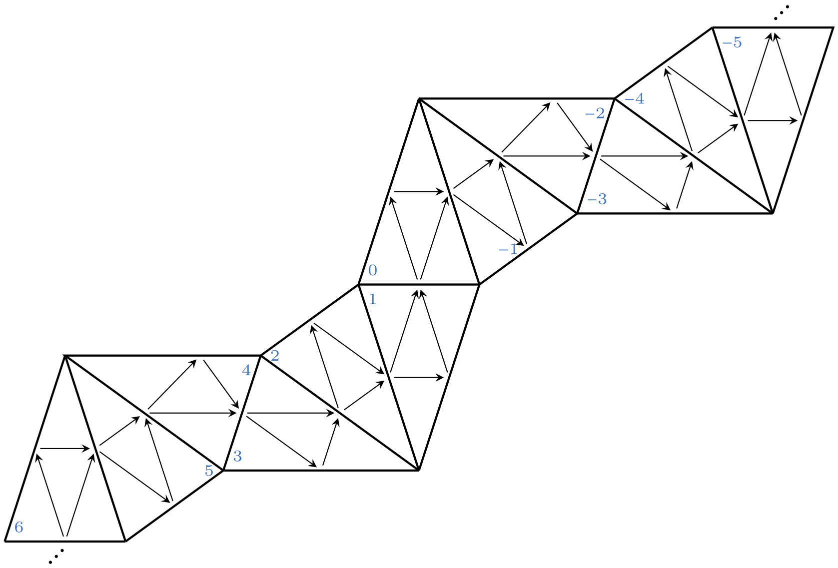

Fig. 13. The initial acyclic belt I.

yield two distinct lines b and b'. Moreover, two different directions of the lines could result in two conflicting definitions of the reference point (and thus, conflicting notions of positivity), and *a priori* the length of the translation vectors in Proposition 6.15 (2) could depend on the acyclic seed. However, we will show in the next section that the direction of the line b, the line b itself, and the length of the translation vector are independent of the choice of the acyclic belt.

6.4. A conserved quantity of mutation

Mutations of geometric realisations admit an interesting invariant.

Definition 6.17 *(Mutation invariant).* Fix a triangle $\Delta = A_1A_2A_3$. For $i \in \{1,2,3\}$ we denote by a_i the side of Δ opposite to A_i. We put

$$T(\Delta) = a_1 \sin(A_2)\sin(A_3).$$

Moreover, suppose now that Δ is an infinite region bounded by two parallel lines a_1 and a_2 and a finite side a_3 that intersects a_1 and a_2 at angles α and $\pi - \alpha$. Then we define

$$T(\Delta) = a_3 \sin(\alpha)\sin(\pi-\alpha) = a_3 \sin^2(\alpha).$$

Proposition 6.18. The quantity T is well-defined, that is, it is independent of the numbering of the vertices of the triangle Δ.

Proof. Suppose that $\Delta = A_1A_2A_3$ a triangle as in Definition 6.17. We have to show

$$a_1 \sin(A_2)\sin(A_3) = a_2 \sin(A_3)\sin(A_1) = a_3 \sin(A_1)\sin(A_2).$$

This is an immediate consequence of the law of sines. □

Proposition 6.19 *(T is mutation-invariant).* Suppose that the geometric realisation $(\mathbf{v}, B)$ is represented by a triangle or an infinite region Δ. Let $k \in \{1,2,3\}$ such that the mutation $\mu_k(\mathbf{v}, B) = (v', B')$ is represented by triangle or an infinite region Δ'. Then $T(\Delta) = T(\Delta')$.

Proof. The statement is true when k is a sink or a source. Assume that k is neither a sink or a source. Without loss of generality we may assume $k = 1$. Consider the formula $T(\Delta) = a_1 \sin(\angle A_1A_2A_3)\sin(\angle A_2A_3A_1)$. The mutation μ_k leaves a_1 and one of the angles invariant and replaces the other angle with its supplementary angle. This means $T(\Delta) = T(\Delta')$. □

Corollary 6.20. Suppose that the triangles Δ and Δ' are similar (i.e. they have the same angles) and are related by a sequence of mutations as in the previous proposition. Then Δ and Δ' are congruent (i.e. they have the same side lengths).

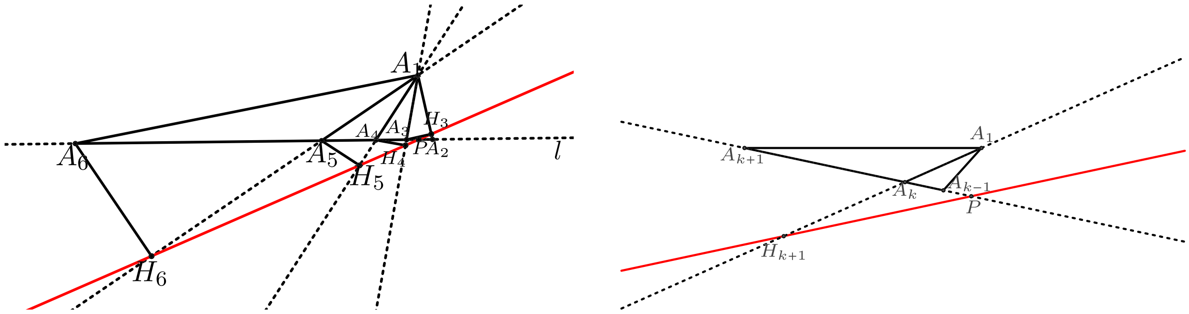

Fig. 14. Mutation of an obtuse-angled triangle.

Proposition 6.21. Let $\Delta = A_1A_2A_3$ be an acute-angled triangle and let H_1, H_2 and H_3 be the feet of the altitudes of Δ. Then the invariant $T(\Delta)$ admits a geometric interpretation as

$$T(\Delta) = \tfrac{1}{2}\left(|H_1H_2| + |H_2H_3| + |H_3H_1|\right).$$

Proof. The semiperimeter of the triangle $H_1H_2H_3$ is equal to $\mathcal{A}/\rho$ where $\mathcal{A}$ is the area and ρ the circumradius of Δ, see Honsberger [16, Section 5]. On the other hand, $\mathcal{A} = \frac{1}{2}a_1a_2\sin(A_3)$. The relation $a_2 = \rho\sin(A_2)$, see Coxeter–Greitzer [3, Theorem 1.1], establishes the claim. □

Note that the points H_1, H_2 and H_3 from the previous proposition constitute the solution to Fagnano's problem. Hence the term $2T(\Delta)$ is the minimum perimeter of a triangle inscribed in a fixed triangle $A_1A_2A_3$.

Definition 6.22 *(Translation vector).* Suppose that a seed $(\mathbf{v}, B)$ is represented by an acute-angled triangle Δ. The vector that relates every sixth triangle in the acyclic belt of $(\mathbf{v}, B)$ is called *translation vector*.

Corollary 6.23. Suppose that the seed $(\mathbf{v}, B)$ is represented by an acute-angled triangle. Then the translation vectors of $(\mathbf{v}, B)$ and the initial seed have the same length $4T(\Delta) = 4T(\Delta_0)$.

Proof. The translation vector is equal to $2\left(|H_1H_2| + |H_2H_3| + |H_3H_1|\right)$, see Fig. 10. □

6.5. The feet of altitudes

Proposition 6.24 *(Feet of altitudes lie on b).* Suppose that the geometric realisation $(\mathbf{v}, B)$ is obtained from the initial geometric realisation by a sequence of mutations and that it is given by a triangle Δ and a quiver Q.

(1) If Δ is acute-angled so that the associated quiver Q is acyclic, then the feet of the two altitudes opposite to the sink and the source lie on b.
(2) If Δ is obtuse-angled, then the feet of the altitudes on the two sides of the obtuse angle lie on b.

Proof. We prove this property by induction on the length of a shortest mutation sequence from $(\mathbf{v}_0, B_0)$ to $(\mathbf{v}, B)$. There are two types of mutations, namely mutations inside an acyclic belt, see Fig. 10, and mutations involving obtuse-angled triangles. For mutations inside an acyclic belt we can use the same arguments as in the proof of Proposition 6.15 because the reference point u lies on the line b.

