

Near optimal spectral gaps for hyperbolic surfaces

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Abstract

We prove that if X is a finite area non-compact hyperbolic surface, then for any $\epsilon > 0$, with probability tending to one as $n \rightarrow \infty$, a uniformly random degree n Riemannian cover of X has no eigenvalues of the Laplacian in $[0, \frac{1}{4} - \epsilon)$ other than those of X , and with the same multiplicities.

As a result, using a compactification procedure due to Buser, Burger, and Dodziuk, we settle in the affirmative the question of whether there exists a sequence of closed hyperbolic surfaces with genera tending to infinity and first non-zero eigenvalue of the Laplacian tending to $\frac{1}{4}$.

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1 Introduction

Let X be a finite-area hyperbolic surface, that is, a smooth surface with Riemannian metric of constant curvature -1 . In this paper, all hyperbolic surfaces are assumed complete and orientable. The Laplacian operator Δ_X on $L^2(X)$ has spectrum in $[0, \infty)$. The bottom of the spectrum is always a discrete eigenvalue at 0 that is simple if and only if X is connected.

If X is non-compact, which is the first focus of our paper, then the spectrum of Δ_X in $[0, \frac{1}{4})$ consists of finitely many discrete eigenvalues and the spectrum is absolutely continuous in $[\frac{1}{4}, \infty)$ [LP81]. As such, the *spectral gap* between 0 and the rest of the spectrum has size at most $\frac{1}{4}$.

We are interested in the size of this spectral gap for random surfaces. The random model we use is the random covering model from [MN20, MNP20, MN21]. For any $n \in \mathbf{N}$, the space of degree n Riemannian covering spaces of X is a finite set that we equip with the uniform probability measure. We say that a family of events, depending on n , holds *asymptotically almost surely* (a.a.s.) if they hold with probability tending to one as $n \rightarrow \infty$. We also note that any eigenvalue of the Laplacian on X will be an eigenvalue of any cover of X , with at least as large multiplicity. Therefore these ‘old eigenvalues’ must always be taken into account. As we explain in §§2.1, a theorem of Dixon [Dix69] implies that a uniformly random degree n covering space of a fixed connected non-compact X will be connected a.a.s.

The first theorem of the paper is the following.

Theorem 1.1. *Let X be a finite-area non-compact hyperbolic surface. Let X_n denote a uniformly random degree n covering space of X . For any $\epsilon > 0$, a.a.s.*

$$\text{spec}(\Delta_{X_n}) \cap \left[0, \frac{1}{4} - \epsilon\right) = \text{spec}(\Delta_X) \cap \left[0, \frac{1}{4} - \epsilon\right)$$

and the multiplicities on both sides are the same.

This theorem is the analog, for finite-area non-compact hyperbolic surfaces of Friedman’s theorem [Fri08] (formerly Alon’s conjecture [Alo86]) stating that random d -regular graphs have almost optimal spectral gaps. Friedman also proposed in [Fri03] that a variant of Alon’s conjecture should hold for random covers of any fixed finite graph, and this extended conjecture was recently proved by Bordenave and Collins [BC19].

If X has Euler characteristic $\chi(X) = -1$, then a result of Otal and Rosas [OR09, Thm. 2] states that $\text{spec}(\Delta_X) \cap [0, \frac{1}{4})$ is just the multiplicity one eigenvalue at 0. Taking such an X as the base surface in Theorem 1.1 we therefore obtain the following corollary.

Corollary 1.2. *There exist connected finite-area non-compact hyperbolic surfaces X_i with $\chi(X_i) \rightarrow -\infty$ and*

$$\inf(\operatorname{spec}(\Delta_{X_i}) \cap (0, \infty)) \rightarrow \frac{1}{4}.$$

In fact one can take $\chi(X_i) = -i$.

The corresponding question for closed surfaces X is as follows. If X is closed then the spectrum of Δ_X consists of eigenvalues

$$0 = \lambda_0(X) \leq \lambda_1(X) \leq \dots \leq \lambda_i(X) \leq \dots$$

with $\lambda_i(X) \rightarrow \infty$ as $i \rightarrow \infty$. Suppose that X_i are a sequence of closed hyperbolic surfaces with genera $g(X_i) \rightarrow \infty$. Huber proved in [Hub74] that in this case,

$$\limsup \lambda_1(X_i) \leq \frac{1}{4}.$$

Therefore $\frac{1}{4}$ is an asymptotically optimal lower bound for $\lambda_1(X_i)$. In the same scenario, it was conjectured in [Bus78] that $\lim_{i \rightarrow \infty} \lambda_1(X_i) \rightarrow 0$. This was corrected in [Bus84] where it was put forward that ‘probably’ there exist a sequence of X_i as above with

$$\lim_{i \rightarrow \infty} \lambda_1(X_i) = \frac{1}{4}.$$

See [WX18, Conj. 5], [Wri20, Problem 10.3] for more recent iterations of the question of whether $\lambda_1(X_i)$ can go to $\frac{1}{4}$ as the genus tends to infinity.

We are able to resolve this question here by combining Corollary 1.2 with the ‘handle lemma’ of Buser, Burger, and Dodziuk [BBD88]; see Brooks and Makover [BM01, Lemma 1.1] for the extraction of the lemma. We obtain

Corollary 1.3. *There exist closed hyperbolic surfaces X_i with genera $g(X_i) \rightarrow \infty$ and $\lambda_1(X_i) \rightarrow \frac{1}{4}$. In fact one can take $g(X_i) = i$.*

1.1 Prior work

The analog of Theorem 1.1 was proved for conformally compact infinite area hyperbolic surfaces by the second named author and Naud in [MN21], following a previous work with an intermediate result [MN20]. The reader should see however [MN21] for details because the spectral theory is more subtle for infinite area surfaces and moreover the results in [MN21] go beyond statements about L^2 eigenvalues.

The very first uniform spectral gap for certain combinatorial models of random surfaces (the ‘Brooks—Makover’ models), yielding both finite-area non-compact and closed surfaces, appears in work of Brooks and Makover [BM04]. This spectral gap is non-explicit.

The analog of Theorem 1.1 for closed hyperbolic surfaces is still unknown. The best known result in the random cover model is due to the second named author, Naud, and Puder [MNP20], building on [MP20], that gives an a.a.s. relative¹ spectral gap of size $\frac{3}{16} - \epsilon$ for random covers of a closed hyperbolic surface. The fact that a Weil—Petersson random closed hyperbolic surface enjoys spectral gap of size $\frac{3}{16} - \epsilon$ with probability tending to one as genus $\rightarrow \infty$ was proved independently by Wu and Xue [WX21] and Lipnowski and Wright [LW21]. The first uniform spectral gap of size ≈ 0.0024 for Weil—Petersson random closed surfaces was proved by Mirzakhani in [Mir13]. Prior to the current work, the first named author proved in [Hid22] that Weil—Petersson random surfaces of genus g with $O(g^\alpha)$ cusps, with $\alpha < \frac{1}{2}$, have an explicit positive uniform spectral gap depending on α as $g \rightarrow \infty$, which coincides (for any fixed $\epsilon > 0$) with $\frac{3}{16} - \epsilon$ if $\alpha = 0$.

The fact that all models of random closed hyperbolic surfaces are currently stuck at $\lambda_1 = \frac{3}{16} - \epsilon$ is an obstacle to proving Corollary 1.3 by the more direct method of proving that random closed surfaces of large genus have almost optimal spectral gaps. This $\frac{3}{16}$ barrier bears explanation; it also appears in the famous $\frac{3}{16}$ Theorem of Selberg [Sel65]. Very roughly speaking, this barrier corresponds to fine control of probability events concerning *simple* closed geodesics in random surfaces and not having as fine control on non-simple geodesics. This splitting of geodesics into simple and non-simple goes back to work of Broder and Shamir [BS87] in the graph theoretic setting. The appearance of $\frac{3}{16}$ in [Sel65] is for different reasons relating to Selberg’s use of Weil’s bounds [Wei48] on Kloosterman sums.

The previous best results towards Corollary 1.2, just for the case of $\chi(X_i) \rightarrow -\infty$, all came from arithmetic hyperbolic surfaces. The records were held by Selberg [Sel65] (spectral gaps of size $\frac{3}{16} = 0.1875$), Gelbart and Jacquet [GJ78] (existence of spectral gaps larger than $\frac{3}{16}$), Luo, Rudnick, and Sarnak [LRS95] (spectral gaps of size $\frac{171}{784} \approx 0.218$), Kim and Shahidi [KS02b, KS02a] (spectral gaps of size $\frac{77}{324} \approx 0.23765$), and currently, Kim and Sarnak [Kim03, Appendix 2] (spectral gaps of size $\frac{975}{4096} \approx 0.23804$).

The history of Corollary 1.3 began with McKean’s paper [McK72, McK74] where it was wrongly asserted that the first eigenvalue of any compact hyperbolic surface is at least $\frac{1}{4}$; this was disproved by Randol in [Ran74] who ac-

¹Here relative refers to disregarding eigenvalues of the base surface.

tually showed that there can be arbitrarily many eigenvalues in $(0, \frac{1}{4})$. Buser proved in [Bus84] using Selberg’s $\frac{3}{16}$ theorem and the Jacquet—Langlands machinery [JL70] that there exist closed hyperbolic surfaces X_i with genera $g(X_i) \rightarrow \infty$ and $\lambda_1(X_i) \geq \frac{3}{16}$. Buser, Burger, and Dodziuk proved in [BBD88] that one can get the slightly weaker result $\lambda_1(X_i) \rightarrow \frac{3}{16}$ but without using Jacquet—Langlands. This turns out to be instrumental in the current work; see Lemma 8.1 below.

Following these works, all progress towards Corollary 1.3 ran parallel to the previously discussed progress to Corollary 1.2. In particular, the previous best known result towards Corollary 1.3 comes from combining the Kim-Sarnak bound with either Jacquet—Langlands or [BBD88] to obtain the first part of Corollary 1.3 with $\frac{1}{4}$ replaced by $\frac{975}{4096}$.

1.2 Overview of proof

The proof of Theorem 1.1 follows the following strategy. We view

$$X = \Gamma \backslash \mathbb{H}$$

where Γ is a discrete torsion free subgroup of $\mathrm{PSL}_2(\mathbf{R})$ and \mathbb{H} is the hyperbolic upper half plane. We explain in §2 that the random degree n covers of X are parameterized by

$$\phi \in \mathrm{Hom}(\Gamma, S_n) \mapsto X_\phi$$

where ϕ is a uniformly random homomorphism of the free group Γ into the symmetric group S_n . Because we disregard eigenvalues and eigenfunctions lifted from X , we restrict our attention to the space $L_{\mathrm{new}}^2(X_\phi)$ of functions that are orthogonal to all lifts of L^2 functions from X . The strategy is then, for any $s_0 > \frac{1}{2}$, to asymptotically almost surely produce a bounded resolvent operator

$$R_{X_\phi}(s) : L_{\mathrm{new}}^2(X_\phi) \rightarrow H_{\mathrm{new}}^2(X_\phi)$$

where $H_{\mathrm{new}}^2(X_\phi) \subset L_{\mathrm{new}}^2(X_\phi)$ is a suitable Sobolev space (see §2.2) such that for any $s \in [s_0, 1]$ the identity

$$(\Delta_{X_\phi} - s(1-s)) R_{X_\phi}(s) = \mathrm{Id}_{L_{\mathrm{new}}^2(X_\phi)},$$

makes sense and holds true; this will forbid new eigenvalues in $[0, s_0(1-s_0)]$.

The way we build our resolvent is by a parametrix construction. The fundamental structure of parametrix construction is to produce a bounded operator

$$\mathbb{M}_\phi(s) : L_{\mathrm{new}}^2(X_\phi) \rightarrow H_{\mathrm{new}}^2(X_\phi)$$

such that

$$(\Delta_{X_\phi} - s(1-s)) \mathbb{M}_\phi(s) = \text{Id}_{L^2_{\text{new}}(X_\phi)} + \mathbb{L}_\phi(s)$$

where we aim to prove $\mathbb{L}_\phi(s)$ has norm less than one as a bounded operator on $L^2_{\text{new}}(X_\phi)$ so that we can obtain our bounded resolvent

$$R_{X_\phi}(s) = \mathbb{M}_\phi(s) \left(\text{Id}_{L^2_{\text{new}}(X_\phi)} + \mathbb{L}_\phi(s) \right)^{-1}.$$

Often in a parametrix construction one also wants the \mathbb{L} term to be compact: this will also be the case here and turns out to be essential for the application of random operator results.

