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# Exchange rates, sunspots and cycles

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## Abstract

The empirical evidence on nominal exchange rate dynamics shows a long-run relationship of this variable with the fundamentals of the economy, although such relationship disappears at shorter horizons (“exchange rate disconnect” puzzle). This apparently contrasting behaviour of the nominal exchange rate can be explained in an overlapping-generations model where the two currencies are not perfect substitutes. In this framework, we show that the nominal exchange rate is pinned down by the fundamentals of the economy at the monetary steady state. However, fluctuations of the nominal exchange rate around its long-run value, which are not driven by shocks to fundamentals, can emerge. Firstly, we prove the existence of endogenous (deterministic) business cycles in the nominal exchange rate. Secondly, we construct stationary sunspot equilibria where random fluctuations of the nominal exchange rate arise as a result of self-fulfilling beliefs.

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# 1 Introduction

Friedman (1953) famously argued that “a flexible exchange rate need not be an unstable exchange rate. If it is, it is primarily because there is underlying instability in the economic conditions governing international trade.” In other words, Friedman’s view was that the exchange rate volatility that we observe between any two currencies is just a symptom of the volatility of the fundamentals of the underlying economies. However, the notion that there is a strong correlation between exchange rates and fundamentals has been widely questioned over the years.

The main lesson that can be drawn from the empirical literature is that while there is evidence that the long-run value of the exchange rate is somewhat tied to fundamentals, it is very difficult to understand and predict its behaviour at shorter horizons.

Obstfeld and Rogoff (2001) coined the term “exchange rate disconnect” to describe the “exceedingly weak relationship (except, perhaps, in the long run) between the exchange rate and virtually any macroeconomic aggregates”. One of the manifestations of this puzzle is that economic models are no more useful than a simple random walk model in forecasting the nominal exchange rate, as seen in the seminal paper of Meese et al. (1983).<sup>1</sup> On the other hand, Mark (1995) has shown that econometricians have more success in forecasting the nominal exchange rate at longer horizons. Moreover, Groen (2000), Mark et al. (2001), Rapach et al. (2002) and Cerra et al. (2010) have all documented the existence of a long-run relationship between exchange rates and monetary fundamentals, such as money supplies and output differentials, using cointegration analysis.

Any macroeconomic model that wishes to capture this puzzling dynamics of the nominal exchange rate should therefore have two main properties: on the one hand, the nominal exchange rate should be a function of the fundamentals at the steady state of the economy; at the same time, there should also exist equilibria where the nominal exchange rate fluctuates around its long-run value but not as a result of randomness in the fundamentals of the economy. The aim of this paper is to propose a model which possesses these two features. In fact, in the main workhorse monetary open economy models under flexible (Lucas, 1982) and sticky prices (Obstfeld et al., 1995), the nominal exchange rate is a function of the fundamentals both in the short and in the long-run.

The most popular framework to study fundamentals-unrelated fluctuations in monetary, closed economies is the overlapping-generations model. The existence of sunspot equilibria and endogenous business cycles has been studied

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<sup>1</sup>Meese et al.’s paper has inspired dozens of empirical studies on the topic. In a recent review of the literature, Rossi (2013) pointed out that the puzzle is still very much alive.

in two-period overlapping-generations economies with one currency and one good in the well-known seminal contributions of respectively Azariadis (1981) and Grandmont (1985), where Azariadis et al. (1986) explored the connections between sunspots and cycles.<sup>2</sup> However, the investigation of endogenous fluctuations and sunspot equilibria in monetary, open economies has proved elusive so far due to Kareken and Wallace (1981)'s result on the indeterminacy of the nominal exchange rate. In their paper, one good is available for consumption purposes and overlapping generations of agents can store two currencies to buy the good when old. However, as the currencies are perfectly substitutable, the nominal exchange rate is indeterminate as well as constant over time.<sup>3</sup> For this reason, the model cannot replicate the previously mentioned stylized facts on exchange rate dynamics.

In this paper, we study a two-country, two-goods and two-currencies overlapping generations economy where agents live for two periods. The timing of trading is structured as follows. The first period of life of each cohort is rather standard. The young are endowed with a country-specific good and after having chosen the optimal consumption bundle for the current period, they choose a portfolio of currencies to fund consumption next period<sup>4</sup>. The main novelties of our model lie in the way the old can use their portfolio of currencies and their access to currency markets. Firstly, we depart from Kareken et al. (1981)'s assumption that the domestic and the foreign currency are perfect substitutes. We assume instead that each currency can only buy the country-specific good as in Lucas (1982). However, we show that this feature alone is not enough to bypass the issue of a constant and indeterminate exchange rate in our model, as the two currencies would still be perfect substitutes as stores of value.<sup>5</sup> Then, we also require that the old have no access to the currency markets, in the sense that they cannot re-trade their portfolio of currencies purchased when young once they are old. In other words, they have to commit to the money holdings acquired in the previous period.