Let us consider mutations involving obtuse-angled triangles, see the left picture in Fig. 14, whose quivers are all cyclic. Hence a mutation at a side a keeps one of the other sides and reflects the other across s. Hence a generic situation is given locally as follows. Points $A_2, A_3, \ldots$ lie on a common line l, and another point A_1 does not lie on the line. The points form triangles $A_1A_kA_{k+1}$ for $k \geq 2$. These triangles all have angles of the same size ψ at A_1 and are obtained from each other by successive mutations. Now we denote by $H_3, H_4, \ldots$ the feet of the altitudes from $A_3, A_4, \ldots$ on the lines $A_1A_2, A_1A_3, \ldots$ We want to prove that $H_3, H_4, \ldots$ all lie on the line b. We apply the induction hypothesis to the triangle $A_1A_kA_{k+1}$ such that $A_1A_kA_{k+1}$ has the shortest mutation distance from the origin among all choices of k. The induction hypothesis implies that the feet P of the perpendicular from A_1 on l lies on b.

Consider the similarity transformation f preserving A_1, which rotates by the angle ψ and stretches with factor $\cos(\psi)$. The transformation f maps A_k to H_k for all $k \geq 2$. Also, f maps the line l to a line b' so that $H_3, H_4, \ldots$ all lie on a common line b'. By construction the line b' intersects l at the angle ψ. The induction hypothesis, applied to the triangle $A_1A_kA_{k+1}$, implies that the feet of the altitudes P and H_k lie on b so that $b = b'$ (see Fig. 14, right). □

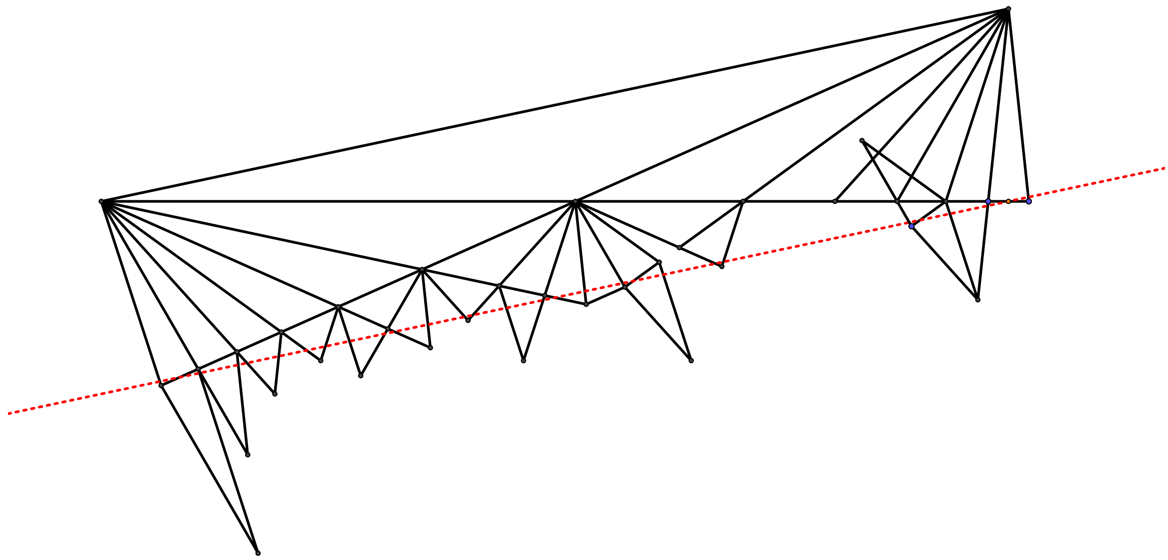

Fig. 15. The belt line b crosses all seeds represented by acute-angled triangles and only them.

Remark 6.25 *(Quivers are oriented towards the reference point).* Let us look at the consequences of Proposition 6.24 for the orientation of the arrows in the associated quivers. To describe the orientations, we refer to the half-planes bounded by b as Π_1 and Π_2. We divide each of the two cases of Proposition 6.24 into two subcases.

(1) First assume that the triangle $\Delta = A_1A_2A_3$ is acute-angled, so that the associated quiver Q is acyclic. Then the belt line b intersects Δ in two points H_1 and H_3, and without loss of generality let us say the side A_1A_2 (containing H_3) corresponds to the source of the quiver; the side A_2A_3 (containing H_1) corresponds to the sink. There are two possibilities.
   (1a) The third side of Δ, namely A_1A_3, lies in Π_1.
   (1b) The third side of Δ, namely A_1A_3, lies in Π_2.
(2) Second assume that the triangle Δ is obtuse-angled so that the associated quiver Q is cyclic. Then the belt line b does not intersect Δ, because the two feet of the altitudes are outside the triangle. Again, there are two possibilities.
   (2a) The triangle Δ lies in Π_1.
   (2b) The triangle Δ lies in Π_2.
   Let us label the vertices $\Delta = A_1A_2A_3$, so that the belt line b intersects the extensions of the sides A_1A_2 and A_2A_3 in H_3 and H_1, respectively, and H_3 lies between H_1 and u.

Now it is easy to see that a sink/source mutation of an acute-angled seed takes case (1a) over to case (1b), and vice versa. Moreover, a mutation of an acute-angled seed at a vertex which is neither sink nor source transports (1a) $\to$ (2a) and (1b) $\to$ (2b). What is more, mutation of a seed in case (2a) produces a seed in cases (1a) or (2a), and mutation of a seed in case (2b) produces a seed in cases (1b) or (2b). All these mutations leave the following structure intact.

(1) If $\Delta = A_1A_2A_3$ is acute-angled, then the triangle is oriented towards the reference point u, that is, the sides are oriented $A_1A_2 \to A_1A_3 \to A_2A_3$.
(2) If $\Delta = A_1A_2A_3$ is obtuse-angled, then the sides are oriented $A_1A_2 \to A_1A_3 \to A_2A_3 \to A_1A_2$. In particular, if two obtuse-angled seeds are located on different sides of the line b, see cases (2a) and (2b), then they have different orientations.

In particular, it follows from Remark 6.25 that the belt line crosses all seeds represented by acute-angled triangles and only them (see Fig. 15 for the illustration).

6.6. Translational symmetries of the exchange graph

Definition 6.26 *(Translated seeds).* Let $w \in \mathbb{E}^2$.

(1) Suppose that the seed $(\mathbf{v}, B)$ is represented by a triangle Δ together with a quiver Q. The *translated seed* $w + (\mathbf{v}, B)$ is given by the translation of Δ in $\mathbb{E}^2$ by the vector w, together with the same quiver Q. In this case, we call w the *translation vector* between the seeds.

(2) Suppose that J is an acyclic belt. We denote its vertices in the exchange graph by J_k with $k \in \mathbb{Z}$. The *translated acyclic belt* $w + J$ is the infinite path graph with vertices $w + J_k$ with $k \in \mathbb{Z}$.

Proposition 6.27. All possible translation vectors between seeds in the mutation class of $(\mathbf{v}_0, B_0)$ are parallel to b.

Proof. Suppose that two seeds are related by a translation w. We assume that the seeds are represented by triangles (or infinite regions) Δ and Δ' together with quivers Q and Q'. By assumption $\Delta' = w + \Delta$. We define H to be the foot of the altitude on the side of Δ that corresponds to the source in Q if Δ is acute-angled, and define H to be the foot of the altitude on the side of Δ that corresponds to the source of the arrow between the two sides of the obtuse angle otherwise. We define H' similarly. Since the triangles Δ and Δ' are congruent and Q is isomorphic to Q', we must have $H' = w + H$. By Proposition 6.24 both H and H' lie on b. □

By construction, the translation of the initial seed $(\mathbf{v}_0, B_0)$ by a translation vector w parallel to b is again an acyclic seed in the acyclic belt $w + (\mathbf{v}_0, B_0)$. For some vectors w, for example integer multiples of the translation vector $T(\Delta_0)$, this coincides with the acyclic belt of $(\mathbf{v}_0, B_0)$.