The way we build $\mathbb{M}_\phi(s)$ is by patching together a ‘cuspidal parametrix’ $\mathbb{M}_\phi^{\text{cusp}}(s)$ based on a model resolvent in the cusps and an interior parametrix $\mathbb{M}_\phi^{\text{int}}(s)$ that only depends on the localization of its argument to a compact part of X_ϕ . We then let

$$\mathbb{M}_\phi(s) = \mathbb{M}_\phi^{\text{int}}(s) + \mathbb{M}_\phi^{\text{cusp}}(s)$$

and we get a resulting splitting

$$\mathbb{L}_\phi(s) = \mathbb{L}_\phi^{\text{int}}(s) + \mathbb{L}_\phi^{\text{cusp}}(s).$$

In §4 we show that for any ϕ , the term $\mathbb{M}_\phi^{\text{cusp}}(s)$ can be designed so that $\|\mathbb{L}_\phi^{\text{cusp}}(s)\| \leq \frac{1}{5}$ (or any small number), so that it will not essentially interfere with our plans of obtaining $\|\mathbb{L}_\phi(s)\| < 1$. The ability to do this is a particular feature of the geometry of hyperbolic cusps.

The term $\mathbb{M}_\phi^{\text{int}}(s)$ is based on averaging the resolvent kernel of the hyperbolic plane over the fundamental group of Γ (suitably twisting by ϕ) to obtain an integral operator on $L^2_{\text{new}}(X_\phi)$. The problem with this is that the averaging will not obviously converge, so we have to multiply the hyperbolic resolvent kernel by a radial cutoff that localizes to radii $\leq T+1$ to get a priori convergence for all $s \in (\frac{1}{2}, 1]$. This gives us that $\mathbb{M}_\phi^{\text{int}}(s)$ is bounded (Lemma 5.5).

The effect of this cutoff is that the error term $\mathbb{L}_\phi^{\text{int}}(s)$ is an integral operator with smooth kernel. We prove that we can unitarily conjugate $\mathbb{L}_\phi^{\text{int}}(s)$ to

$$\sum_{\gamma \in \Gamma} a_\gamma(s) \otimes \phi(\gamma)$$

acting on $L^2(F) \otimes V_n^0$, where F is a Dirichlet fundamental domain for Γ and V_n^0 is the standard $n-1$ dimensional irreducible representation of S_n .

The $a_\gamma(s)$ are compact operators on $L^2(F)$ and there are only finitely many $\gamma \in \Gamma$ for which $a_\gamma(s)$ is non-zero.

If instead, the $a_\gamma(s)$ were elements of $\text{End}(\mathbf{C}^r)$ for some fixed finite r , because Γ is free we would now be exactly in the situation to apply the breakthrough results of Bordenave and Collins from [BC19]. These random operator results, combined with a linearization trick of Pisier [Pis18], would tell us that for any $\epsilon > 0$, a.a.s.

$$\|\mathbb{L}_\phi^{\text{int}}(s)\| \leq \left\| \sum_{\gamma \in \Gamma} a_\gamma(s) \otimes \rho_\infty(\gamma) \right\| + \epsilon \quad (1.1)$$

where $\rho_\infty : \Gamma \rightarrow \text{End}(\ell^2(\Gamma))$ is the right regular representation. Because the $a_\gamma(s)$ are in reality compact operators on Hilbert spaces we can approximate by finite rank operators to the same effect.

The key point (which is a sticking point in other approaches to this problem using e.g. transfer operators) is that we understand the operator in the right hand side of (1.1) well: it can be unitarily conjugated to an operator on $L^2(\mathbb{H})$ that is the composition of multiplication with a cutoff (with norm ≤ 1) and an integral operator with real-valued radial kernel that localizes to radii in $[T, T + 1]$.

The key is that as this latter operator is self-adjoint we can use the theory of the Selberg transform to estimate its norm in Lemma 5.2. By choosing T sufficiently large in the beginning, we can force the norm in the right hand side of (1.1) to be as small as we like, given any $s_0 > \frac{1}{2}$, for all $s \geq s_0$.

Assembling these arguments, for any $s \geq s_0 > \frac{1}{2}$, a.a.s. $1 + \mathbb{L}_\phi(s)$ can be inverted which rules out X_ϕ having a new eigenvalue at $s(1 - s)$. To be able to get this result for all $s \geq s_0$ with probability tending to one, we make sure that $\|\mathbb{L}_\phi(s)\| \leq \frac{3}{5}$ at a fine enough net of $s \in [s_0, 1]$ and then use a deviations estimate (Lemma 6.1) to show (deterministically) that $\|\mathbb{L}_\phi(s)\|$ fluctuates at most by $\frac{1}{5}$ from point to point.

A historical remark: a similar parametrix method in the context of hyperbolic surfaces (albeit for completely different purposes) goes back to work of Guillopé and Zworski [GZ95] who used the method to give sharp upper bounds on the number of resonances of geometrically finite hyperbolic surfaces in balls. In turn this method is based on Vodev's ‘impressive refinement of the Fredholm determinant method’ (*ibid.*) [Vod91, Vod92, Vod94] for the control of scattering poles of perturbed Laplacians in Euclidean spaces.

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2 Random covers

2.1 Construction

Let X be a finite area non-compact connected hyperbolic surface. We view X as

$$X = \Gamma \backslash \mathbb{H}$$

where

$$\mathbb{H} = \{x + iy : x, y \in \mathbf{R}, y > 0\}$$

with metric

$$\frac{dx^2 + dy^2}{y^2}$$

where Γ is a discrete torsion-free subgroup of $\mathrm{PSL}_2(\mathbf{R})$ that acts, via Möbius transformations, by orientation preserving isometries on \mathbb{H} .

For $n \in \mathbf{N}$ let $[n] \stackrel{\mathrm{def}}{=} \{1, \dots, n\}$ and S_n denote the group of permutations of $[n]$. For any homomorphism $\phi \in \mathrm{Hom}(\Gamma, S_n)$ we construct a hyperbolic surface as follows. Let Γ act on $\mathbb{H} \times [n]$ by

$$\gamma(z, x) \stackrel{\mathrm{def}}{=} (\gamma z, \phi(\gamma)[x])$$

and let

$$X_\phi \stackrel{\mathrm{def}}{=} \Gamma \backslash_\phi (\mathbb{H} \times [n])$$

denote the quotient by this action. If we choose ϕ uniformly at random in $\mathrm{Hom}(\Gamma, S_n)$, the resulting X_ϕ is a uniformly random degree n covering space of X . Note that Γ is a free group freely generated by some

$$\gamma_1, \dots, \gamma_d \in \Gamma$$

and choosing ϕ is the same as choosing

$$\sigma_i \stackrel{\mathrm{def}}{=} \phi(\gamma_i), \quad i = 1, \dots, d$$

independently and uniformly at random in S_n .

The surface X_ϕ is connected if and only if Γ acts transitively on $[n]$ via ϕ . By a theorem of Dixon [Dix69], two independent and uniformly random permutations in S_n generate S_n or A_n a.a.s and it follows that a uniformly random cover X_ϕ is connected a.a.s.

Let $V_n \stackrel{\text{def}}{=} \ell^2([n])$ and $V_n^0 \subset V_n$ the subspace of functions on $[n]$ with zero mean. The representation of S_n on $\ell^2([n])$ is its standard representation by 0-1 matrices and the subspace V_n^0 is an irreducible subspace of dimension $(n-1)$: we write

$$\rho_\phi : \Gamma \rightarrow \text{End}(V_n^0)$$

for the random representation of Γ induced by the random ϕ .

2.2 Function spaces

We define

$$L_{\text{new}}^2(X_\phi)$$

to be the space of L^2 functions on X_ϕ that are orthogonal to all pullbacks of L^2 functions from X . The elements $f \in L_{\text{new}}^2(X_\phi)$ have mean value 0 fiber-wise in the sense that for almost every $x \in X$ we have $\sum_{i=1}^n f(x_i) = 0$ where x_i are the lifts of x to X_ϕ .

We have

$$L^2(X_\phi) \cong L_{\text{new}}^2(X_\phi) \oplus L^2(X).$$

This induces a multiplicity respecting inclusion $\text{spec}(\Delta_X) \subset \text{spec}(\Delta_{X_\phi})$. All other eigenvalues of Δ_{X_ϕ} (with multiplicities) arise from eigenfunctions in the subspace $L_{\text{new}}^2(X_\phi)$.

Let F denote a Dirichlet fundamental domain for X , that is, for some fixed $o \in \mathbb{H}$,

$$F \stackrel{\text{def}}{=} \bigcap_{\gamma \in \Gamma \setminus \{\text{id}\}} \{z \in \mathbb{H} : d(o, z) < d(z, \gamma o)\}.$$

This choice will be convenient for the proof of Lemma 5.4. Let $C^\infty(\mathbb{H}; V_n^0)$ denote the smooth V_n^0 -valued functions on \mathbb{H} . There is an isometric linear isomorphism between

$$C^\infty(X_\phi) \cap L_{\text{new}}^2(X_\phi)$$

and the space of smooth V_n^0 -valued functions on \mathbb{H} satisfying

$$f(\gamma z) = \rho_\phi(\gamma)f(z) \tag{2.1}$$

for all $\gamma \in \Gamma$, with finite norm

$$\|f\|_{L^2(F)}^2 \stackrel{\text{def}}{=} \int_F \|f(z)\|_{V_n^0}^2 d\mathbb{H} < \infty$$

We will denote $C_\phi^\infty(\mathbb{H}; V_n^0) \subset C^\infty(\mathbb{H}; V_n^0)$ for these functions. Under this isomorphism, the Laplacian on $C^\infty(X_\phi) \cap L_{\text{new}}^2(X_\phi)$ is intertwined with the Laplacian that acts on $C_\phi^\infty(\mathbb{H}; V_n^0)$ in the obvious way (this can be defined by choosing any basis of V_n^0 and letting the Laplacian act coordinatewise on V_n^0 -valued functions). The completion of $C_\phi^\infty(\mathbb{H}; V_n^0)$ with respect to $\|\bullet\|_{L^2(F)}$ is denoted by $L_\phi^2(\mathbb{H}; V_n^0)$; the isomorphism above extends to one between $L_{\text{new}}^2(X_\phi)$ and $L_\phi^2(\mathbb{H}; V_n^0)$.

Let $C_{c,\phi}^\infty(\mathbb{H}; V_n^0)$ denote the elements of $C_\phi^\infty(\mathbb{H}; V_n^0)$ that are compactly supported when restricted to \bar{F} (in other words, compactly supported modulo Γ).

We also consider the following Sobolev spaces. Let $H^2(\mathbb{H})$ denote the completion of $C_c^\infty(\mathbb{H})$ with respect to the norm

$$\|f\|_{H^2(\mathbb{H})}^2 \stackrel{\text{def}}{=} \|f\|_{L^2(\mathbb{H})}^2 + \|\Delta f\|_{L^2(\mathbb{H})}^2.$$

We let $H_\phi^2(\mathbb{H}; V_n^0)$ denote the completion of $C_{c,\phi}^\infty(\mathbb{H}; V_n^0)$ with respect to the norm

$$\|f\|_{H_\phi^2(\mathbb{H}; V_n^0)}^2 \stackrel{\text{def}}{=} \|f\|_{L^2(F)}^2 + \|\Delta f\|_{L^2(F)}^2.$$

We let $H^2(X_\phi)$ denote the completion of $C_c^\infty(X_\phi)$ with respect to the norm

$$\|f\|_{H^2(X_\phi)}^2 \stackrel{\text{def}}{=} \|f\|_{L^2(X_\phi)}^2 + \|\Delta f\|_{L^2(X_\phi)}^2.$$

Viewing $H^2(X_\phi)$ as a subspace of $L^2(X_\phi)$ in the obvious way, we let

$$H_{\text{new}}^2(X_\phi) \stackrel{\text{def}}{=} H^2(X_\phi) \cap L_{\text{new}}^2(X_\phi).$$

Similarly to before there is an isometric isomorphism between $H_{\text{new}}^2(X_\phi)$ and $H_\phi^2(\mathbb{H}; V_n^0)$ that intertwines the two relevant Laplacian operators.

3 Background random matrix theory

In this section we give an account of the breakthrough result of Bordenave and Collins [BC19] and its amplification through a linearization trick that appears in Pisier [Pis18, Cor. 14] (see also, historically, [Pis96, HT05], and [BC19, §6]).

Let $\text{std}_n : S_n \rightarrow \text{End}(V_n^0)$ denote the linear action of S_n on V_n^0 . We let $\rho_\infty : \Gamma \rightarrow \text{End}(\ell^2(\Gamma))$ denote the right regular representation of Γ . We are going to use the following direct consequence of [BC19, Thm. 2].