We show that these two features imply that agents' portfolios can be pinned down as a result of agents' future demand for the two goods. As a consequence, the dynamics of the nominal exchange rate is tightly linked to the dynamics of the prices of the two goods. For instance, if the domestic good is expected to

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<sup>2</sup>The literature has also shown that the existence of endogenous cycles is not an exclusive feature of two-period overlapping-generations models. Bhattacharya et al. (2003) has shown the existence of two-period cycles in a model where agents live for three periods, while Reichlin (1992) has proved the existence of periodic cycles in a model with longer but uncertain lifetime spans.

<sup>3</sup>Sargent (1987) has pointed out that Kareken and Wallace's indeterminacy result is related to the presence of an extra asset in the model, and not specifically due to the demographic structure.

<sup>4</sup>The assumption that agents are only endowed with a country-specific good is only made to simplify the notation. Allowing partial instead of complete specialization would not change our main results. We also restrict our attention to "Samuelsonian economies", where the value of the endowment when old is sufficiently small to generate a positive demand for money (Gale, 1973).

<sup>5</sup>The interested reader will find further details on this issue in the Supplementary Material.

be more expensive in the following period, the domestic currency has a lower purchasing power. If the elasticity of substitution is sufficiently high, then this implies a lower global demand for the domestic currency and an exchange rate depreciation.

One of the difficulties of our framework is that its dynamical system is much more complex than in the case of a one-good, one-currency economy. Standard geometric tools which have been extensively used to study the properties of monetary economies with overlapping-generations, such as the offer curve, cannot be easily adapted to a two-goods, two-currencies framework<sup>6</sup>. However, we show that for a CES utility function which is additively separable across the two goods (as well as intertemporally separable), the dynamics of the two goods' prices can be studied independently. As a result, the dynamic behaviour of the world economy can be fully characterized by i) two separate difference equations in the two prices; ii) an equation for the nominal exchange rate, which is pinned down by the expected prices of the two goods, and finally iii) two dynamic inequality constraints in the price of both goods and the nominal exchange rate, which guarantee positive demand of the two currencies.

In this context, we prove that the nominal exchange rate is determined by the fundamentals of the economy at the monetary steady state. Two among the determinants of the exchange rate, relative money supplies and aggregate endowments, are common to other monetary models of exchange rate determinacy (e.g. Frankel, 1979; Lucas, 1982). In our setting, the nominal exchange rate also depends on how much of each good is saved by the young, as currencies serve the function of stores of value. In particular, higher savings of a particular good are associated with an appreciation of the domestic currency. As the supply of a good by the young increases, its price falls and hence the purchasing power of the domestic currency in units of the domestic good increases. As the domestic currency is worth more, then its demand increases hence the appreciation in equilibrium. We also show that Lucas' exchange rate equation can be retrieved by imposing that the endowment of the old is zero. Crucially, this equivalence only holds in the long-run, as exchange rate and fundamentals are disconnected outside the monetary steady state in our framework.

We then demonstrate that our model is able to explain why econometricians struggle to find a strong correlation between the exchange rate and the fundamentals of the economy at shorter horizons. We prove that randomness in the fundamentals is not required to generate exchange rate fluctuations around the steady state. Persistent endogenous fluctuations in the exchange rate can indeed emerge either as a consequence of periodic deterministic cycles or as a result of sunspot equilibria once extrinsic uncertainty is added to the model.

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<sup>6</sup>See Woodford (1984) for a literature review of one-good, one-currency OLG economies.

Under conditions on preferences which are alike to the one-good one-currency case, periodic deterministic cycles may exist for an open set of the endowments. We show that one can construct many possible scenarios each of which can give rise to a cyclical behaviour of the nominal exchange rate around its long-run value. In particular, we provide an example where the endowment of one of the two countries is such that the domestic price is determinate while the other one is such that a two-period cycle emerge. This results in a two-period cycle of the nominal exchange rate. Alternatively, one could suppose that one of the two prices has cyclical behaviour while the other price is indeterminate. In this case, the path of the nominal exchange rate is indeterminate but would always converge to a two-period cycle. Another example would be one where both prices exhibit cyclical behaviour. Depending on the initial prices selected, the two values that the exchange rate can assume in the cycle will be different. Only under very special circumstances, the nominal exchange rate can be constant over time: this is only possible when the fundamentals of the two economies, as well as the initial price levels, are identical.