Proposition 6.28. Let Q be a quiver with three vertices and let $n_1, n_2, n_3 \in [1, 2n]$ be natural numbers such that $n_1\alpha + n_2\alpha + n_3\alpha = \pi$ and $\gcd(n_1, n_2, n_3) = 1$. Then, up to translation by vectors parallel to b, the mutation class of $(\mathbf{v}_0, B_0)$ contains exactly two seeds with quiver Q whose triangles have angles $n_1\alpha$, $n_2\alpha$ and $n_3\alpha$ (corresponding to the vertices of Q in this order).

Proof. The mutation class of $(\mathbf{v}_0, B_0)$ must contain a triangle with angles $n_1\alpha$, $n_2\alpha$ and $n_3\alpha$ by Proposition 6.3. The conserved quantity $T(\Delta) = T(\Delta_0)$ from Proposition 6.19 is homogeneous of degree 1, which implies that the angles of the triangle Δ suffice to determine its size. In particular, we can reconstruct the triangle up to congruence uniquely when given its angles.

Proposition 6.24 shows that the belt line b intersects every triangle corresponding to a seed in Γ in the feet of two altitudes. Moreover, the given quiver Q and the given angles tell us which altitudes we have to consider. Namely, for acyclic quivers (with acute-angled triangles as in case (1) of Proposition 6.24) we have to consider the altitudes on the sides that correspond to the source and the sink. The quiver must be oriented towards the reference point by Remark 6.25, which determines the location of the triangles up to translations by vectors parallel to b and reflection across b.

For cyclic quivers (with obtuse-angled triangles as in case (2) of Proposition 6.24) we have to consider the altitudes on the sides of the obtuse angle. The triangle must lie either in Π_1 or in Π_2, where Π_1 and Π_2 are the half-planes bounded by b as in Remark 6.25. This location determines the orientation of the triangle by Remark 6.25 as well as the relative position of H_1 and H_3 on b.

Hence in all cases, up to translation by vectors parallel to b, there are exactly two triangles with prescribed angles in the mutation class of $(\mathbf{v}_0, B_0)$. □

Proposition 6.29. Let $w \in \mathbb{E}^2$. Assume that there exist two seeds $(\mathbf{v}_1, B_1)$ and $(\mathbf{v}_2, B_2)$ in Γ such that $(\mathbf{v}_2, B_2) = w + (\mathbf{v}_1, B_1)$. Then the map $T_w : \Gamma \to \Gamma$ with $(\mathbf{v}, B) \mapsto w + (\mathbf{v}, B)$ is an automorphism of graphs.

Proof. Suppose that $(\mathbf{v}, B)$ is a seed in Γ and that the triangles of $(\mathbf{v}, B)$ and $w + (\mathbf{v}, B)$ are given by Δ and $w + \Delta$. Since w is parallel to b, a side of Δ has the same sign as the corresponding side of $w + \Delta$. Hence, a mutation at these sides yields triangles which are related to each other by a translation by w. □

Proposition 6.30. The choice of $V^+ \subseteq V$ is compatible with every seed.

Proof. We can verify the statement by drawing the sectors of non-compatible reference points, separately for acute-angled and obtuse-angled triangles, like we draw them for the initial triangle in Fig. 11. An inspection based on Remark 6.25 shows that the choice of the reference point u infinitely far on the line b is compatible for every triangle. □

6.7. On the lengths of sides and translation vectors

We denote the length of the larger side in the initial triangle by d_1, see Fig. 9.

Suppose that the seed $(\mathbf{v}, B)$ is obtained from $(\mathbf{v}_0, B_0)$ by a sequence of mutations. Assume that $(\mathbf{v}, B)$ is represented by a triangle Δ. Recall that $T(\Delta) = T(\Delta_0)$. Hence, if a is a side of Δ and β and γ are angles in Δ incident with a, then β and γ are rational multiples of $\alpha = \pi/(2n+1)$ and

$$a = d_1 \frac{\sin(\alpha)\sin(n\alpha)}{\sin(\beta)\sin(\gamma)}. \tag{1}$$

Definition 6.31 *(Translation vector lengths).* For every $k \in [1, n]$ with $\gcd(k, 2n+1) = 1$ we put

$$s_k = d_1 \frac{\sin(\alpha)\sin(n\alpha)}{\sin^2(k\alpha)}.$$

Proposition 6.32. Suppose that $k \in [1, n]$ is coprime to $2n+1$. Furthermore, let $w_k \in \mathbb{E}^2$ be a vector parallel to the line b whose length is equal to s_k, where s_k is as in Definition 6.31. Then the exchange graph of $(\mathbf{v}_0, B_0)$ contains the translated acyclic belt $I + w_k$.

Proof. We consider an infinite region Δ that is given by two parallel half-lines that intersect the third (finite) side a of the region at angles $k\alpha$ and $\pi - k\alpha$. By Proposition 6.3 the mutation class of $(\mathbf{v}_0, B_0)$ contains a seed $(\mathbf{v}, B)$ whose domain is equal to Δ. Its quiver must be oriented cyclically since $\pi - k\alpha$ is obtuse. The mutation of Δ at one of the parallel sides yields an infinite region Δ' which is obtained from Δ by translation along a. Note that a is parallel to the line b by Proposition 6.27. The equation $T(\Delta) = T(\Delta_0)$ implies that the length of a is equal to s_k.

Then there exists a mutation sequence that takes Δ to a triangle I_k in the initial acyclic belt I in the same way as Δ' is mapped to the translation of I_k along a. □

7. The algebraic number theory of sines and cosines of occurring angles

7.1. Cyclotomic fields

In this section we study the number theoretic properties of the cosines of the angles that occur in the triangles. The material in this section is classical and can be found for example in Lang's books about cyclotomic fields.

Definition 7.1 *(Cyclotomic polynomials).* For every $d \geq 1$ the polynomial

$$\Phi_d = \prod_{\substack{k \in [1,d] \\ \gcd(k,d)=1}} (x - e^{2\pi k i/d}) \in \mathbb{C}[x]$$

is called the d-th *cyclotomic polynomial.*

Cyclotomic polynomials satisfy many properties. In particular, for any given $d \geq 1$, the polynomial $\Phi_d \in \mathbb{Z}[x]$ is a monic polynomial with integer coefficients. In fact, it is the minimal polynomial of any primitive d-th root of unity $e^{2\pi k i/d}$ with $\gcd(k, d) = 1$. Moreover, Φ_d is irreducible over $\mathbb{Z}$ and for every $m \geq 1$ we have

$$x^m - 1 = \prod_{d|m} \Phi_d.$$

The degree of the cyclotomic polynomial $\deg(\Phi_d) = \varphi(d)$ is given by Euler's totient function. We consider the primitive root of unity $\zeta = e^{2\alpha i}$ with $\alpha = \frac{\pi}{2n+1}$. Since $2n+1$ is odd, the zeros of Φ_{2n+1} can be grouped into pairs of complex conjugates. In other words the set of complex zeros of Φ_{2n+1} is equal to $\left\{ \zeta^k, \zeta^{-k} \mid k \in U \right\}$ where

$$U = \{ k \in [1, n] \mid \gcd(k, 2n+1) = 1 \}.$$

In particular, the polynomial Φ_{2n+1} is palindromic, i.e. $\Phi_{2n+1}(1/x) = x^{-\varphi(2n+1)}\Phi_{2n+1}(x)$. Hence there is a polynomial $\Psi_{2n+1} \in \mathbb{Z}[x]$ of degree $\varphi(2n+1)/2$ such that

$$\Psi_{2n+1}(x + x^{-1}) = x^{-\varphi(2n+1)/2}\Phi_{2n+1}(x).$$

This polynomial must be irreducible over $\mathbb{Z}$, because any non-trivial factorization would yield a non-trivial factorization of Φ_{2n+1}. Furthermore, $2\cos(2k\alpha)$ is a root of Ψ_{2n+1} for every $k \in U$. As the cardinality of the set U is equal to the degree of Ψ_{2n+1}, the polynomial Ψ_{2n+1} cannot have roots other than the ones from above. Moreover, Ψ_{2n+1} is monic.