Theorem 3.1 (Bordenave—Collins). *Suppose that $r \in \mathbf{N}$ and $a_0, a_1, \dots, a_d \in \text{Mat}_{r \times r}(\mathbf{C})$. Suppose $a_0^* = a_0$. Then for any $\epsilon > 0$, with probability tending to one as $n \rightarrow \infty$, we have*

$$\begin{aligned} & \left\| a_0 \otimes \text{Id}_{V_n^0} + \sum_{i=1}^d a_i \otimes \text{std}_n(\sigma_i) + a_i^* \otimes \text{std}_n(\sigma_i^{-1}) \right\|_{\mathbf{C}^r \otimes V_n^0} \\ & \leq \left\| a_0 \otimes \text{Id}_{\ell^2(\Gamma)} + \sum_{i=1}^d a_i \otimes \rho_\infty(\gamma_i) + a_i^* \otimes \rho_\infty(\gamma_i^{-1}) \right\|_{\mathbf{C}^r \otimes \ell^2(\Gamma)} + \epsilon. \end{aligned}$$

The norm on the top is the operator norm on $\mathbf{C}^r \otimes V_n^0$. The norm on the bottom is the operator norm on $\mathbf{C}^r \otimes \ell^2(\Gamma)$.

Remark 3.2. Note in the above that $\text{std}_n(\sigma_i) = \rho_\phi(\gamma_i)$.

This will be combined with the following result of Pisier [Pis18, Cor. 14 and following remark].

Proposition 3.3 (Pisier). *The result of Theorem 3.1 implies that for any $R, k \in \mathbf{N}$, any non-commutative polynomials*

$$P_1, P_2, \dots, P_k$$

in the variables $\sigma_1, \dots, \sigma_d, \sigma_1^{-1}, \dots, \sigma_d^{-1}$ with complex coefficients, and any $a_1, \dots, a_k \in \text{Mat}_{R \times R}(\mathbf{C})$ we have the following. For any $\epsilon > 0$, with probability tending to one as $n \rightarrow \infty$, we have

$$\begin{aligned} & \left\| \sum_{i=1}^k a_i \otimes \text{std}_n(P_i(\sigma_1, \dots, \sigma_d, \sigma_1^{-1}, \dots, \sigma_d^{-1})) \right\|_{\mathbf{C}^R \otimes V_n^0} \tag{3.1} \\ & \leq \left\| \sum_{i=1}^k a_i \otimes \rho_\infty(P_i(\gamma_1, \dots, \gamma_d, \gamma_1^{-1}, \dots, \gamma_d^{-1})) \right\|_{\mathbf{C}^R \otimes \ell^2(\Gamma)} + \epsilon. \end{aligned}$$

This type of ‘linearization’ is obviously extremely powerful and has played a role in many of the major breakthroughs in random operators in recent years [HT05, CM14, BC19]. Notice that one way to obtain an operator as in (3.1) is as

$$\sum_{\gamma \in \Gamma} a_\gamma \otimes \rho_\phi(\gamma)$$

where all $a_\gamma \in \text{Mat}_{R \times R}(\mathbf{C})$ and there are only finitely many non-zero a_γ . Therefore we obtain combining Theorem 3.1 and Proposition 3.3 the following:

Corollary 3.4. *For any $r \in \mathbf{N}$ and any finitely supported map $\gamma \in \Gamma \mapsto a_\gamma \in \text{Mat}_{r \times r}(\mathbf{C})$, for any $\epsilon > 0$, with probability tending to one as $n \rightarrow \infty$ we have*

$$\left\| \sum_{\gamma \in \Gamma} a_\gamma \otimes \rho_\phi(\gamma) \right\|_{\mathbf{C}^r \otimes V_n^0} \leq \left\| \sum_{\gamma \in \Gamma} a_\gamma \otimes \rho_\infty(\gamma) \right\|_{\mathbf{C}^r \otimes \ell^2(\Gamma)} + \epsilon.$$

No originality is claimed here: Corollary 3.4 is essentially already contained in [BC19] and in fact, [BC19, Thm. 3] is the version of this corollary when all a_i are scalar.

4 Cusp parametrix

To simplify notation, we assume there is just one cusp of the finite-area non compact hyperbolic surface X . This does not affect the arguments. It follows from the Collar Lemmas [Bus92, Thm. 4.4.6] that this single cusp can be identified with

$$\mathcal{C} \stackrel{\text{def}}{=} (1, \infty) \times S^1$$

with the metric

$$\frac{dr^2 + dx^2}{r^2}; \tag{4.1}$$

here the x coordinate is in $S^1 \stackrel{\text{def}}{=} \mathbf{R}/\mathbf{Z}$ and the r coordinate is in $[1, \infty)$. In the following, $\chi_{\mathcal{C}}^+, \chi_{\mathcal{C}}^- : \mathcal{C} \rightarrow [0, 1]$ will be functions that are identically zero in a neighborhood of $\{1\} \times S^1$, identically equal to 1 in a neighborhood of $\{\infty\} \times S^1$, and such that

$$\chi_{\mathcal{C}}^+ \chi_{\mathcal{C}}^- = \chi_{\mathcal{C}}^-. \tag{4.2}$$

We will extend both these functions by zero to functions on X , and define

$$\chi_{\mathcal{C}, \phi}^\pm \stackrel{\text{def}}{=} \chi_{\mathcal{C}}^\pm \circ \pi_\phi : X_\phi \rightarrow [0, 1]$$

where $\pi_\phi : X_\phi \rightarrow X$ is the covering map. Indeed, the cusp of X splits in X_ϕ into several regions of the form

$$(1, \infty) \times (\mathbf{R}/m\mathbf{Z}), \tag{4.3}$$

with $m \in \mathbf{N}$, and with the same metric (4.1). In these coordinates the covering map sends

$$\pi_\phi : (r, x + m\mathbf{Z}) \mapsto (r, x + \mathbf{Z}).$$

In particular, it preserves the r coordinate.

Lemma 4.1. *For any $\epsilon > 0$, it is possible to choose $\chi_{\mathcal{C}}^\pm$ as above so that for any ϕ*

$$\|\nabla\chi_{\mathcal{C},\phi}^+\|_\infty, \|\Delta\chi_{\mathcal{C},\phi}^+\|_\infty \leq \epsilon.$$

Proof. Given $\epsilon > 0$, let $\chi_{\mathcal{C}}^+ : [0, \infty) \rightarrow [0, 1]$ be a function such that $\chi_{\mathcal{C}}^+(\tau) \equiv 0$ for τ in $[0, 1]$, $\chi_{\mathcal{C}}^+(\tau) \equiv 1$ for $\tau \geq \tau_0$, for some $\tau_0 > 0$, and such that we have the following bounds for the derivatives of $\chi_{\mathcal{C}}^+$

$$\sup_{[0,\infty)} |(\chi_{\mathcal{C}}^+)'|, \sup_{[0,\infty)} |(\chi_{\mathcal{C}}^+)''| \leq \frac{\epsilon}{2}$$

(this can be achieved by scaling some fixed cutoff).

Let \mathcal{C}' be any cusp region of X_ϕ as in (4.3). Using the change of coordinates $r = e^\tau$ we view \mathcal{C}' as

$$(0, \infty)_\tau \times \mathbf{R}/m\mathbf{Z}$$

with the metric $(d\tau)^2 + e^{-2\tau}(dx)^2$; x being the coordinate in $\mathbf{R}/m\mathbf{Z}$. In these coordinates, one calculates directly from the formula for the metric that

$$\|\nabla\chi_{\mathcal{C},\phi}^+\|(\tau, x + m\mathbf{Z}) = |[\chi_{\mathcal{C}}^+]'(\tau)| \leq \epsilon$$

and

$$|\Delta\chi_{\mathcal{C},\phi}^+|(\tau, x + m\mathbf{Z}) = |[\chi_{\mathcal{C}}^+]''(\tau) - [\chi_{\mathcal{C}}^+]'(\tau)| \leq \epsilon$$

as required. Finally, one can easily choose $\chi_{\mathcal{C}}^-$ to be a function with $\chi_{\mathcal{C}}^-(\tau) \equiv 0$ for $\tau \leq \tau_0$ and $\chi_{\mathcal{C}}^-(\tau) \equiv 1$ for $\tau \geq 2\tau_0$. This will fulfill (4.2). \square

Let \mathcal{C}_ϕ denote the subset of X_ϕ that covers \mathcal{C} . It is convenient, to avoid complicated discussions about Sobolev spaces, to extend \mathcal{C} to the parabolic cylinder

$$\tilde{\mathcal{C}} \stackrel{\text{def}}{=} (0, \infty) \times S^1$$

with the same metric (4.1), and let $\tilde{\mathcal{C}}_\phi$ be the corresponding extension of \mathcal{C}_ϕ (the union of extensions of (4.3) to $(0, \infty) \times \mathbf{R}/m\mathbf{Z}$).

Let $H^2(\tilde{\mathcal{C}}_\phi)$ denote the completion of $C_c^\infty(\tilde{\mathcal{C}}_\phi)$ with respect to the given norm

$$\|f\|_{H^2}^2 \stackrel{\text{def}}{=} \|f\|_{L^2}^2 + \|\Delta f\|_{L^2}^2.$$

The Laplacian $\Delta = \Delta_{\tilde{\mathcal{C}}_\phi}$ extends uniquely from $C_c^\infty(\tilde{\mathcal{C}}_\phi)$ to a self-adjoint unbounded operator on $L^2(\tilde{\mathcal{C}}_\phi)$ with domain $H^2(\tilde{\mathcal{C}}_\phi)$.

Lemma 4.2. *For any $f \in H^2(\tilde{\mathcal{C}}_\phi)$, we have $\langle \Delta f, f \rangle \geq \frac{1}{4} \|f\|^2$.*

Proof. This is similar to [Mag15, Lemma 3.2]. It suffices to prove this for $\tilde{\mathcal{C}}_\phi$ replaced by $(0, \infty) \times (\mathbf{R}/m\mathbf{Z})$ with the metric (4.1) i.e. with only one connected component. Then changing coordinates to τ we are working in the region $(-\infty, \infty) \times (\mathbf{R}/m\mathbf{Z})$ with the metric $(d\tau)^2 + e^{-2\tau}(dx)^2$. The corresponding volume form is $e^{-\tau} d\tau \wedge dx$ and the Laplacian is given by $\Delta = -e^\tau \frac{\partial}{\partial \tau} e^{-\tau} \frac{\partial}{\partial \tau} - e^{2\tau} \frac{\partial^2}{\partial \theta^2}$. Now suppose $f \in C_c^\infty((-\infty, \infty) \times (\mathbf{R}/m\mathbf{Z}))$. We calculate

$$e^{-\tau/2} \Delta e^{\tau/2} = -\frac{\partial^2}{\partial \tau^2} + \frac{1}{4} - e^{2\tau} \frac{\partial^2}{\partial \theta^2}$$

so if $g = e^{-\tau/2} f \in C_c^\infty((-\infty, \infty) \times (\mathbf{R}/m\mathbf{Z}))$ we have

$$\begin{aligned} \int \Delta[f] \bar{f} e^{-\tau} d\tau \wedge dx &= \int_{-\infty}^{\infty} \int_0^n [e^{-\tau/2} \Delta e^{\tau/2}] (g) \bar{g} d\tau \wedge dx \\ &\geq \frac{1}{4} \int_{-\infty}^{\infty} \int_0^n g \bar{g} d\tau \wedge dx = \frac{1}{4} \int_{-\infty}^{\infty} \int_0^n f \bar{f} e^{-\tau} d\tau \wedge dx. \end{aligned}$$

The inequality here used integrating by parts. The inequality obtained now extends to $H^2(\tilde{\mathcal{C}}_\phi)$ by density of $C_c^\infty(\tilde{\mathcal{C}}_\phi)$ therein and continuity of $\langle \Delta f, f \rangle$. \square

Lemma 4.2 implies that the resolvent operator

$$R_{\tilde{\mathcal{C}}_\phi}(s) \stackrel{\text{def}}{=} (\Delta - s(1-s))^{-1} : L^2(\tilde{\mathcal{C}}_\phi) \rightarrow H^2(\tilde{\mathcal{C}}_\phi)$$

is a holomorphic family of bounded operators in $\text{Re}(s) > \frac{1}{2}$, each a bijection to their image. This gives an a priori bound for the resolvent: using

$$(\Delta - s(1-s)) R_{\tilde{\mathcal{C}}_\phi}(s) f = f$$

and Lemma 4.2 we obtain that for $f \in L^2(\tilde{\mathcal{C}}_\phi)$ and $s \in (\frac{1}{2}, \infty)$

$$\|R_{\tilde{\mathcal{C}}_\phi}(s) f\|_{L^2} \leq \left(\frac{1}{4} - s(1-s) \right)^{-1} \|f\|_{L^2} \quad (4.4)$$

and as

$$\Delta R_{\tilde{\mathcal{C}}_\phi}(s)f = f + s(1-s)R_{\tilde{\mathcal{C}}_\phi}(s)f$$

we obtain for $s \in (\frac{1}{2}, \infty)$

$$\begin{aligned} \|\Delta R_{\tilde{\mathcal{C}}_\phi}(s)f\|_{L^2} &\leq \|f\|_{L^2} + s(1-s)\|R_{\tilde{\mathcal{C}}_\phi}(s)f\|_{L^2} \\ &\leq \left(1 + \frac{s(1-s)}{\frac{1}{4} - s(1-s)}\right) \|f\|_{L^2} \\ &= \frac{1}{1 - 4s(1-s)} \|f\|_{L^2}. \end{aligned} \quad (4.5)$$