Nominal exchange rate fluctuations can also emerge as a result of stationary sunspot equilibria. In fact, we prove that, under some conditions, the randomness of the nominal exchange rate that we observe in the data can be generated by self-fulfilling beliefs that prices are stochastic.

As Azariadis et al. (1986) argued, focusing on stationarity sunspot equilibria is important for two reasons: firstly, because stable beliefs can be the asymptotic outcome of learning processes; secondly, this is a first step towards understanding dynamical sunspot behaviour. In a one-currency one-good economy, Azariadis (1981) showed that sufficient conditions for the existence of stationary sunspot equilibria are the complementarity between consumption and leisure and the local stability (indeterminacy) of the monetary steady state. In our framework, we prove that if similar conditions hold, then stationary sunspot equilibria exist. It is enough that agents believe that the price of one of the two goods fluctuates over time. If agents believe that the real interest rate in one country (measured as the change over time in the purchasing power of the domestic currency in units of the domestic good) is subject to some degree of volatility, then the nominal exchange rate follows a stochastic process merely dictated by such beliefs. In particular, an increase in the real interest rate is associated with the appreciation of the domestic currency and viceversa.

Interestingly, we do not need any heterogeneity across agents to generate these fundamentals-unrelated fluctuations, as sunspot equilibria are supported by agents' sharing the same beliefs. On the contrary, recent contributions in the literature have attempted to explain the exchange rate disconnect puzzle assuming that private investors possess heterogeneous information about

the fundamentals of the economy<sup>7</sup>. In such setting, a common information framework would not be able to explain the puzzle.

This paper shows that it is then possible to construct equilibria which replicate the two main features of the dynamics of the nominal exchange rates: while our model can easily rationalise the “exchange rate disconnect” puzzle, empirical evidence which points at the higher predictability of the nominal exchange rate in the long-run can also be reconciled within this framework.

The paper is structured as follows. In section 2, we describe the model and show how the dynamic equilibrium system can be simplified. We characterise the monetary steady state in section 3 and the dynamics around the monetary steady state in section 4. In section 5, we investigate the existence of non-fundamentals related persistent fluctuations: endogenous deterministic cycles and stationary sunspot equilibria. Section 6 concludes. All the proofs can be found in Appendix B.

## 2 The model

We study the following two-country pure exchange overlapping-generations economy. Time is discrete and a generic date is indicated with  $t$ . At each  $t$ , an agent is born in each country with a two-period lifetime, where  $a = 1, 2$  refers to age. We indicate with  $h = 1, 2$  the agent living in country  $h$  while  $\ell = 1, 2$  refers to the good or the currency. For instance,  $c_{ah,t}^\ell$  is the consumption of good  $\ell$  of the agent born in country  $h$  in period of life  $a$  at time  $t$ . Agents are endowed with a country-specific good in both periods of life: agents born in country 1 (2) are endowed with good 1 (2). Since our objective is to show that there can exist fluctuations in the nominal exchange rate not related to fluctuations in the fundamentals, we assume that endowments are stationary. Hence,  $(y_{a1}^1, y_{a1}^2) = (y_a^1, 0)$  and  $(y_{a2}^1, y_{a2}^2) = (0, y_a^2)$  for every  $a$ .

There also exists a generation that lives only in period 0. These agents are “the old” at time 0 and are endowed with some units of the domestic currency and the domestic good. The total endowment of the two currencies is indicated with  $M^1$  and  $M^2$  and the monetary authorities are inactive in the following periods. As in standard overlapping-generations economies, money is demanded by the young to transfer wealth across periods as long as there is a motive for saving. We indicate with  $m_{h,t}^\ell$  the demand of currency  $\ell$  of the (young) agent born in country  $h$  at time  $t$ .