Definition 7.2 *(Chebyshev polynomials of the first kind).* Define $T_m \in \mathbb{Z}[x]$ recursively by $T_0 = 1$, $T_1 = x$ and $T_{m+1} = 2xT_m - T_{m-1}$ for $m \geq 1$.

It is well-known that $T_m(\cos(x)) = \cos(mx)$ for all $m \in \mathbb{N}$ and $x \in \mathbb{R}$. Note that every T_m is a monic integer polynomial in $t = 2x$. Moreover, a result of Watkins–Zeitlin [27, Equation (2)] asserts that if $m = 2n+1$ is odd, then

$$T_{n+1} - T_n = 2^n \prod_{d|m} \Psi_d.$$

The field $\mathbb{Q}(\zeta)$ is known as the *cyclotomic field*. We also consider the following number field.

Definition 7.3 *(Maximal real subfield).* Put $K = \mathbb{Q}(2\cos(2\alpha)) = \mathbb{Q}(\zeta + \zeta^{-1})$.

Then $K \subseteq \mathbb{Q}(\zeta)$ is the maximal real subfield of the cyclotomic field. Notice that K contains the element $2\cos(k\alpha)$ for every $k \in U$.

Recall that an element $x \in \mathbb{C}$ is called an *algebraic integer* if it is a root of a monic polynomial with integer coefficients. The set of algebraic integers in a field extension $F/\mathbb{Q}$ is a subring of F and is denoted $\mathcal{O}_F$. It is known that $\mathcal{O}_{\mathbb{Q}(\zeta)} = \mathbb{Z}[\zeta]$. Note that $2\cos(2k\alpha) \in \mathcal{O}_K$ for every $k \in U$ because the elements are roots of the monic integer polynomial Ψ_{2n+1}. In fact, the set $\{2\cos(2k\alpha) \mid k \in U\}$ is a basis of the $\mathbb{Z}$-module $\mathcal{O}_K$.

Note that for every $k \in U$ the equality $\cos(2k\alpha) = -\cos((2n+1-2k)\alpha)$ holds. If $k > n/2$, then the number $2n+1-2k$ is an odd integer in the interval $[1, n]$. In particular,

$$\mathcal{O}_K = \langle 2\cos(2k\alpha) \mid k \in U \rangle_{\mathbb{Z}} = \langle 2\cos(k\alpha) \mid k \in U \rangle_{\mathbb{Z}} . \tag{2}$$

We are also interested in the groups of units $\mathcal{O}_K^\times \subseteq \mathcal{O}_{\mathbb{Q}(\zeta)}^\times$ because they contain elements which are relevant for our geometric discussions. Elements of these groups are known as *cyclotomic units* in the literature.

Proposition 7.4. Let $k \in [1, n]$.

(1) We have

$$(a) \quad \frac{1-\zeta^k}{1-\zeta}, \frac{1-\zeta^k}{1-\zeta^n} \in \mathcal{O}_{\mathbb{Q}(\zeta)}, \qquad (b) \quad \frac{\sin(k\alpha)}{\sin(\alpha)}, \frac{\sin(k\alpha)}{\sin(n\alpha)} \in \mathcal{O}_K.$$

(2) If in addition k is coprime to $2n+1$, then

$$(a) \quad \frac{1-\zeta^k}{1-\zeta}, \frac{1-\zeta^k}{1-\zeta^n} \in \mathcal{O}_{\mathbb{Q}(\zeta)}^\times, \qquad (b) \quad \frac{\sin(k\alpha)}{\sin(\alpha)}, \frac{\sin(k\alpha)}{\sin(n\alpha)} \in \mathcal{O}_K^\times.$$

Proof. (1a) Using the geometric series we see that the element $(1-\zeta^k)/(1-\zeta) = \sum_{l=0}^{k-1} \zeta^l$ belongs to $\mathcal{O}_{\mathbb{Q}(\zeta)} = \mathbb{Z}[\zeta]$. For the second claim note that n is coprime to $2n+1$. Hence ζ^n is a generator of the cyclic group $\{\zeta^k \mid k \in [0, 2n]\}$. Hence there is an integer l such that $\zeta^k = \zeta^{nl}$. Then $(1-\zeta^{nl})/(1-\zeta^n) \in \mathcal{O}_{\mathbb{Q}(\zeta)} = \mathbb{Z}[\zeta]$ using the geometric series as before.

(1b) By the previous part of the proposition $(1-\zeta^k)/(1-\zeta)$ is an algebraic integer. The same is true for the $(4n+2)$-th root of unity $\zeta^{\frac{1-k}{2}}$. Hence the product of the two numbers is an algebraic integer as well. The identities $2\sin(k\alpha) = \zeta^{k/2} - \zeta^{-k/2}$ and $2\sin(\alpha) = \zeta^{1/2} - \zeta^{-1/2}$ imply

$$\zeta^{\frac{1-k}{2}} \cdot \frac{1-\zeta^k}{1-\zeta} = \frac{\sin(k\alpha)}{\sin(\alpha)}.$$

Now $\sin(k\alpha)/\sin(\alpha) \in \mathcal{O}_K$ because it is an algebraic integer and it belongs to $K = \mathbb{Q}(\zeta) \cap \mathbb{R}$.

(2) Now suppose that $\gcd(k, 2n+1) = 1$. First we want to show that $(1-\zeta)/(1-\zeta^k) \in \mathcal{O}_{\mathbb{Q}(\zeta)}$. The assumption implies that ζ^k is a generator of the cyclic group $\{\zeta^l \mid l \in [0, 2n]\}$ and $(1-\zeta^{kl})/(1-\zeta^k) \in \mathcal{O}_{\mathbb{Q}(\zeta)} = \mathbb{Z}[\zeta]$ using the geometric series as above. The claim $(1-\zeta^n)/(1-\zeta^k) \in \mathcal{O}_{\mathbb{Q}(\zeta)}$ can be shown analogously. The claim $(1-\zeta)/(1-\zeta^k) \in \mathcal{O}_{\mathbb{Q}(\zeta)}$ also implies that $\sin(\alpha)/\sin(k\alpha) \in \mathcal{O}_K$. Similarly $\sin(n\alpha)/\sin(k\alpha) \in \mathcal{O}_K$. □

7.2. The Galois group

The field extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ is Galois. Its Galois group is isomorphic to the multiplicative group $(\mathbb{Z}/(2n+1)\mathbb{Z})^\times$. Explicitly, an isomorphism is given by

$$(\mathbb{Z}/(2n+1)\mathbb{Z})^\times \to \mathrm{Gal}(\mathbb{Q}(\zeta), \mathbb{Q}),\ l + (2n+1)\mathbb{Z} \mapsto m_l$$

where $m_l \colon \mathbb{Q}(\zeta) \to \mathbb{Q}(\zeta)$ maps $\zeta^r \mapsto \zeta^{rl}$ for every $r \in [0, 2n]$.

Notice that every m_l (with l coprime to $2n+1$) leaves the subfield K invariant. Hence the restriction defines a homomorphism of fields

$$\sigma_l \colon K \to K,\ 2\cos(2r\alpha) \mapsto 2\cos(2rl\alpha) \tag{3}$$

for every $r \in [0, 2n]$. These homomorphisms are actually automorphisms since they define a permutation on the basis $\{2\cos(2k\alpha) \mid k \in U\}$ thanks to the coprimality of l and $2n+1$. However, they are not pairwise different. If $l \in [1, n]$ is coprime to $2n+1$, then $2n+1-l$ is coprime to $2n+1$ as well and $m_l = m_{2n+1-l}$ because $(2l + 2(2n+1-l))\alpha = 2\pi$. In fact, the map

$$(\mathbb{Z}/(2n+1)\mathbb{Z})^\times/\{\pm 1\} \to \mathrm{Aut}(K)$$
$$l \mapsto m_l$$

is an isomorphism of groups. In particular, the automorphism group of K is abelian.

Proposition 7.5. Suppose that $r, l \in [1, n]$ such that $\gcd(l, 2n+1) = 1$. Then

$$\sigma_l\left(\frac{1}{\sin^2(r\alpha)}\right) = \frac{1}{\sin^2(rl\alpha)}$$

where σ is defined as in equation (3).