We now define the cusp parametrix as

$$\mathbb{M}_\phi^{\text{cusp}}(s) \stackrel{\text{def}}{=} \chi_{\mathcal{C},\phi}^+ R_{\tilde{\mathcal{C}}_\phi}(s) \chi_{\mathcal{C},\phi}^-. \quad (4.6)$$

Here,

- (multiplication by) $\chi_{\mathcal{C},\phi}^-$ is viewed as an operator from $L^2(X_\phi)$ to $L^2(\tilde{\mathcal{C}}_\phi)$ by mapping first to $L^2(\mathcal{C}_\phi)$ and then extending by zero. This is a bounded linear operator.
- $R_{\tilde{\mathcal{C}}_\phi}(s)$ is a bounded operator from $L^2(\tilde{\mathcal{C}}_\phi)$ to $H^2(\tilde{\mathcal{C}}_\phi)$.
- (multiplication by) $\chi_{\mathcal{C},\phi}^+$ is viewed as an operator from $H^2(\tilde{\mathcal{C}}_\phi)$ to $H^2(X_\phi)$, using that $\chi_{\mathcal{C},\phi}^+$ localizes to \mathcal{C}_ϕ and then extending by zero. This operator is bounded because derivatives of $\chi_{\mathcal{C},\phi}^+$ are bounded and compactly supported.

Hence

$$\mathbb{M}_\phi^{\text{cusp}}(s) : L^2(X_\phi) \rightarrow H^2(X_\phi)$$

is a bounded operator.

The covering map $\mathcal{C}_\phi \rightarrow \mathcal{C}$ extends in an obvious way to a covering map $\tilde{\mathcal{C}}_\phi \rightarrow \tilde{\mathcal{C}}$ that intertwines the two Laplacian operators. This, together with the fact that multiplication by $\chi_{\mathcal{C},\phi}^-$ and $\chi_{\mathcal{C},\phi}^+$ leave invariant the subspaces of functions lifted through the covering map, one sees that

$$\mathbb{M}_\phi^{\text{cusp}}(s)(L_{\text{new}}^2(X_\phi)) \subset H_{\text{new}}^2(X_\phi).$$

Because $\chi_{\mathcal{C},\phi}^+ \chi_{\mathcal{C},\phi}^- = \chi_{\mathcal{C},\phi}^-$,

$$(\Delta - s(1-s))\mathbb{M}_\phi^{\text{cusp}}(s) = \chi_{\mathcal{C},\phi}^- + [\Delta, \chi_{\mathcal{C},\phi}^+] R_{\tilde{\mathcal{C}}_\phi}(s) \chi_{\mathcal{C},\phi}^- = \chi_{\mathcal{C},\phi}^- + \mathbb{L}_\phi^{\text{cusp}}(s) \quad (4.7)$$

where

$$\mathbb{L}_\phi^{\text{cusp}}(s) \stackrel{\text{def}}{=} [\Delta, \chi_{\mathcal{C},\phi}^+] R_{\tilde{\mathcal{C}}_\phi}(s) \chi_{\mathcal{C},\phi}^-$$

and $[A, B] \stackrel{\text{def}}{=} AB - BA$ denotes the commutator of linear maps. Here again we view $\chi_{\mathcal{C},\phi}^-$ and $R_{\tilde{\mathcal{C}}_\phi}(s)$ as above, and $[\Delta, \chi_{\mathcal{C},\phi}^+] : H^2(\tilde{\mathcal{C}}_\phi) \rightarrow L^2(X_\phi)$. This means that $\mathbb{L}_\phi^{\text{cusp}}(s)$ is an operator on $L^2(X_\phi)$. By similar arguments to before, using that $[\Delta, \chi_{\mathcal{C},\phi}^+]$ only involves radial derivatives (since $\chi_{\mathcal{C},\phi}^+$ is radial), we obtain

$$\mathbb{L}_\phi^{\text{cusp}}(s) (L_{\text{new}}^2(X_\phi)) \subset L_{\text{new}}^2(X_\phi).$$

Lemma 4.3. *Given $s_0 > \frac{1}{2}$, there exists $C(s_0) > 0$ such that for $s \in [s_0, 1]$, the operator $\mathbb{L}_\phi^{\text{cusp}}(s)$ is a self-adjoint, bounded operator on $L^2(X_\phi)$ with operator norm*

$$\|\mathbb{L}_\phi^{\text{cusp}}(s)\|_{L^2} \leq \left(\|(\Delta \chi_{\mathcal{C},\phi}^+)\|_\infty + 2\|\nabla \chi_{\mathcal{C},\phi}^+\|_\infty \right) C(s_0).$$

Proof. As an operator on $H^2(\tilde{\mathcal{C}}_\phi)$

$$[\Delta, \chi_{\mathcal{C},\phi}^+] = (\Delta \chi_{\mathcal{C},\phi}^+) - 2(\nabla \chi_{\mathcal{C},\phi}^+) \cdot \nabla.$$

The first summand is a multiplication operator; for $f \in H^2(\tilde{\mathcal{C}}_\phi)$ we have

$$\|(\Delta \chi_{\mathcal{C},\phi}^+) f\|_{L^2} \leq \|(\Delta \chi_{\mathcal{C},\phi}^+)\|_\infty \|f\|_{L^2} \quad (4.8)$$

and by Schwarz inequality if $\|f\|_{H^2} \leq 1$ then

$$\begin{aligned} \|(\nabla \chi_{\mathcal{C},\phi}^+) \cdot \nabla f\|_{L^2} &\leq \|\nabla \chi_{\mathcal{C},\phi}^+\|_\infty \|\nabla f\|_{L^2} \\ &= \|\nabla \chi_{\mathcal{C},\phi}^+\|_\infty \langle \Delta f, f \rangle^{\frac{1}{2}} \leq \|\nabla \chi_{\mathcal{C},\phi}^+\|_\infty \|\Delta f\|^{\frac{1}{2}} \|f\|^{\frac{1}{2}} \\ &\leq \|\nabla \chi_{\mathcal{C},\phi}^+\|_\infty. \end{aligned} \quad (4.9)$$

The two estimates (4.8), (4.9) hence show that $[\Delta, \chi_{\mathcal{C},\phi}^+]$ has norm bounded by $\|(\Delta \chi_{\mathcal{C},\phi}^+)\|_\infty + 2\|\nabla \chi_{\mathcal{C},\phi}^+\|_\infty$ as a map $H^2(\tilde{\mathcal{C}}_\phi) \rightarrow L^2(X_\phi)$. Since multiplication by $\chi_{\mathcal{C},\phi}^-$ has norm ≤ 1 from L^2 to L^2 , and $R_{\tilde{\mathcal{C}}_\phi}(s)$ has norm $\leq C(s_0)$ from $L^2(\tilde{\mathcal{C}}_\phi)$ to $H^2(\tilde{\mathcal{C}}_\phi)$ by (4.4) and (4.5), we obtain the stated result. \square

We can now obtain the following key proposition by combining Lemmas 4.1 and 4.3.

Proposition 4.4. *Given any $s_0 > \frac{1}{2}$, we can choose χ_C^+ and χ_C^- (depending on s_0) such that for any $n \in \mathbf{N}$ and $\phi \in \text{Hom}(\Gamma, S_n)$ and any $s \in [s_0, 1]$, we have for the operator norm of $\mathbb{L}_\phi^{\text{cusp}}(s)$ on $L^2(X_\phi)$*

$$\|\mathbb{L}_\phi^{\text{cusp}}(s)\|_{L^2} \leq \frac{1}{5}.$$

We assume that χ_C^- and χ_C^+ have now been fixed to satisfy Proposition 4.4 in the rest of the paper. Hence all constants may depend on these functions.

5 Interior parametrix

5.1 Background on spectral theory of hyperbolic plane

Resolvent

Let

$$\begin{aligned} R_{\mathbb{H}}(s) &: L^2(\mathbb{H}) \rightarrow L^2(\mathbb{H}), \\ R_{\mathbb{H}}(s) &\stackrel{\text{def}}{=} (\Delta_{\mathbb{H}} - s(1-s))^{-1}. \end{aligned}$$

Since $\text{spec}(\Delta_{\mathbb{H}}) \subset [\frac{1}{4}, \infty)$ [Bor16, Thm. 4.3], this is well defined as a bounded operator for $\text{Re}(s) > \frac{1}{2}$.

Given $x, y \in \mathbb{H}$, we write $r = d_{\mathbb{H}}(x, y)$ where $d_{\mathbb{H}}$ is hyperbolic distance and

$$\sigma \stackrel{\text{def}}{=} \cosh^2\left(\frac{r}{2}\right).$$

The operator $R_{\mathbb{H}}(s)$ is an integral operator with kernel [Bor16, Prop. 4.2]

$$R_{\mathbb{H}}(s; x, y) = \frac{1}{4\pi} \int_0^1 \frac{t^{s-1}(1-t)^{s-1}}{(\sigma-t)^s} dt. \quad (5.1)$$

For $\sigma > 1$ and $t \in (0, 1)$ we have

$$\begin{aligned} \frac{\partial}{\partial r} \frac{t^{s-1}(1-t)^{s-1}}{(\sigma-t)^s} &= -s \sinh\left(\frac{r}{2}\right) \cosh\left(\frac{r}{2}\right) \frac{t^{s-1}(1-t)^{s-1}}{(\sigma-t)^{s+1}}, \\ \frac{\partial}{\partial s} \frac{t^{s-1}(1-t)^{s-1}}{(\sigma-t)^s} &= \log\left(\frac{t(1-t)}{(\sigma-t)}\right) \frac{t^{s-1}(1-t)^{s-1}}{(\sigma-t)^s}, \\ \frac{\partial^2}{\partial s \partial r} \frac{t^{s-1}(1-t)^{s-1}}{(\sigma-t)^s} &= -\sinh\left(\frac{r}{2}\right) \cosh\left(\frac{r}{2}\right) \frac{t^{s-1}(1-t)^{s-1}}{(\sigma-t)^{s+1}} \\ &\quad \cdot \left[1 + s \log\left(\frac{t(1-t)}{(\sigma-t)}\right)\right]. \end{aligned}$$

Each of these are smooth in $(s, r, t) \in [\frac{1}{2}, 1] \times [1, \infty) \times (0, 1)$. Because for s, r in a fixed compact set of $[\frac{1}{2}, 1] \times [1, \infty)$, these all have absolute values majorized by integrable functions of $t \in (0, 1)$, we can interchange derivatives and integrals to get corresponding bounds for $R_{\mathbb{H}}$ as follows. We define

$$R_{\mathbb{H}}(s; r) \stackrel{\text{def}}{=} R_{\mathbb{H}}(s; x, y).$$

Firstly, there is a constant $C > 0$ such that for $r_0 \geq 1$ and $s \in [\frac{1}{2}, 1]$ we have

$$|R_{\mathbb{H}}(s; r_0)|, \left| \frac{\partial R_{\mathbb{H}}}{\partial r}(s; r_0) \right| \leq C e^{-sr_0}. \quad (5.2)$$

Secondly, for any $T > 1$ and r_0 in the compact region $[1, T + 1]$ we obtain for constant $c = c(T) > 0$, for all $s_0 \in [\frac{1}{2}, 1]$,

$$\left| \frac{\partial R_{\mathbb{H}}}{\partial s}(s_0; r_0) \right|, \left| \frac{\partial^2 R_{\mathbb{H}}}{\partial s \partial r}(s_0; r_0) \right| \leq c(T). \quad (5.3)$$

Integral operators

If $k_0 : [0, \infty) \rightarrow \mathbf{R}$ is smooth and compactly supported, which will suffice here, then one can construct a kernel

$$k(x, y) \stackrel{\text{def}}{=} k_0(d_{\mathbb{H}}(x, y))$$

with corresponding integral operator $C^\infty(\mathbb{H}) \rightarrow C^\infty(\mathbb{H})$

$$K[f](x) \stackrel{\text{def}}{=} \int_{y \in \mathbb{H}} k(x, y) f(y) d\mathbb{H}(y)$$

where $d\mathbb{H}$ is the hyperbolic area form on \mathbb{H} . Such an operator commutes with the Laplacian on \mathbb{H} and hence preserves its generalized eigenspaces. If $f \in C^\infty(\mathbb{H})$ is a generalized eigenfunction of Δ with eigenvalue $\frac{1}{4} + \xi^2$, $\xi \geq 0$, then by [Ber16, Thm. 3.7, Lemma 3.9] (cf. also Selberg's original article [Sel56])

$$K[f] = h(\xi) f$$

where

$$h(\xi) = \sqrt{2} \int_{-\infty}^{\infty} e^{i\xi u} \int_{|u|}^{\infty} \frac{k_0(\rho) \sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}} d\rho du.$$

By our assumptions on k_0 the integral above is convergent. It will be estimated more precisely for particular choice of k_0 in the proof of Lemma 5.2.

For now note that if k_0 is real valued then h is the Fourier transform of a real valued even function and hence real-valued. Since $L^2(\mathbb{H})$ has a generalized eigenbasis of C^∞ eigenfunctions of the Laplacian, by Borel functional calculus K extends from e.g. $C_c^\infty(\mathbb{H})$ to a (possibly unbounded) self-adjoint operator on $L^2(\mathbb{H})$ with operator norm

$$\|K\|_{L^2(\mathbb{H})} = \sup_{\xi \geq 0} |h(\xi)|. \quad (5.4)$$

5.2 Interior parametrix on hyperbolic plane

Let $\chi_0 : \mathbf{R} \rightarrow [0, 1]$ denote a smooth function with $\chi_0|_{(-\infty, 0]} \equiv 1$ and $\chi_0|_{[1, \infty)} = 0$. Then let

$$\chi_T(t) \stackrel{\text{def}}{=} \chi_0(t - T)$$

so χ_T is supported on $(-\infty, T + 1]$; note for later that all derivatives of χ_T are supported in $[T, T + 1]$ and have uniform bounds independent of T .

We define

$$R_{\mathbb{H}}^{(T)}(s; r) \stackrel{\text{def}}{=} \chi_T(r) R_{\mathbb{H}}(s; r).$$

Let $R_{\mathbb{H}}^{(T)}(s)$ denote the corresponding integral operator. In radial coordinates the Laplacian on \mathbb{H} is given by [Bor16, pg. 65]

$$-\frac{\partial^2}{\partial r^2} - \frac{1}{\tanh r} \frac{\partial}{\partial r} - \frac{1}{\sinh^2 r} \frac{\partial^2}{\partial \theta^2}.$$

We now perform the following calculation, writing Δ_x for the Laplacian acting on coordinate x :

$$\begin{aligned} [\Delta_x - s(1 - s)] R_{\mathbb{H}}^{(T)}(s; r) &= [\Delta_x - s(1 - s)] (\chi_T(r) R_{\mathbb{H}}(s; r)) \\ &= \left[-\frac{\partial^2}{\partial r^2} - \frac{1}{\tanh r} \frac{\partial}{\partial r}, \chi_T \right] R_{\mathbb{H}}(s; r) + \delta_{r=0} \end{aligned} \quad (5.5)$$

which is understood in a distributional sense. We further calculate

$$\left[-\frac{\partial^2}{\partial r^2} - \frac{1}{\tanh r} \frac{\partial}{\partial r}, \chi_T \right] = -\frac{\partial^2}{\partial r^2} [\chi_T] - 2 \frac{\partial}{\partial r} [\chi_T] \frac{\partial}{\partial r} - \frac{1}{\tanh r} \frac{\partial}{\partial r} [\chi_T]. \quad (5.6)$$

Combining (5.5) and (5.6) we expect an identity of operators

$$[\Delta - s(1 - s)] R_{\mathbb{H}}^{(T)}(s) = 1 + \mathbb{L}_{\mathbb{H}}^{(T)}(s) \quad (5.7)$$

where $\mathbb{L}_{\mathbb{H}}^{(T)}(s)$ is the integral operator with real-valued radial kernel

$$\mathbb{L}_{\mathbb{H}}^{(T)}(s; r_0) \stackrel{\text{def}}{=} \left(-\frac{\partial^2}{\partial r^2} [\chi_T] - \frac{1}{\tanh r} \frac{\partial}{\partial r} [\chi_T] \right) R_{\mathbb{H}}(s; r_0) - 2 \frac{\partial}{\partial r} [\chi_T] \frac{\partial R_{\mathbb{H}}}{\partial r}(s; r_0). \quad (5.8)$$

The identity (5.7) will be established in Lemma 5.3 below.

We note the following properties of this kernel that are straightforward consequences of the way χ_T was chosen and (5.2), (5.3).

Lemma 5.1. *We have*

1. For $T > 0$ and $s \in [\frac{1}{2}, 1]$, $\mathbb{L}_{\mathbb{H}}^{(T)}(s; \bullet)$ is smooth and supported in $[T, T + 1]$.
2. There is a constant $C > 0$ such that for any $T > 0$ and $s \in [\frac{1}{2}, 1]$ we have

$$|\mathbb{L}_{\mathbb{H}}^{(T)}(s; r_0)| \leq C e^{-sr_0}$$

3. For any $T > 0$ there is a constant $c(T) > 0$ such that for any $r_0 \in [T, T + 1]$

$$\left| \frac{\partial \mathbb{L}_{\mathbb{H}}^{(T)}}{\partial s}(s_0; r_0) \right| \leq c(T).$$

We can now show the following.

Lemma 5.2. *There is a constant $C > 0$ such that for any $T > 0$ and $s \in [\frac{1}{2}, 1]$ the operator $\mathbb{L}_{\mathbb{H}}^{(T)}(s)$ extends to a bounded operator on $L^2(\mathbb{H})$ with operator norm*

$$\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2} \leq C T e^{(\frac{1}{2}-s)T}.$$

Proof. We do this using (5.4) which tells us

$$\begin{aligned}
\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2} &= \sup_{\xi \geq 0} \left| \sqrt{2} \int_{-\infty}^{\infty} e^{i\xi u} \int_{|u|}^{\infty} \frac{\mathbb{L}_{\mathbb{H}}^{(T)}(s; \rho) \sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}} d\rho du \right| \\
&\leq \sqrt{2} \int_{-\infty}^{\infty} \int_{|u|}^{\infty} \frac{|\mathbb{L}_{\mathbb{H}}^{(T)}(s; \rho)| \sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}} d\rho du \\
&\leq 2\sqrt{2}C \int_0^{T+1} \int_{\max(|u|, T)}^{T+1} \frac{e^{-s\rho} \sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}} d\rho du \\
&\leq C' e^{-sT} \int_0^{T+1} \int_{\max(|u|, T)}^{T+1} \frac{\sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}} d\rho du \\
&= C' e^{-sT} \int_0^{T+1} \int_{\cosh \max(|u|, T)}^{\cosh(T+1)} \frac{dy}{\sqrt{y - \cosh(u)}} du \\
&= C'' e^{-sT} \int_0^{T+1} \left[\sqrt{\cosh(T+1) - \cosh u} \right. \\
&\quad \left. - \sqrt{\cosh \max(|u|, T) - \cosh |u|} \right] du \\
&\leq C''' T e^{(\frac{1}{2}-s)T}
\end{aligned}$$

where the third inequality used Lemma 5.1. \square

The implication of this is that for any $s_0 > \frac{1}{2}$ we can choose $T = T(s_0)$ such that for all $s \in [s_0, 1]$ we have

$$\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2} \leq \frac{1}{5}.$$

We will do this later. The following lemma shows that smoothly cutting off $R_{\mathbb{H}}(s)$ at radius T does not significantly affect its mapping properties.

Lemma 5.3. *For any $T > 0$ and $s \in [\frac{1}{2}, 1]$, for any compact $K \subset \mathbb{H}$, there is $C = C(s, K, T) > 0$ such that:*

1. *For any $f \in C_c^\infty(\mathbb{H})$ with $\text{supp}(f) \subset K$ we have $R_{\mathbb{H}}^{(T)}(s)f \in H^2(\mathbb{H})$ and*

$$\|R_{\mathbb{H}}^{(T)}(s)f\|_{H^2} \leq C(s, K, T)\|f\|_{L^2}.$$

2. *Furthermore, with f as above*

$$(\Delta - s(1-s))R_{\mathbb{H}}^{(T)}(s)[f] = f + \mathbb{L}_{\mathbb{H}}^{(T)}(s)[f]$$

in the sense of equivalence of L^2 functions.

Proof. Suppose that compact K is given and $f \in C_c^\infty(\mathbb{H})$ with $\text{supp}(f) \subset K$. For $y \in K$ we have $R_{\mathbb{H}}^{(T)}(s; x, y) = 0$ unless

$$x \in K'(T, K) \stackrel{\text{def}}{=} \{x : d(x, K) \leq T + 1\}$$

with K' compact. Therefore using the usual Hilbert-Schmidt inequality we obtain

$$\begin{aligned} & \|R_{\mathbb{H}}^{(T)}(s)[f]\|_{L^2(\mathbb{H})}^2 \\ &= \int_{x \in K'} \left| \int_{y \in K} R_{\mathbb{H}}^{(T)}(s; x, y) f(y) d\mathbb{H}(y) \right|^2 d\mathbb{H}(x) \\ &\leq \int_{x \in K'} \left(\int_{y \in K} R_{\mathbb{H}}^{(T)}(s; x, y)^2 d\mathbb{H}(y) \right) \left(\int_{y \in K} |f(y)|^2 d\mathbb{H}(y) \right) d\mathbb{H}(x). \end{aligned} \quad (5.9)$$

Recall that we write $r = d_{\mathbb{H}}(x, y)$, hence the inner integral can be written in polar coordinates as

$$\begin{aligned} \int_{y \in K} R_{\mathbb{H}}^{(T)}(s; x, y)^2 d\mathbb{H}(y) &= \int_0^{2\pi} \int_0^\infty R_{\mathbb{H}}^{(T)}(s; r)^2 \sinh r \, dr \, d\theta \\ &\leq 2\pi \int_0^M R_{\mathbb{H}}^{(T)}(s; r)^2 \sinh r \, dr \end{aligned} \quad (5.10)$$

for $M = M(K, T)$. Because $\chi_T \equiv 1$ near 0, the type of singularity that $R_{\mathbb{H},n}^{(T)}(s; r)$ has at $r = 0$ is exactly the same as the type of singularity of $R_{\mathbb{H}}(s; r)$ near $r = 0$; namely by [Bor16, (4.2)]

$$R_{\mathbb{H}}^{(T)}(s; r) = -\frac{1}{4\pi} \log\left(\frac{r}{2}\right) + O(1) \quad (5.11)$$

as $r \rightarrow 0$. The function $R_{\mathbb{H}}^{(T)}(s; r)$ is smooth away from 0. Hence, since $(\log(\frac{r}{2}))^2 \sinh r \rightarrow 0$ as $r \rightarrow 0$, $R_{\mathbb{H}}^{(T)}(s; r)$ is in particular square integrable on $[0, M]$. This gives from (5.9)

$$\|R_{\mathbb{H}}^{(T)}(s)[f]\|_{L^2(\mathbb{H})}^2 \leq \int_{x \in K'} C(s, K, T) \|f\|_{L^2(\mathbb{H})}^2 d\mathbb{H}(x) \leq C'(s, K, T) \|f\|_{L^2(\mathbb{H})}^2. \quad (5.12)$$

We now aim for a bound on $\|\Delta R_{\mathbb{H}}^{(T)}(s)[f]\|_{L^2(\mathbb{H})}^2$ so as to prove $R_{\mathbb{H}}^{(T)}(s)[f] \in H^2(\mathbb{H})$ and bound its H^2 -norm.