Proof. The claim follows from the identity $1/\sin^2(r\alpha) = 2/(1-\cos(2r\alpha))$ because σ_l is a map of fields. □

8. Linear independence of translation vectors

8.1. An estimate of the coefficients

In this section we want to show that the set

$$\left\{ \frac{1}{\sin^2(k\alpha)} \mid k \in U \right\}$$

is linearly independent over $\mathbb{Q}$. This set is equal to the set of all translation vectors from Definition 6.31, scaled by inverse of the common factor $d_1 \sin(\alpha)\sin(n\alpha)$. Since the cardinality of the set $U = \{k \in [1, n] \mid \gcd(k, 2n+1) = 1\}$ is equal to the degree of the field extension $K/\mathbb{Q}$, this implies that this set of translation vectors is a basis of K as a vector space over $\mathbb{Q}$.

The following result is a special case of a Verlinde formula. A more general formula and a proof can be found in the article by Zagier, see [28]. Our formula can be obtained from Zagier's $D(g, k)$ by putting $k = 2$.

Proposition 8.1. For every $n \geq 1$ we have

$$\sum_{k=1}^{n} \frac{1}{\sin^2(k\alpha)} = \frac{2}{3}n(n+1).$$

Clearly, the first term in the sum is larger than every other summand. In fact, the first summand is larger than the sum of the remaining terms as the following lemma shows.

Lemma 8.2. For every $n \geq 1$ we have

$$\sum_{k=2}^{n} \frac{1}{\sin^2(k\alpha)} < \frac{1}{\sin^2(\alpha)}.$$

Proof. By Proposition 8.1 the claim is equivalent to the inequalities

$$\frac{2}{3}n(n+1) < \frac{2}{\sin^2(\alpha)} \Leftrightarrow \sin^2\left(\frac{\pi}{2n+1}\right) < \frac{3}{n(n+1)}.$$

It is well known that $\sin(x) < x$ for all $x > 0$. Moreover, using Archimedes' bound $\pi < 22/7$ it is easy to see that $\pi < 2\sqrt{3}$ or $\pi^2 < 12$. We conclude that

$$\sin^2\left(\frac{\pi}{2n+1}\right) < \frac{\pi^2}{4n^2+4n+1} < \frac{12}{4n^2+4n} = \frac{3}{n(n+1)},$$

which finishes the proof of the claim. □

8.2. Determinants of group characters

In this subsection we recall the definition of a group character and present a classical result about a determinant constructed from group characters. The determinant will play a crucial role in the proof of the linear independence of the translation vectors.

Definition 8.3 *(Group character).* Let G be a finite group. A *character* is a group homomorphism from G to $\mathbb{C}^\times$.

Since every element g in a finite group G has finite order, the image of every character χ must lie in the unit circle $S^1 \subseteq \mathbb{C}^\times$.

The following theorem has a long and colourful history. In the special case of cyclic groups the theorem yields a factorisation of the determinant of a circulant matrix which was first proved by Catalan. Our formulation is due to Dedekind although Burnside proved a related statement. The theorem was generalised to all finite groups by Frobenius.

Theorem 8.4 *(Dedekind).* Let $G = \{g_1, \ldots, g_t\}$ be a finite abelian group of order t. Suppose that $R = \mathbb{C}[X_g \mid g \in G]$ is the polynomial ring in t variables indexed by the elements of G. We define a matrix $M \in \mathrm{Mat}_{t\times t}(R)$ by putting $M_{ij} = X_{g_i g_j^{-1}}$. Then the determinant is given by

$$\det(M) = \prod_{\chi\,:\,G\to S^1} \left(\sum_{g\in G} \chi(g) X_g\right),$$

where the sum runs over all characters of the group G. In particular, every factor on the right hand side is an irreducible homogeneous polynomial of degree 1.

8.3. Linear independence of translation vectors

Recall that a $\mathbb{Q}$-basis B of a Galois extension $F/\mathbb{Q}$ is called *normal* if there exists an element $a \in F$ such that

$$B = \{\sigma(a) \mid \sigma \in \mathrm{Gal}(F, \mathbb{Q})\}.$$

The Normal Basis Theorem asserts that every Galois extension has a normal basis. The following statement is a variation of a well-known argument which plays a role in the proof of the Normal Basis Theorem. Note that we do not assume that the field extension is Galois.

Lemma 8.5. Let $F/\mathbb{Q}$ be a field extension. Assume that $\phi_1, \ldots, \phi_r \colon F \to F$ are isomorphisms of fields. Let $a \in K$. We define a matrix $M \in \mathrm{Mat}_{r\times r}(K)$ by $M_{ij} = \phi_i^{-1}(\phi_j(a))$. Suppose that $\det(M) \neq 0$. Then $\phi_1(a), \ldots, \phi_r(a)$ are linearly independent over $\mathbb{Q}$.

Proof. Suppose that $\lambda_1, \ldots, \lambda_r \in \mathbb{Q}$ such that $\lambda_1\phi_1(a) + \ldots + \lambda_r\phi_r(a) = 0$. Note that every field automorphism $\phi \in \mathrm{Aut}(K)$ restricts to the identity on $\mathbb{Q} \subseteq F$ since $\phi(1) = 1$. We apply $\phi_1^{-1}, \ldots, \phi_r^{-1}$ to the above equation and obtain

$$M \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{pmatrix} = 0.$$

As M is invertible, we can conclude that $\lambda_1 = \ldots = \lambda_r = 0$. □

Theorem 8.6. Let $n \geq 1$ and $\alpha = \frac{\pi}{2n+1}$. The set

$$\left\{ \frac{1}{\sin^2(k\alpha)} \mid k \in [1, n], \gcd(k, 2n+1) = 1 \right\}$$

is linearly independent over $\mathbb{Q}$.

Proof. Recall the abbreviation $U = \{k \in [1, n] \mid \gcd(k, 2n+1) = 1\}$. We apply Lemma 8.5 to $a = 1/\sin^2(\alpha)$ and the field automorphisms $\sigma_l \in \mathrm{Aut}(K)$ with $l \in U$ from Section 7.2. It is sufficient to prove that the determinant of the matrix $M \in \mathrm{Mat}_{U\times U}(K)$ with

$$M_{rs} = (\sigma_r^{-1} \circ \sigma_s)(a) \qquad (r, s \in U)$$

is not equal to zero. We compute the determinant by Theorem 8.4 of Dedekind. More precisely, we apply the theorem to $G = \mathrm{Aut}(K)$ and substitute $X_m = \sigma(a)$ for all $m \in G$. We obtain

$$\det(M) = \prod_{\chi\,:\,G\to S^1} \left(\sum_{l\in U} \chi(\sigma_l)\sigma_l(a)\right).$$

We show that every factor in the right hand side of the equation is non-zero. Let $\chi : G \to S^1$ be a character. Notice that $1 \in U$ because it is always coprime to $2n+1$. We have $\sigma_1(a) = a$ because σ_1 is the neutral element in G. The triangle inequality, the fact that the image $\mathrm{im}(\chi)$ is contained in the unit circle, and Lemma 8.2 yield the chain

$$\left|\sum_{\substack{l\in U\\ l\neq 1}} \chi(\sigma_l)\sigma_l(a)\right| \leq \sum_{\substack{l\in U\\ l\neq 1}} \sigma_l(a)|\chi(\sigma_l)| = \sum_{\substack{l\in U\\ l\neq 1}} \sigma_l(a) \leq \sum_{l=2}^{n} \sigma_l(a) < a = |\chi(\sigma_1)\sigma_1(a)|$$

of inequalities. We can conclude that

$$\sum_{\substack{l\in U\\ l\neq 1}} \chi(\sigma_l)\sigma_l(a) \neq -\chi(\sigma_1)\sigma_1(a)$$

because those are two complex numbers with different absolute values. □

9. The structure of the exchange graph

9.1. The lattice of translation vectors

Let us introduce the following lattices.

Definition 9.1 *(Length lattices).*

(1) (Translation lattice from infinite regions) Put

$$R = \left\langle d_1 \frac{\sin(\alpha)\sin(n\alpha)}{\sin^2(k\alpha)} \mid k \in U \right\rangle_{\mathbb{Z}}.$$

(2) (Translation lattice) Let L be the $\mathbb{Z}$-module spanned by the Euclidean lengths of all vectors $w \in \mathbb{E}^2$ such that the exchange graph of $(\mathbf{v}_0, B_0)$ contains two seeds $(\mathbf{v}_1, B_1)$ and $(\mathbf{v}_2, B_2)$ with $(\mathbf{v}_1, B_1) = (\mathbf{v}_2, B_2) + w$.

(3) (Side lattice) Let L' be the $\mathbb{Z}$-module spanned by the Euclidean lengths of sides in triangles Δ such that the exchange graph of $(\mathbf{v}_0, B_0)$ contains a seed $(\mathbf{v}, B)$ that is represented by the triangle Δ.

Proposition 9.2.

(1) There exists a natural number d such that there are inclusions:

$$\begin{array}{ccc} L & \subseteq & \frac{1}{d}\mathbb{Z}[2\cos(\alpha)]d_1 \\ \cup\!\mid & & \cup\!\mid \\ R & \subseteq & \mathbb{Z}[2\cos(\alpha)]d_1 \end{array}$$

(2) The ranks of the lattices are equal to $\mathrm{rk}_{\mathbb{Z}}(R) = \mathrm{rk}_{\mathbb{Z}}(L) = \mathrm{rk}_{\mathbb{Z}}(\mathbb{Z}[2\cos(\alpha)]d_1) = \varphi(2n+1)/2$.

Proof. The inclusion $R \subseteq \mathbb{Z}[2\cos(\alpha)]d_1$ follows from Proposition 7.4 (2b) and closure of $\mathbb{Z}[2\cos(\alpha)] = \mathcal{O}_K$ under multiplication. The inclusion $R \subseteq L$ was established in Proposition 6.32.

To construct the integer d note that

$$L' \subseteq \left\langle d_1 \frac{\sin(\alpha)\sin(n\alpha)}{\sin(k_1\alpha)\sin(k_2\alpha)} \mid k_1, k_2 \in [1, n] \right\rangle_{\mathbb{Z}}$$

by virtue of formula (1), see Subsection 6.7. The formula $2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$ for $x, y \in \mathbb{R}$ implies that the numerators and the denominators of the generators belong to the field $K = \langle 2\cos(k\alpha) \mid k \in U \rangle_{\mathbb{Q}}$. Hence $L' \subseteq Kd_1$. Let $e \in \mathbb{N}$ be the common denominator of all the fractions that occur as coefficients in $\mathbb{Q}$-linear combinations of the generators of L' in the basis $\{2\cos(k\alpha)d_1 \mid k \in U\}$ of d_1K. Then $L' \subseteq \frac{1}{e}\mathbb{Z}[2\cos(\alpha)]d_1$.

Let $w \in L$ be a translation vector. Hence, there exists a sequence (v_i, B_i), $i \in [1, k]$ of seeds such that $(v_1, B_1) = (v_k, B_k) + w$ and two adjacent elements in the sequence are related to each other by a single mutation. We denote by Δ_1 and Δ_k the triangles associated with the first and the last seed, and by $w_1, w_k \in \mathbb{E}^2$ two vertices of Δ_1 and Δ_k with $w_1 = w_k + w$. From this we can conclude that there are vectors u_i with $i \in [1, l]$ with $w = \sum_{i=1}^{l} u_i$ such that $|u_i| \in L'$ is a side length in a triangle given by one of the seeds in the mutation sequence and angle between u_i and b is an integer multiple of α for every $i \in [1, l]$. So we may write $w = \sum_{i=1}^{l} |u_i| \cos(m_i\alpha)$ with $m_i \in \mathbb{Z}$. The formula $2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$ and

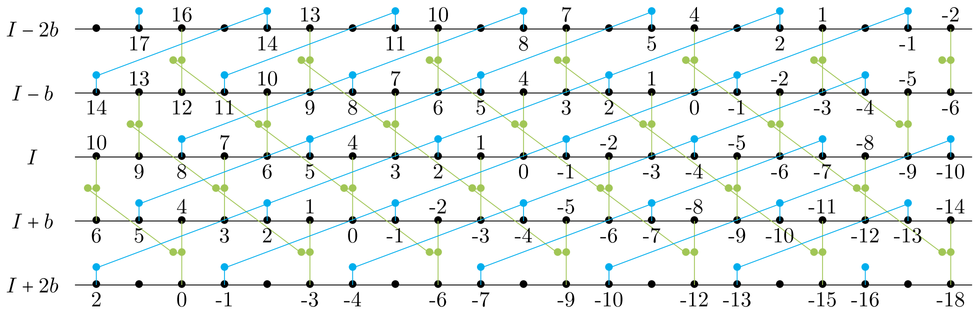

Fig. 16. The exchange graph for the case $\alpha = \pi/5$.

the inclusion $L' \subseteq \frac{1}{e}\mathbb{Z}[2\cos(\alpha)]d_1$ imply that $L \subseteq \frac{1}{d}\mathbb{Z}[2\cos(\alpha)]d_1$ for $d = 2e$. The inclusion $\mathbb{Z}[2\cos(\alpha)]d_1 \subseteq \frac{1}{d}\mathbb{Z}[2\cos(\alpha)]d_1$ is automatic.

The rank of the lattice R is equal to $\varphi(2n+1)/2$ by Theorem 8.6. The lattice $\frac{1}{d}\mathbb{Z}[2\cos(\alpha)]d_1$ has the same rank. From this we can conclude that all intermediate lattices must have the same rank as well. □

Question 9.3. Does the equality $R = L$ hold?

9.2. The exchange graph

In this subsection we describe the structure of the exchange graph of $(\mathbf{v}_0, B_0)$. We can summarise the previous discussion as follows.

Theorem 9.4. The exchange graph of $(\mathbf{v}_0, B_0)$ has the following structure.

(1) The acyclic belt $I + w$ is a full subgraph of the exchange graph for every $w \in L$.
(2) Suppose that $n_1, n_2, n_3 \in [0, 2n+1]$ are natural numbers such that $\gcd(n_1, n_2, n_3) = 1$, so that there exists a Euclidean triangle Δ with angles $n_1\alpha$, $n_2\alpha$, $n_3\alpha$. Furthermore, let Q be a quiver corresponding to Δ, which is acyclic if Δ is acute-angled and cyclic if Δ is obtuse-angled. Then there exist exactly two seeds in the exchange graph whose triangle is congruent to Δ and whose quiver is Q. Furthermore, two such seeds are related to each other by a translation by a vector in L or a reflection across the belt line b.

Let us illustrate the result by an example.

Example 9.5. Suppose that $n = 2$, i.e. $\alpha = \pi/5$. The exchange graph of $(\mathbf{v}_0, B_0)$ is shown in Fig. 16. We denote by $w \in \mathbb{E}^2$ the generator of the 1-dimensional lattice L. The exchange graph has the following properties.

(1) The acyclic belt $I + nw$ is a full subgraph of the exchange graph for every $n \in \mathbb{Z}$.
(2) (Lines with negative slope, green when in colour) For every $(n, m) \in \mathbb{Z} \times 3\mathbb{Z}$ the exchange graph contains additional vertices $R_{n,m}$ and $S_{n,m}$ such that the following conditions hold:
  - The vertex $R_{n,m}$ is adjacent to $(I + nw)_m$, $(I + (n-1)w)_{m+4}$ and $S_{n,m}$.
  - The vertex $S_{n,m}$ is adjacent to $R_{n,m}$, $S_{n-1,m+6}$ and $S_{n+1,m-6}$.
(3) (Lines with positive slope, blue when in colour) For every $(n, m) \in \mathbb{Z} \times (2 + 3\mathbb{Z})$ the exchange graph contains an additional vertex $T_{n,m}$ such that $T_{n,m}$ adjacent to $(I + nw)_m$, $T_{n-1,m}$ and $T_{n+1,m}$.