Let $g \in C_c^\infty(\mathbb{H})$ be a test function and $f \in C_c^\infty(\mathbb{H})$ with support as above. Consider

$$\langle R_{\mathbb{H}}^{(T)}(s)[f], \Delta g \rangle = \int_{x \in \mathbb{H}} \left(\int_{y \in \mathbb{H}} R_{\mathbb{H}}^{(T)}(s; x, y) f(y) d\mathbb{H}(y) \right) \Delta \bar{g}(x) d\mathbb{H}(x).$$

Because f and g are compactly supported and the singularity of $R_{\mathbb{H}}^{(T)}(x, y)$ is locally L^1 using (5.11), we can use Fubini to get

$$\langle R_{\mathbb{H}}^{(T)}(s)[f], \Delta g \rangle = \int_{y \in \mathbb{H}} \left(\int_{x \in \mathbb{H}} R_{\mathbb{H}}^{(T)}(s; x, y) \Delta \bar{g}(x) d\mathbb{H}(x) \right) f(y) d\mathbb{H}(y). \quad (5.13)$$

We use hyperbolic polar coordinates for the inner integral, writing $r = d(x, y)$ and θ for polar angle, $G_y(r, \theta) \stackrel{\text{def}}{=} \bar{g}(x)$, and the inner integral is understood as an improper integral as follows:

$$\begin{aligned} & \int_{x \in \mathbb{H}} R_{\mathbb{H}}^{(T)}(s; x, y) \Delta \bar{g}(x) d\mathbb{H}(x) \\ &= - \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\infty} \int_0^{2\pi} R_{\mathbb{H}}^{(T)}(s; \tilde{r}) \left(\frac{\partial}{\partial r} \left(\sinh r \frac{\partial G_y}{\partial r} \right) (\tilde{r}, \tilde{\theta}) + \frac{1}{(\sinh r)^2} \frac{\partial^2 G_y}{\partial \theta^2} (\tilde{r}, \tilde{\theta}) \right) d\tilde{\theta} d\tilde{r} \\ &= - \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\infty} \int_0^{2\pi} R_{\mathbb{H}}^{(T)}(s; \tilde{r}) \left(\frac{\partial}{\partial r} \left(\sinh r \frac{\partial G_y}{\partial r} \right) (\tilde{r}, \tilde{\theta}) \right) d\tilde{\theta} d\tilde{r} \\ &= - \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} \int_{\epsilon}^{\infty} R_{\mathbb{H}}^{(T)}(s; \tilde{r}) \left(\frac{\partial}{\partial r} \left(\sinh r \frac{\partial G_y}{\partial r} \right) (\tilde{r}, \tilde{\theta}) \right) d\tilde{r} d\tilde{\theta} \\ &= \lim_{\epsilon \rightarrow 0} R_{\mathbb{H}}^{(T)}(s; \epsilon) \sinh \epsilon \int_0^{2\pi} \frac{\partial G_y}{\partial r}(\epsilon, \tilde{\theta}) d\tilde{\theta} \\ &\quad + \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} \int_{\epsilon}^{\infty} \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}(s; \tilde{r}) \sinh \tilde{r} \frac{\partial G_y}{\partial r}(\tilde{r}, \tilde{\theta}) d\tilde{r} d\tilde{\theta} \\ &= \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} \int_{\epsilon}^{\infty} \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}(s; \tilde{r}) \sinh \tilde{r} \frac{\partial G_y}{\partial r}(\tilde{r}, \tilde{\theta}) d\tilde{r} d\tilde{\theta} \end{aligned} \quad (5.14)$$

where the last equality used (5.11) with smoothness of G_y . Now a similar calculation gives

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} \int_{\epsilon}^{\infty} \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}(s; \tilde{r}) \sinh \tilde{r} \frac{\partial G_y}{\partial r}(\tilde{r}, \tilde{\theta}) d\tilde{r} d\tilde{\theta} \\ &= - \lim_{\epsilon \rightarrow 0} \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}(s; \epsilon) \sinh \epsilon \int_0^{2\pi} G_y(\epsilon, \tilde{\theta}) d\tilde{\theta} \\ &\quad - \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} \int_{\epsilon}^{\infty} \frac{\partial}{\partial r} \left(\sinh r \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r} \right) (s; \tilde{r}) G_y(\tilde{r}, \tilde{\theta}) d\tilde{r} d\tilde{\theta} \\ &= \bar{g}(y) - \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} \int_{\epsilon}^{\infty} \frac{\partial}{\partial r} \left(\sinh r \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r} \right) (s; \tilde{r}) G_y(\tilde{r}, \tilde{\theta}) d\tilde{r} d\tilde{\theta}. \end{aligned} \quad (5.15)$$

The second equality used [Bor16, pg. 66]

$$\frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}(s; \epsilon) = -\frac{1}{2\pi\epsilon} + O(1)$$

as $\epsilon \rightarrow 0$ with $\lim_{\epsilon \rightarrow 0} G_y(\epsilon, \tilde{\theta}) = \bar{g}(y)$. Now we note

$$-\frac{1}{\sinh r} \frac{\partial}{\partial r} \left(\sinh r \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r} \right) (s; \tilde{r}) = \Delta R_{\mathbb{H}}^{(T)}(s; \tilde{r})$$

and so using (5.5), (5.6) and (5.8) we get, for $\tilde{r} > 0$,

$$-\frac{1}{\sinh r} \frac{\partial}{\partial r} \left(\sinh r \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r} \right) (s; \tilde{r}) = s(1-s)R_{\mathbb{H}}^{(T)}(s; \tilde{r}) + \mathbb{L}_{\mathbb{H}}^{(T)}(s; \tilde{r}).$$

Therefore

$$\begin{aligned} & -\lim_{\epsilon \rightarrow 0} \int_0^{2\pi} \int_{\epsilon}^{\infty} \frac{\partial}{\partial r} \left(\sinh r \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r} \right) (s; \tilde{r}) G_y(\tilde{r}, \tilde{\theta}) d\tilde{r} d\tilde{\theta} \\ &= \lim_{\epsilon \rightarrow 0} \int_{d(x,y) > \epsilon} \left(s(1-s)R_{\mathbb{H}}^{(T)}(s; \tilde{r}) + \mathbb{L}_{\mathbb{H}}^{(T)}(s; d(x,y)) \right) \bar{g}(x) d\mathbb{H}(x) \\ &= \int_{x \in \mathbb{H}} \left(s(1-s)R_{\mathbb{H}}^{(T)}(s; d(x,y)) + \mathbb{L}_{\mathbb{H}}^{(T)}(s; d(x,y)) \right) \bar{g}(x) d\mathbb{H}(x) \quad (5.16) \end{aligned}$$

and this last integral is easily seen to converge by working in polar coordinates centered at y and using $g \in C_c^\infty(\mathbb{H})$ and (5.11).

Now combining (5.14), (5.15), and (5.16) gives, for (5.13),

$$\begin{aligned} & \langle R_{\mathbb{H}}^{(T)}(s)[f], \Delta g \rangle \\ &= \int_{y \in \mathbb{H}} f(y) \bar{g}(y) d\mathbb{H}(y) \\ &+ \int_{y \in \mathbb{H}} \left(\int_{x \in \mathbb{H}} \left(s(1-s)R_{\mathbb{H}}^{(T)}(s; d(x,y)) + \mathbb{L}_{\mathbb{H}}^{(T)}(s; d(x,y)) \right) \bar{g}(x) d\mathbb{H}(x) \right) f(y) d\mathbb{H}(y) \\ &= \langle f, g \rangle + \langle s(1-s)R_{\mathbb{H}}^{(T)}(s)[f], g \rangle + \langle \mathbb{L}_{\mathbb{H}}^{(T)}(s)[f], g \rangle. \end{aligned}$$

Note that by (5.12) and Lemma 5.2 all functions above are in $L^2(\mathbb{H})$. This identity now clearly extends to any $g \in H^2(\mathbb{H})$ and now self-adjointness of $\Delta_{\mathbb{H}}$ on $H^2(\mathbb{H})$ (see [Str83]) gives that $R_{\mathbb{H}}^{(T)}(s)[f] \in H^2(\mathbb{H})$ and moreover

$$(\Delta - s(1-s))R_{\mathbb{H}}^{(T)}(s)[f] = f + \mathbb{L}_{\mathbb{H}}^{(T)}(s)[f]$$

in the sense of elements of $L^2(\mathbb{H})$. *This proves part 2 of the lemma.*

We now rewrite this identity as

$$\Delta R_{\mathbb{H}}^{(T)}(s)[f] = f + s(1-s)R_{\mathbb{H}}^{(T)}(s)[f] + \mathbb{L}_{\mathbb{H}}^{(T)}(s)[f]$$

and using Lemma 5.2 and (5.12) now gives

$$\|\Delta R_{\mathbb{H}}^{(T)}(s)[f]\|_{L^2(\mathbb{H})} \leq c(s, K, T)\|f\|_{L^2(\mathbb{H})}.$$

Combining this with (5.12), *this proves part 1 of the lemma.* \square

5.3 Automorphic kernels

At two points in the rest of the paper we use the following geometric lemma. Recall that F is a Dirichlet fundamental domain for Γ in \mathbb{H} .

Lemma 5.4. *For any compact set $K \subset \bar{F}$, and $T > 0$, there is another compact set $K' = K'(K, T) \subset \bar{F}$ and a finite set $S = S(K, T) \subset \Gamma$ such that for all $x, y \in \mathbb{H}$ with*

$$x \in \bar{F}, y \in \bigcup_{\gamma \in \Gamma} \gamma(K), \quad d(x, y) \leq T,$$

we have

$$x \in K', \quad y \in \bigcup_{\gamma \in S} \gamma(K).$$

Proof. Suppose that $y = \gamma y'$ with $y' \in K$, $x \in \bar{F}$, $\gamma \in \Gamma$ and $d(x, y) \leq T$. Then $d(\gamma y', x) = d(y', \gamma^{-1}x) \leq T$ implies $\gamma^{-1}F$ intersects the compact set

$$\{z : d(z, K) \leq T\}.$$

Since we took F to be a Dirichlet fundamental domain, by [Bea83, Thm. 9.4.2] there are only a finite number of γ for which this is possible. Let S be this finite subset of Γ . Then furthermore, if we let o denote a fixed point in K we have

$$\begin{aligned} d(x, o) &\leq d(x, \gamma y') + d(\gamma y', \gamma o) + d(\gamma o, o). \\ &\leq T + d(y', o) + d(\gamma o, o) \leq C'(K, T). \end{aligned}$$

This implies that x must be in some compact set $K' = K'(K, T) \subset \bar{F}$. \square

Define

$$\begin{aligned} R_{\mathbb{H},n}^{(T)}(s; x, y) &\stackrel{\text{def}}{=} R_{\mathbb{H}}^{(T)}(s; x, y)\text{Id}_{V_n^0}, \\ \mathbb{L}_{\mathbb{H},n}^{(T)}(s; x, y) &\stackrel{\text{def}}{=} \mathbb{L}_{\mathbb{H}}^{(T)}(s; x, y)\text{Id}_{V_n^0}, \end{aligned}$$

and $R_{\mathbb{H},n}^{(T)}(s), \mathbb{L}_{\mathbb{H},n}^{(T)}(s)$ the corresponding integral operators. In the next lemma we describe the mapping properties of these integral operators and describe a functional identity that they satisfy that arises from (5.7). In the following we regard the function $\chi_{\bar{C}}$ defined in §4 as a Γ -invariant C^∞ function on \mathbb{H} . The action of multiplication by $\chi_{\bar{C},\phi}$ on new functions in $L_{\text{new}}^2(X_\phi)$ corresponds to multiplication by the scalar-valued invariant function $\chi_{\bar{C}}$ on $L_\phi^2(\mathbb{H}; V_n^0)$.

Lemma 5.5. *For all $s \in [\frac{1}{2}, 1]$,*

1. *The integral operator $R_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}}^-)$ is well-defined on $C_{c,\phi}^\infty(\mathbb{H}; V_n^0)$ and extends to a bounded operator*

$$R_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}}^-) : L_\phi^2(\mathbb{H}; V_n^0) \rightarrow H_\phi^2(\mathbb{H}; V_n^0).$$

2. *The integral operator $\mathbb{L}_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}}^-)$ is well-defined on $C_{c,\phi}^\infty(\mathbb{H}; V_n^0)$ and extends to a bounded operator on $L_\phi^2(\mathbb{H}; V_n^0)$.*

3. *We have*

$$[\Delta - s(1 - s)] R_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}}^-) = (1 - \chi_{\bar{C}}^-) + \mathbb{L}_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}}^-) \quad (5.17)$$

as an identity of operators on $L_\phi^2(\mathbb{H}; V_n^0)$.

Proof. Suppose first that $f \in C_{c,\phi}^\infty(\mathbb{H}; V_n^0)$ (i.e. automorphic, smooth, and compactly supported modulo Γ). We have for $x \in F$

$$R_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}}^-)[f](x) \stackrel{\text{def}}{=} \int_{y \in \mathbb{H}} R_{\mathbb{H}}^{(T)}(s; x, y)(1 - \chi_{\bar{C}}^-(y))f(y)d\mathbb{H}(y). \quad (5.18)$$

The integrand here is non-zero unless $d(x, y) \leq T + 1$ and y is in the support of $1 - \chi_{\bar{C}}^-$, which is a union of the Γ -translates of a compact set K of \bar{F} . Applying Lemma 5.4 tells us then that there is compact $K_1 = K_1(T) \subset \bar{F}$ and finite set $S = S(T)$ such that the integrand in (5.18) is supported on the compact set $K_2 \stackrel{\text{def}}{=} \cup_{\gamma \in S} \gamma^{-1}K$ and the whole integral is zero unless $x \in K_1$ (given $x \in \bar{F}$ to begin with).