9.3. Quasi-isometries

Definition 9.6 *(Quasi-isometries).* Suppose that (M_1, d_1) and (M_2, d_2) are metric spaces. A map $f \colon M_1 \to M_2$ is called a *quasi-isometry* if the following conditions hold.

(1) There exist real numbers $a \geq 1$ and $b \geq 0$ such that

$$\frac{1}{a}d_1(x, y) - b \leq d_2(f(x), f(y)) \leq a d_1(x, y) + b$$

for all $x, y \in M_1$.
(2) There exists a real number $c \geq 0$ such that for every $z \in M_2$ there exists $x \in M_1$ with $d_2(z, f(x)) \leq c$.

We say that (M_1, d_1) and (M_2, d_2) are *quasi-isometric* if there exists a quasi-isometry $f: M_1 \to M_2$.

It is known that the composition of two quasi-isometries is again a quasi-isometry, and that if $f: M_1 \to M_2$ is a quasi-isometry with constants a, b, c, then the map $g: M_2 \to M_1$, where $g(z) = x$ for an arbitrary $x \in M_1$ such that $d_2(z, f(x)) \le c$, is again a quasi-isometry. It follows that quasi-isometry is an equivalence relation on metric spaces.

Every isometry is a quasi-isometry, but the converse is false in general. Other examples of quasi-isometries are the inclusions $\mathbb{Z}^k \hookrightarrow \mathbb{R}^k$ and $k\mathbb{Z} \hookrightarrow \mathbb{Z}$ for each $k \ge 1$.

Let $\Gamma = (V_0, V_1)$ be a simple graph with vertex set V_0 and edge set V_1. Recall that the distance d between two vertices in Γ is the number of edges in a shortest path between the two vertices. In this way, (V_0, d) becomes a metric space. Also we may view (V_0, V_1) as a 1-dimensional cell complex, and the distance d induces a metric on (V_0, V_1). Notice that the embedding $(V_0, d) \hookrightarrow ((V_0, V_1), d)$ is a quasi-isometry.

Another source of quasi-isometries is group theory. Recall that a generating set S of a group G is called *symmetric* if $S = S^{-1}$ and S does not contain the identity.

Notation 9.7 *(Cayley graph).* Assume that a group G is generated by a finite symmetric set $S \subseteq G$. By $\mathrm{Cay}_S(G)$ we denote the *Cayley graph* of G (i.e. the graph with vertex set G and an edge between $g, h \in G$ if $gh^{-1} \in S$).

It is well known that if a graph has two finite, symmetric generating sets $S, S' \subseteq G$, then $\mathrm{id}: G \to G$ induces a quasi-isomorphism between $\mathrm{Cay}_S(G)$ and $\mathrm{Cay}_{S'}(G)$.

Suppose that G is a group with identity e and M is a set. Recall that a *group action* of G on M is a map $G \times M \to M$, $(g, m) \mapsto gm$ such that $ex = x$ for all $x \in M$ and $(gh)x = g(hx)$ for all $g, h \in M$ and $x \in X$.

Definition 9.8 *(Group actions on metric spaces).* Assume that the group G acts on a metric space (M, d).

(1) We say the group action is *isometric* if $d(gx, gy) = d(x, y)$ for all $g \in G$ and $x, y \in M$.
(2) We say the group action is *properly discontinuous* if the set $\{g \in G \mid d(x, gx) \le r\}$ is finite for every $x \in M$ and $r \ge 0$.
(3) We say the group action is *cocompact* if the orbit space M/G is compact with respect to the quotient topology.

Recall that a metric space (M, d) is called *proper* if the closed ball $B_r(x) = \{y \in M \mid d(x, y) \le r\}$ is compact for every $x \in M$ and $r > 0$. It is called *geodesic* if for all $x, y \in M$ there exists a *geodesic* between x and y, that is, an isometric embedding $p: [0, d] \to M$ such that $p(0) = x$ and $p(d) = y$ where $d = d(x, y)$.

Theorem 9.9 *(Schwarz [22] and Milnor [20]).* Suppose that a group G acts on a proper, geodesic metric space (M, d) and that the group action is isometric, properly discontinuous and cocompact. Then G is finitely generated. Moreover, (M, d) is quasi-isometric to $\mathrm{Cay}_S(G)$ for every finite, symmetric generating set $S \subseteq G$. To be concrete, fix $x \in M$. Then the map $G \to M$ with $g \mapsto gx$ is a quasi-isomorphism.

Corollary 9.10. The exchange graph Γ is quasi-isometric to the Cayley graph of the lattice L, and a quasi-isomorphism is given by the translation of the initial seed i.e. by the map

$$L \to \Gamma,\ w \mapsto w + (\mathbf{v}_0, B_0).$$

Proof. The graph Γ (viewed as a 1-dimensional cell complex) is a geodesic metric space. It is proper because every ball $B_r(x)$ in Γ is sequentially compact and hence compact.

We consider the translation map $L \times \Gamma \to \Gamma$ with $(w, (\mathbf{v}, B)) \mapsto w + (\mathbf{v}, B)$. The definition of L implies that the map is well-defined, that is, the image $w + (\mathbf{v}, B)$ belongs to Γ for all $w \in L$ and all seeds $(\mathbf{v}, B)$. It is a group action by definition. Proposition 6.29 implies that the group action is isometric. Let $x = (\mathbf{v}, B)$ be a vertex of Γ and $r \ge 0$. Then the number of vertices y of Γ with $d(x, y) \le r$ is finite. For every such y there is at most one $w \in L$ such that $y = w + x$. We see that the group action is properly discontinuous. By construction every seed $x = (\mathbf{v}, B)$ is given by a quiver and a triangle such that the angles are given by $n_1\alpha$, $n_2\alpha$ and $n_3\alpha$ for some natural numbers satisfying $n_1 + n_2 + n_3 = 2n + 1$. By Proposition 6.28 the exchange graph contains at most 2 triangles of that kind for every quiver Q and every triple (n_1, n_2, n_3) up to translation. Hence Γ/L is a finite graph. In particular, it is compact so that the group action is cocompact. The claim follows from the theorem of Schwarz and Milnor. □

Choose a basis $B = \{l_1, \dots, l_r\}$ of the lattice L where $r = \mathrm{rk}_{\mathbb{Z}}(L) = \varphi(2n + 1)/2$ denotes the rank. Then the Cayley graph of L with respect to B is isomorphic to $\mathbb{Z}^r$ with two lattice points $x, y \in \mathbb{Z}^r$ being connected by an edge if and only if their Euclidean distance is equal to 1. In particular, Γ is quasi-isometric to $\mathbb{Z}^r$.

9.4. The growth rate of the exchange graph

We use the abbreviation $x_0 = (\mathbf{v}_0, B_0)$ for the initial seed. Recall that $\Gamma = (V_0, V_1)$ is the exchange graph of x_0.

Definition 9.11 *(Growth function).* The *growth function* $gr\colon \mathbb{N} \to \mathbb{N}$ is defined by

$$gr(n) = |\{x \in V_0 \mid d(x_0, x) \leq n\}|.$$

Definition 9.12 *(Polynomial growth).* We say that Γ has *polynomial growth* if $gr(n) = \mathcal{O}(n^r)$ for some $r \geq 0$. If this happens to be the case, then we call the smallest natural number r such that $gr(n) = \mathcal{O}(n^r)$ the *polynomial growth rate* of Γ.

For example, let us consider $\mathbb{Z}^r$. Then $f(n)$ is equal to the number of points $\mathbf{x} \in \mathbb{Z}^r$ such that $\sum_{i=1}^n |x_i| \leq n$. The sequence $gr(n)$ is also known as the *crystal ball sequence* in the literature. It is well known that $gr(n) = \lambda n^r + \mathcal{O}(n^{r-1})$ where $\lambda = 2^r/r!$ is a constant (depending on r but not on n). In particular, $\mathbb{Z}^r$ has polynomial growth with growth rate r.

Corollary 9.13. The exchange graph Γ has polynomial growth and its polynomial growth rate is equal to $\varphi(2n+1)/2$.