Let ψ be a smooth function that is $\equiv 1$ in $K_2 \cup K$, valued in $[0, 1]$ and compactly supported. Let $\{e_i : i \in [n-1]\}$ denote an orthonormal basis for V_n^0 and let

$$f_i \stackrel{\text{def}}{=} \langle f, e_i \rangle \in C^\infty(\mathbb{H}).$$

The above shows that for $x \in F$ we have

$$\begin{aligned} R_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}})[f](x) &= R_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}})[\psi f](x) \\ &= \sum_{i=1}^{n-1} R_{\mathbb{H}}^{(T)}(s) [(1 - \chi_{\bar{C}}) \psi f_i](x) e_i \end{aligned} \quad (5.19)$$

hence

$$\begin{aligned} \|R_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}})[f](x)\|_{V_n^0}^2 &= \sum_{i=1}^{n-1} \left| R_{\mathbb{H}}^{(T)}(s) [(1 - \chi_{\bar{C}}) \psi f_i](x) \right|^2, \\ \|\Delta R_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}})[f](x)\|_{V_n^0}^2 &= \sum_{i=1}^{n-1} \left| \Delta R_{\mathbb{H}}^{(T)}(s) [(1 - \chi_{\bar{C}}) \psi f_i](x) \right|^2. \end{aligned}$$

Each function $(1 - \chi_{\bar{C}}) \psi f_i$ is smooth here and has compact support depending only on T and $\chi_{\bar{C}}$.

Now using Lemma 5.3 Part 1, the fact that $(1 - \chi_{\bar{C}})$ is valued in $[0, 1]$, using ψ is supported only on finitely many Γ -translates of F , together with automorphy of f , we get by integrating over F

$$\begin{aligned} \|R_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}})[f]\|_{L^2(F)}^2 &\leq C \sum_i \|\psi f_i\|_{L^2(\mathbb{H})}^2 \leq C' \|f\|_{L^2(F)}^2, \\ \|\Delta R_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}})[f]\|_{L^2(F)}^2 &\leq C \sum_i \|\psi f_i\|_{L^2(\mathbb{H})}^2 \leq C' \|f\|_{L^2(F)}^2, \end{aligned}$$

where C, C' depend on s, T . Now this bound clearly extends to $f \in L_\phi^2(\mathbb{H}; V_n^0)$. *This proves the first statement of the lemma.*

The statement that $\mathbb{L}_{\mathbb{H},n}^{(T)}(s)$ is well-defined and bounded on $L_\phi^2(\mathbb{H}; V_n^0)$ is just an easier version of the previous proof using Lemma 5.2 instead of Lemma 5.3. *This proves the second statement of the lemma.* We note that we also obtain

$$\mathbb{L}_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\bar{C}})[f] = \sum_{i=1}^{n-1} \mathbb{L}_{\mathbb{H}}^{(T)}(s) [(1 - \chi_{\bar{C}}) \psi f_i] e_i \quad (5.20)$$

analogously to (5.19).

Now going back to (5.19) and using Lemma 5.3 Part 2 gives, considered by restriction as equivalence classes of measurable functions on \overline{F} ,

$$\begin{aligned}
& (\Delta - s(1-s))R_{\mathbb{H},n}^{(T)}(s)(1-\chi_{\overline{C}})[f] \\
&= \sum_{i=1}^{n-1} (\Delta - s(1-s))R_{\mathbb{H}}^{(T)}(s) [(1-\chi_{\overline{C}})\psi f_i] e_i \\
&= \sum_{i=1}^{n-1} \left((1-\chi_{\overline{C}})\psi f_i + \mathbb{L}_{\mathbb{H}}^{(T)}(s) [(1-\chi_{\overline{C}})\psi f_i] \right) e_i \\
&= (1-\chi_{\overline{C}})f + \mathbb{L}_{\mathbb{H},n}^{(T)}(s)(1-\chi_{\overline{C}})[f].
\end{aligned}$$

On the other hand, the fact that all functions at the two ends of the string of equalities above satisfy the automorphy equation (2.1) almost everywhere implies that indeed

$$(\Delta - s(1-s))R_{\mathbb{H},n}^{(T)}(s)(1-\chi_{\overline{C}})[f] = (1-\chi_{\overline{C}})f + \mathbb{L}_{\mathbb{H},n}^{(T)}(s)(1-\chi_{\overline{C}})[f]$$

as equivalence classes of measurable functions on \mathbb{H} . *This proves the final part of the lemma.* \square

Lemma 5.5 allows us to define our interior parametrix

$$\mathbb{M}_{\phi}^{\text{int}}(s) : L_{\text{new}}^2(X_{\phi}) \rightarrow H_{\text{new}}^2(X_{\phi})$$

to be the operator corresponding under $L_{\text{new}}^2(X_{\phi}) \cong L_{\phi}^2(\mathbb{H}; V_n^0)$ and $H_{\text{new}}^2(X_{\phi}) \cong H_{\phi}^2(\mathbb{H}; V_n^0)$ to the integral operator $R_{\mathbb{H},n}^{(T)}(s)(1-\chi_{\overline{C}})$. The equation (5.17) becomes for $s > \frac{1}{2}$

$$(\Delta_{X_{\phi}} - s(1-s))\mathbb{M}_{\phi}^{\text{int}}(s) = (1-\chi_{\overline{C},\phi}) + \mathbb{L}_{\phi}^{\text{int}}(s) \quad (5.21)$$

where

$$\mathbb{L}_{\phi}^{\text{int}}(s) : L_{\text{new}}^2(X_{\phi}) \rightarrow L_{\text{new}}^2(X_{\phi})$$

is the operator corresponding under $L_{\text{new}}^2(X_{\phi}) \cong L_{\phi}^2(\mathbb{H}; V_n^0)$ to the integral operator $\mathbb{L}_{\mathbb{H},n}^{(T)}(s)(1-\chi_{\overline{C}})$ (the dependence on T is suppressed here but remembered).

Therefore if we define

$$\mathbb{M}_{\phi}(s) \stackrel{\text{def}}{=} \mathbb{M}_{\phi}^{\text{int}}(s) + \mathbb{M}_{\phi}^{\text{cusp}}(s),$$

we have

$$\mathbb{M}_{\phi}(s) : L_{\text{new}}^2(X_{\phi}) \rightarrow H_{\text{new}}^2(X_{\phi})$$

and combining (4.7) and (5.21) we obtain

$$\begin{aligned} (\Delta_{X_\phi} - s(1-s)) \mathbb{M}_\phi(s) &= (1 - \chi_{\mathcal{C},\phi}^-) + \mathbb{L}_\phi^{\text{int}}(s) + \chi_{\mathcal{C},\phi}^- + \mathbb{L}_\phi^{\text{cusp}}(s) \\ &= 1 + \mathbb{L}_\phi^{\text{int}}(s) + \mathbb{L}_\phi^{\text{cusp}}(s). \end{aligned} \quad (5.22)$$

The aim is to show that, with high probability, we can invert the right hand side of (5.22) to define a bounded resolvent in $s \geq s_0 > \frac{1}{2}$. Since we have already suitably bounded $\mathbb{L}_\phi^{\text{cusp}}(s)$ in Proposition 4.4, it remains to bound the operator $\mathbb{L}_\phi^{\text{int}}(s)$; this random operator will be studied in detail in the next section.

6 Random operators

6.1 Set up

Suppose that $f \in C_\phi^\infty(\mathbb{H}; V_n^0)$ with $\|f\|_{L^2(F)}^2 < \infty$. We have

$$\begin{aligned} \mathbb{L}_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\mathcal{C}}^-)[f](x) &= \int_{y \in \mathbb{H}} \mathbb{L}_{\mathbb{H},n}^{(T)}(s; x, y)(1 - \chi_{\mathcal{C}}^-(y))f(y) \\ &= \sum_{\gamma \in \Gamma} \int_{y \in F} \mathbb{L}_{\mathbb{H},n}^{(T)}(s; \gamma x, y) \rho_\phi(\gamma^{-1})(1 - \chi_{\mathcal{C}}^-(y))f(y). \end{aligned} \quad (6.1)$$

Using that $\mathbb{L}_{\mathbb{H},n}^{(T)}(s; x, y)$ localizes to $d(x, y) \leq T + 1$, Lemma 5.4 implies that there is a compact set $K = K(T) \subset F$ and a finite set $S = S(T) \subset \Gamma$ such that for $x, y \in \bar{F}$ and $\gamma \in \Gamma$

$$\mathbb{L}_{\mathbb{H},n}^{(T)}(s; \gamma x, y) \rho_\phi(\gamma^{-1})(1 - \chi_{\mathcal{C}}^-(y)) = 0$$

unless $x, y \in K$ and $\gamma \in S$.

The point of view we take in the rest of this section is that there is an isomorphism of Hilbert spaces

$$\begin{aligned} L_\phi^2(\mathbb{H}; V_n^0) &\cong L^2(F) \otimes V_n^0; \\ f &\mapsto \sum_{e_i} \langle f|_F, e_i \rangle_{V_n^0} \otimes e_i \end{aligned}$$

where e_i is an orthonormal basis of V_n^0 (the choice of this basis does not matter). When conjugated by this isomorphism, (6.1) shows that

$$\mathbb{L}_{\mathbb{H},n}^{(T)}(s)(1 - \chi_{\mathcal{C}}^-) \cong \sum_{\gamma \in S} a_\gamma^{(T)}(s) \otimes \rho_\phi(\gamma^{-1})$$

where

$$a_\gamma^{(T)}(s) : L^2(F) \rightarrow L^2(F)$$

$$a_\gamma^{(T)}(s)[f](x) \stackrel{\text{def}}{=} \int_{y \in F} \mathbb{L}_{\mathbb{H}}^{(T)}(s; \gamma x, y)(1 - \chi_C^-(y))f(y)d\mathbb{H}(y).$$

Again, this sum is supported on a finite set $S = S(T) \subset \Gamma$ depending on T (and the fixed χ_C^-).

Since $\mathbb{L}_{\mathbb{H}}^{(T)}(s; \gamma x, y)$ is bounded depending on T and smooth we have

$$\int_{x, y \in F} |\mathbb{L}_{\mathbb{H}}^{(T)}(s; \gamma x, y)|^2 (1 - \chi_C^-(y))^2 d\mathbb{H}(x)d\mathbb{H}(y) =$$

$$\int_{x, y \in K} |\mathbb{L}_{\mathbb{H}}^{(T)}(s; \gamma x, y)|^2 (1 - \chi_C^-(y))^2 d\mathbb{H}(x)d\mathbb{H}(y) < \infty.$$

This shows that each integral operator $a_\gamma^{(T)}(s)$ is bounded, and in fact Hilbert-Schmidt, so compact. We also produce the following deviations estimate for the $a_\gamma^{(T)}(s)$.

Lemma 6.1. *For fixed $T > 0$, there is a constant $c_1 = c_1(T) > 0$ such that for $s_1, s_2 \in [\frac{1}{2}, 1]$ and $\gamma \in S(T)$ we have*

$$\|a_\gamma^{(T)}(s_1) - a_\gamma^{(T)}(s_2)\|_{L^2(F)} \leq c_1 |s_1 - s_2|.$$

Proof. Suppose $s_1, s_2 \in [\frac{1}{2}, 1]$, and γ is fixed. The operator $a_\gamma^{(T)}(s_1) - a_\gamma^{(T)}(s_2)$ is a Hilbert-Schmidt operator on $L^2(F)$ with kernel

$$[\mathbb{L}_{\mathbb{H}}^{(T)}(s_1; \gamma x, y) - \mathbb{L}_{\mathbb{H}}^{(T)}(s_2; \gamma x, y)](1 - \chi_C^-(y));$$

once again this is zero unless x, y are in compact set $K(T)$.

We have by Lemma 5.1 Part 3 that for all $x, y \in K$

$$\left| \mathbb{L}_{\mathbb{H}}^{(T)}(s_1; \gamma x, y) - \mathbb{L}_{\mathbb{H}}^{(T)}(s_2; \gamma x, y) \right| \leq c(T) |s_1 - s_2|.$$

It follows that there is a constant $c_1 = c_1(T)$ such that, writing $\|\bullet\|_{\text{H.S.}}$ for Hilbert-Schmidt norm of a Hilbert-Schmidt operator on $L^2(F)$, we have

$$\|a_\gamma^{(T)}(s_1) - a_\gamma^{(T)}(s_2)\|_{L^2(F)} \leq \|[\mathbb{L}_{\mathbb{H}}^{(T)}(s_1) - \mathbb{L}_{\mathbb{H}}^{(T)}(s_2)](1 - \chi_C^-)\|_{\text{H.S.}}$$

$$\leq c_1(T) |s_1 - s_2|.$$

Finally, because γ is in the finite set $S(T)$, the estimate can be taken uniformly over all $\gamma \in S(T)$ by taking the maximal value of c_1 . \square

6.2 The operator on the free group

We are momentarily going to apply Corollary 3.4 to the random operator

$$\mathcal{L}_{s,\phi}^{(T)} \stackrel{\text{def}}{=} \sum_{\gamma \in S} a_{\gamma}^{(T)}(s) \otimes \rho_{\phi}(\gamma^{-1})$$

Up to an intermediate step, where we approximate the coefficients $a_{\gamma}^{(T)}(s)$ by finite rank operators, we expect to learn that the operator norm of $\mathcal{L}_{s,\phi}^{(T)}$ is close to the operator norm of

$$\mathcal{L}_{s,\infty}^{(T)} \stackrel{\text{def}}{=} \sum_{\gamma \in S} a_{\gamma}^{(T)}(s) \otimes \rho_{\infty}(\gamma^{-1})$$

on $L^2(F) \otimes \ell^2(\Gamma)$, where

$$\rho_{\infty} : \Gamma \rightarrow \text{End}(\ell^2(\Gamma))$$

is the right regular representation of Γ . So we must understand the operator $\mathcal{L}_{s,\infty}^{(T)}$. Under the isomorphism

$$\begin{aligned} L^2(F) \otimes \ell^2(\Gamma) &\cong L^2(\mathbb{H}), \\ f \otimes \delta_{\gamma} &\mapsto f \circ \gamma^{-1} \end{aligned}$$

(with $f \circ \gamma^{-1}$ extended by zero from a function on γF) the operator $\mathcal{L}_{s,\infty}^{(T)}$ is conjugated to none other than

$$\mathbb{L}_{\mathbb{H}}^{(T)}(s)(1 - \chi_{\mathcal{C}^-}) : L^2(\mathbb{H}) \rightarrow L^2(\mathbb{H})$$

from §§5.2. Since $(1 - \chi_{\mathcal{C}^-})$ is valued in $[0, 1]$, multiplication by it has operator norm ≤ 1 on $L^2(\mathbb{H})$. Hence we obtain the following corollary of Lemma 5.2.

Corollary 6.2. *For any $s_0 > \frac{1}{2}$ there is $T = T(s_0) > 0$ such that for any $s \in [s_0, 1]$ we have*

$$\|\mathcal{L}_{s,\infty}^{(T)}\|_{L^2(F) \otimes \ell^2(\Gamma)} \leq \frac{1}{5}.$$

6.3 Probabilistic bounds for operator norms

The aim of this section is to prove the following result as a consequence of Corollary 3.4.

Proposition 6.3. *For any $s_0 > \frac{1}{2}$ there is $T = T(s_0) > 0$ such that for s fixed in $[s_0, 1]$, with probability tending to one as $n \rightarrow \infty$,*

$$\|\mathcal{L}_{s,\phi}^{(T)}\|_{L^2(F)\otimes V_n^0} \leq \frac{2}{5}.$$

Proof. Let T be the one provided by Corollary 6.2 for the given s_0 , so that

$$\|\mathcal{L}_{s,\infty}^{(T)}\|_{L^2(F)\otimes \ell^2(\Gamma)} \leq \frac{1}{5}. \quad (6.2)$$

Fix $s \in [s_0, 1]$. Recall that the coefficients $a_\gamma^{(T)}(s)$ are supported on a finite set $S = S(T) \subset \Gamma$. Because the $a_\gamma^{(T)}(s)$ are Hilbert-Schmidt, and hence compact, we can find a *finite-dimensional* subspace

$$W \subset L^2(F)$$

and operators $b_\gamma^{(T)}(s) : W \rightarrow W$ such that

$$\|b_\gamma^{(T)}(s) - a_\gamma^{(T)}(s)\|_{L^2(F)} \leq \frac{1}{20|S(T)|}$$

for all $\gamma \in S(T)$. Therefore, since each $\rho_\phi(\gamma)$ is unitary on V_n^0 we get

$$\|\mathcal{L}_{s,\phi}^{(T)} - \sum_{\gamma \in S} b_\gamma^{(T)}(s) \otimes \rho_\phi(\gamma^{-1})\|_{L^2(F)\otimes V_n^0} \leq \frac{1}{20}. \quad (6.3)$$

We now apply Corollary 3.4 to the operator $\sum_{\gamma \in S} b_\gamma^{(T)}(s) \otimes \rho_\phi(\gamma^{-1})$ to obtain that with probability $\rightarrow 1$ as $n \rightarrow \infty$,

$$\|\sum_{\gamma \in S} b_\gamma^{(T)}(s) \otimes \rho_\phi(\gamma^{-1})\|_{W \otimes V_n^0} \leq \|\sum_{\gamma \in S} b_\gamma^{(T)}(s) \otimes \rho_\infty(\gamma^{-1})\|_{W \otimes \ell^2(\Gamma)} + \frac{1}{20}. \quad (6.4)$$

(Above, $\frac{1}{20}$ could be replaced by any $\epsilon > 0$ but this is not needed here.)

On the other hand, we also have by the same argument as led to (6.3)

$$\|\mathcal{L}_{s,\infty}^{(T)} - \sum_{\gamma \in S} b_\gamma^{(T)}(s) \otimes \rho_\infty(\gamma^{-1})\|_{L(F)\otimes \ell^2(\Gamma)} \leq \frac{1}{20}. \quad (6.5)$$

Then combining (6.2), (6.3), (6.4), and (6.5) gives the result. \square

7 Proof of Theorem 1.1

Given $\epsilon > 0$ let $s_0 = \frac{1}{2} + \sqrt{\epsilon}$ so that $s_0(1 - s_0) = \frac{1}{4} - \epsilon$. We assume χ_C^\pm were chosen as in Proposition 4.4 for this s_0 so that

$$\|\mathbb{L}_\phi^{\text{cusp}}(s)\|_{L^2} \leq \frac{1}{5} \quad (7.1)$$

for all $s \in (s_0, 1]$.

Let $T = T(s_0)$ be the value provided by Proposition 6.3 for this s_0 . To control all values of $s \in [s_0, 1]$ we use a finite net. For $s_1, s_2 \in [s_0, 1]$ we have

$$\mathcal{L}_{s_1, \phi}^{(T)} - \mathcal{L}_{s_2, \phi}^{(T)} = \sum_{\gamma \in S} [a_\gamma^{(T)}(s_1) - a_\gamma^{(T)}(s_2)] \otimes \rho_\phi(\gamma^{-1}) \quad (7.2)$$

where $S \subset \Gamma$ is a finite set depending on T . Lemma 6.1 tells us that for $c_1 = c_1(T) > 0$ we have

$$\|a_\gamma^{(T)}(s_1) - a_\gamma^{(T)}(s_2)\|_{L^2(F)} \leq c_1 |s_1 - s_2|$$

for all $\gamma \in S$ and $s_1, s_2 \in [s_0, 1]$; hence we obtain from (7.2) and as each $\rho_\phi(\gamma^{-1})$ is unitary that

$$\|\mathcal{L}_{s_1, \phi}^{(T)} - \mathcal{L}_{s_2, \phi}^{(T)}\| \leq |S|c_1 |s_1 - s_2|. \quad (7.3)$$

We choose a finite set Y , depending on s_0 , of points in $[s_0, 1]$ such that every point of $[s_0, 1]$ is within

$$\frac{1}{5|S|c_1}$$

of some element of Y . Now, applying Proposition 6.3 to each point of Y we obtain that with probability tending to one as $n \rightarrow \infty$ we have

$$\|\mathcal{L}_{s, \phi}^{(T)}\|_{L^2(F) \otimes V_n^0} \leq \frac{2}{5}$$

for all $s \in Y$. Hence also using (7.3) we obtain that with probability tending to one as $n \rightarrow \infty$ we have

$$\|\mathcal{L}_{s, \phi}^{(T)}\|_{L^2(F) \otimes V_n^0} \leq \frac{3}{5}$$

for all $s \in [s_0, 1]$.

Recall that we defined

$$\mathbb{M}_\phi(s) \stackrel{\text{def}}{=} \mathbb{M}_\phi^{\text{int}}(s) + \mathbb{M}_\phi^{\text{cusp}}(s).$$

By Lemma 5.5 and the discussion after (4.6), for $s > \frac{1}{2}$ $\mathbb{M}_\phi(s)$ is a bounded operator from $L_{\text{new}}^2(X_\phi)$ to $H_{\text{new}}^2(X_\phi)$. We also proved in (5.22)

$$(\Delta_{X_\phi} - s(1-s)) \mathbb{M}_\phi(s) = 1 + \mathbb{L}_\phi^{\text{int}}(s) + \mathbb{L}_\phi^{\text{cusp}}(s)$$

on $L_{\text{new}}^2(X_\phi)$. The operator $\mathbb{L}_\phi^{\text{int}}(s)$ is unitarily conjugate to $\mathcal{L}_{s,\phi}^{(T)}$ as in §§6.1 and hence a.a.s. has operator norm at most $\frac{3}{5}$. Hence also using the deterministic bound (7.1) we obtain that

$$\|\mathbb{L}_\phi^{\text{int}}(s) + \mathbb{L}_\phi^{\text{cusp}}(s)\|_{L_{\text{new}}^2(X_\phi)} \leq \frac{4}{5}$$

and hence a.a.s.

$$\left(1 + \mathbb{L}_\phi^{\text{int}}(s) + \mathbb{L}_\phi^{\text{cusp}}(s)\right)^{-1}$$

exists as a bounded operator from $L_{\text{new}}^2(X_\phi)$ to itself. We now get

$$(\Delta_{X_\phi} - s(1-s)) \mathbb{M}_\phi(s) \left(1 + \mathbb{L}_\phi^{\text{int}}(s) + \mathbb{L}_\phi^{\text{cusp}}(s)\right)^{-1} = 1$$

which shows there is a bounded right inverse to $(\Delta_{X_\phi} - s(1-s))$; this shows that $(\Delta_{X_\phi} - s(1-s))$ maps $H_{\text{new}}^2(X_\phi)$ onto $L_{\text{new}}^2(X_\phi)$ and since it is self-adjoint for $s \in [\frac{1}{2}, 1]$, cannot have any kernel in $L_{\text{new}}^2(X_\phi)$. This shows a.a.s. that Δ_{X_ϕ} has no new eigenvalues λ with $\lambda \leq s_0(1-s_0) = \frac{1}{4} - \epsilon$. This concludes the proof of Theorem 1.1.

8 Proof of Corollary 1.3

Take X to be a once-punctured torus or thrice-punctured sphere and apply Corollary 1.2 to obtain a sequence of connected orientable finite area non-compact hyperbolic surfaces X_i with $\chi(X_i) = -i$ and

$$\inf(\text{spec}(\Delta_{X_i}) \cap (0, \infty)) \rightarrow \frac{1}{4}.$$

Suppose X_i has genus g_i and n_i cusps. We have

$$-i = \chi(X_i) = 2 - 2g_i - n_i$$

which shows

$$i \equiv \chi(X_i) \equiv n_i \pmod{2}.$$

In particular, for even i , we have an even number of cusps in X_i . We use the following result of Buser, Burger, and Dodziuk [BBD88]. See Brooks and Makover [BM01, Lemma 1.1] for the extraction of this lemma from [BBD88].

Lemma 8.1 (Handle Lemma). *Let X be a finite area hyperbolic surface with an even number of cusps $\{C_i\}$. It is possible to deform the surface X in a certain way to a finite area hyperbolic surface with boundary, where each cusp becomes a bounding geodesic of length t , and then glue the geodesic corresponding to C_{2i-1} to the one corresponding to C_{2i} to form a family of closed hyperbolic surfaces $X^{(t)}$ such that*

$$\limsup_{t \rightarrow 0} \lambda_1(X^{(t)}) \geq \inf(\operatorname{spec}(\Delta_X) \cap (0, \infty)).$$

In particular, since each X_{2k} has an even number of cusps, we can construct a *closed* hyperbolic surface \tilde{X}_{2k} of Euler characteristic $-2k$ and

$$\lambda_1(\tilde{X}_{2k}) \geq \inf(\operatorname{spec}(\Delta_{X_{2k}}) \cap (0, \infty)) - \frac{1}{k}.$$

On the other hand, the upper bound of Huber [Hub74] now implies $\lambda_1(\tilde{X}_{2k}) \rightarrow \frac{1}{4}$ as required. \square

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