Proof. By Corollary 9.10, Γ is quasi-isometric to the Cayley graph of L where L is a lattice of rank $\varphi(2n+1)/2$. The Cayley graph of L is isomorphic to $\mathbb{Z}^{\varphi(2n+1)/2}$ and therefore has polynomial growth with growth rate $r = \varphi(2n+1)/2$.

Recall that $x_0 = (\mathbf{v}_0, B_0)$ denotes the initial seed. We consider the map $f\colon L \to \Gamma$ given by $f(w) = w + x_0$. Corollary 9.10 asserts that f is a quasi-isomorphism. Note that $f(0) = x_0$.

According to the definition of a quasi-isomorphism we can pick $a \geq 1$ and $b \geq 0$ such that $\frac{1}{a} d_L(u, w) - b \leq d_\Gamma(f(u), f(w)) \leq a d_L(u, w) + b$ for all $u, w \in L$. Moreover, we can pick $c \geq 0$ such that for all $x \in V_0$ there exists $w \in L$ such that $d_\Gamma(x, f(w)) \leq c$. Here, the subscripts indicate the metric spaces of the distance functions. Notice that for every $w \in L$ there are only finitely many $x \in V_0$ satisfying $d_\Gamma(x, f(w)) \leq c$. In fact, since Γ is a 3-regular graph, the number of such x can be bounded by 3^c. Notice that this constant does not depend on x.

Suppose that $x \in V_0$ is a seed in Γ. We choose a $w \in L$ such that $d_\Gamma(x, f(w)) \leq c$. The triangle inequality implies

$$d_L(w, 0) \leq a[d_\Gamma(f(w), f(0)) + b] \leq a[d_\Gamma(f(w), x) + d_\Gamma(x, x_0) + b] \leq a d_\Gamma(x, x_0) + a(b+c); \tag{4}$$

$$d_L(w, 0) \geq \frac{1}{a}[d_\Gamma(f(w), f(0)) - b] \geq \frac{1}{a}[-d_\Gamma(f(w), x) + d_\Gamma(x, x_0) - b] \geq \frac{1}{a} d_\Gamma(x, x_0) - \frac{b+c}{a}. \tag{5}$$

From inequality (4) we can conclude that

$$gr^\Gamma(n) \leq 3^c gr^L(an + a(b+c)) = \mathcal{O}(n^r).$$

Here the superscripts indicate the metric space of the growth function. In particular, Γ has polynomial growth, and its growth rate is at most r. For any $n \gg 0$ there are $\lambda n^r + \mathcal{O}(n^{r-1})$ pairwise distinct elements $w \in L$ such that $d_L(w, 0) \leq n$. Notice that the quasi-isomorphism f is injective by construction. Application of f yields $\lambda n^r + \mathcal{O}(n^{r-1})$ pairwise distinct seeds x in Γ such that $d_\Gamma(x, x_0) \leq an + b + c = \mathcal{O}(n)$. $\square$

10. Exchange graphs for even least common denominators

10.1. The structure of the exchange graphs

In this section we consider geometric mutations of seeds that are given by triangles in the Euclidean plane whose angles are rational multiples of π where the least common denominator of the three rational multiples is *even*.

Fix a natural number $n \geq 2$, and put $\alpha = \pi/2n$. Suppose that the initial seed $(\mathbf{v}_0, B_0)$ is given by a triangle $\Delta_0 = A_1^{(0)} A_2^{(0)} A_3^{(0)}$ with angles $A_1^{(0)} = \alpha$, $A_2^{(0)} = (n-1)\alpha$, and $A_3^{(0)} = n\alpha$. For $i \in \{1, 2, 3\}$ we denote the side of Δ opposite to $A_i^{(0)}$ by $a_i^{(0)}$. To construct a seed we introduce a quiver with vertices $a_1^{(0)}$, $a_2^{(0)}$, and $a_3^{(0)}$, and arrows $a_1^{(0)} \to a_3^{(0)}$ and $a_3^{(0)} \to a_2^{(0)}$. As before, we assume that the reference point lies infinitely far away on the line b, which is constructed from the initial triangle by the billiard geometry. It is easy to check that this is the unique compatible choice of it.

The geometric considerations in Section 6.2 do not depend on the parity of the common denominator. In particular, there is a line $b \subseteq \mathbb{E}^2$ that contains the feet of two altitudes of every triangle Δ in the exchange graph. In this case, the line b contains the vertex with the right angle of the initial triangle. Moreover, the quantity $T(\Delta) = a_1 \sin(A_2)\sin(A_3)$ is conserved and hence is the same for every triangle $\Delta = A_1 A_2 A_3$ in Γ.

Some statements in Section 7 undergo a slight change when we switch to an even common denominator. Quintessentially, equation (2) does not hold anymore. We put $K = \mathbb{Q}(2\cos(2\alpha))$ and consider the ring of integers $\mathcal{O}_K$. Then $2\cos(\alpha) \notin K$. (For example, when $n = 2$, then $K = \mathbb{Q}(2\cos(2\alpha)) = \mathbb{Q}$ but $2\cos(\alpha) = \sqrt{2} \notin \mathbb{Q}$.) In particular,

$$\mathcal{O}_K = \langle\, 2\cos(2k\alpha) \mid k \in [1,n],\ \gcd(k,2n) = 1\, \rangle_{\mathbb{Z}} \subsetneqq \langle\, 2\cos(k\alpha) \mid k \in [1,n],\ \gcd(k,2n) = 1\, \rangle_{\mathbb{Z}}\,.$$

Notice that $\mathrm{rk}_{\mathbb{Z}}(\mathbb{Z}[2\cos(\alpha)]) = \varphi(2n)$ is twice as large as $\mathrm{rk}_{\mathbb{Z}}(\mathcal{O}_K) = \varphi(2n)/2$. The lattice

$$R = \left\langle d_1 \frac{\sin(\alpha)\sin(n\alpha)}{\sin^2(k\alpha)} \mid k \in [1,n],\ \gcd(k,2n) = 1 \right\rangle_{\mathbb{Z}}$$

is generated by the lengths of all the finite sides bounding infinite regions in Γ. For every element $w \in R$ the translation map $(\mathbf{v}, B) \mapsto w + (\mathbf{v}, B)$ induces a symmetry of the exchange graph. We can show that $\mathrm{rk}_{\mathbb{Z}}(R) = \varphi(2n)/2$ similar to Section 7.2. As before, let L denote the lattice of all elements $w \in \mathbb{E}^2$ such that the translation map $(\mathbf{v}, B) \mapsto w + (\mathbf{v}, B)$ induces a symmetry of the exchange graph. As in Proposition 9.2 there exists a natural number d such that there are inclusions:

$$\begin{array}{ccc} L & \subseteq & \frac{1}{d}\mathbb{Z}[2\cos(\alpha)]d_1 \\ \cup\!\mid & & \cup\!\mid \\ R & \subseteq & \mathbb{Z}[2\cos(\alpha)]d_1 \end{array}$$

However, in this situation the lower left corner of the diagram does not have the same rank as the upper right corner as before. We conclude with the following theorem.

Theorem 10.1.

(1) For every $(n_1, n_2, n_3) \in [0, 2n+1]$ with $\gcd(n_1, n_2, n_3) = 1$ and every $w \in L$ the exchange graph contains at most two additional vertices represented by (finite or infinite) triangles with angles $n_1\alpha$, $n_2\alpha$, $n_3\alpha$ (with two different orientations). For a fixed triple (n_1, n_2, n_3) together with a fixed orientation all these triangles are related to each other by translations by vectors in L.
(2) We have $\mathrm{rk}_{\mathbb{Z}}(L) = \varphi(2n)/2$ or $\mathrm{rk}_{\mathbb{Z}}(L) = \varphi(2n)$.
(3) The exchange graph Γ is quasi-isometric to the Cayley graph of the lattice L, and a quasi-isomorphism is given by the map $L \to \Gamma$ with $w \mapsto w + (\mathbf{v}_0, B_0)$ where $(\mathbf{v}_0, B_0)$ is the initial seed of Γ.

Data availability

No data was used for the research described in the article.

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