An agent born in country  $h$  chooses consumption allocations  $\mathbf{c}_{1h,t} := (c_{1h,t}^1, c_{1h,t}^2)$ ,  $\mathbf{c}_{2h,t+1} := (c_{2h,t+1}^1, c_{2h,t+1}^2)$  and a portfolio allocation  $\mathbf{m}_{h,t} := (m_{h,t}^1, m_{h,t}^2)$  to

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<sup>7</sup>We refer to the “microstructure” literature by Evans et al. (2002), Bacchetta et al. (2006) and Evans (2010).



maximise her intertemporal utility function:

$$\max_{\mathbf{c}_{1h,t}, \mathbf{c}_{2h,t+1}, \mathbf{m}_{h,t}} \frac{c_{1h,t}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \frac{c_{1h,t}^2^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \beta \left[ \frac{c_{2h,t+1}^1^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \frac{c_{2h,t+1}^2^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \right] \quad (1)$$

$$s.t. \quad p_t^1 c_{1h,t}^1 + p_t^2 e_t c_{1h,t}^2 + m_{h,t}^1 + e_t m_{h,t}^2 = w_{1h,t}, \quad (2)$$

$$p_{t+1}^1 (c_{2h,t+1}^1 - y_{2h}^1) = m_{h,t}^1, \quad (3)$$

$$p_{t+1}^2 (c_{2h,t+1}^2 - y_{2h}^2) = m_{h,t}^2, \quad (4)$$

where  $\sigma > 0$  is the elasticity of substitution and  $\beta \in (0, 1]$  is the discount factor<sup>8</sup>. We indicate with  $p_t^\ell$  the price of good  $\ell$  in units of the domestic currency and with  $e_t$  the nominal exchange rate or the price of currency 2 in units of currency 1, where the latter is the numéraire currency. Since agents are only endowed with a country-specific good, the wealth of the two agents when young is respectively  $w_{11,t} := p_t^1 y_1^1$  and  $w_{12,t} := p_t^2 e_t y_1^2$ .

The budget constraint of the young is rather standard: each agent uses her wealth to buy the two consumption goods and a portfolio of currencies. The novelty of our model with respect to Kareken et al. (1981) is that the two currencies are not perfect substitutes, in the sense that each of them cannot be used to buy any good. Firstly, we assume that currency 1 (2) can only buy good 1 (2). In addition, the old are not allowed to re-adjust their portfolio composition before spending the currencies in the respective goods' markets. These two features jointly ensure that the two currencies are not perfect substitutes as stores of value, in contrast with Kareken et al. (1981)<sup>9</sup>. As the old face two separate budget constraints, the two currencies have different rates of return which depend on the expected purchasing power of the two currencies. As a result, both the nominal exchange rate as well as the agents' portfolios can be pinned down in equilibrium.

Let  $\lambda_{1h,t}$  be the Lagrange multiplier associated to (2) while  $\lambda_{2h,t+1}^1$  and  $\lambda_{2h,t+1}^2$  are the multipliers associated to the budget constraint of the old (3) and (4). The Lagrangian function of agent  $h$  born at time  $t$  is:<sup>10</sup>

$$\begin{aligned} \mathbb{L} &= \frac{c_{1h,t}^1^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \frac{c_{1h,t}^2^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \beta \left[ \frac{c_{2h,t+1}^1^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \frac{c_{2h,t+1}^2^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \right] + \\ &+ \lambda_{1h,t} [w_{1h,t} - p_t^1 c_{1h,t}^1 - p_t^2 e_t c_{1h,t}^2 - m_{h,t}^1 - e_t m_{h,t}^2] + \\ &+ \sum_{\ell} \lambda_{2h,t+1}^\ell [m_{h,t}^\ell - p_{t+1}^\ell (c_{2h,t+1}^\ell - y_{2h}^\ell)] \end{aligned} \quad (5)$$

<sup>8</sup>This utility function implies that preferences across the two goods are separable. We adopt this specification for tractability reasons, since we can study analytically the dynamics around the monetary steady state (see section 3). We also assume that agents assign the same weight to the domestic and the foreign good to make notation less cumbersome, but our main results hold if home bias is allowed.

<sup>9</sup>See the Supplementary material for more details on the point that both ingredients are necessary.

<sup>10</sup>Since we are interested in studying fluctuations of the nominal exchange rate around the monetary steady state, we only study equilibria where money holdings are strictly positive.

The first-order conditions are necessary and sufficient for a maximum and can be found in Appendix A. Rearranging the first-order conditions, it can be shown that the optimal demand for the two currencies is:<sup>11</sup>

$$m_{1,t}^1 = \frac{\beta^\sigma p_{t+1}^{1-\sigma} p_t^1 y_1^1 - p_{t+1}^1 y_2^1 [p_t^{1-\sigma} + (p_t^2 e_t)^{1-\sigma} + \beta^\sigma (p_{t+1}^2 e_t)^{1-\sigma}]}{A_t} \quad (6)$$

$$m_{1,t}^2 = \frac{\beta^\sigma (p_{t+1}^2 e_t)^{1-\sigma} [p_t^1 y_1^1 + p_{t+1}^1 y_2^1]}{e_t A_t} \quad (7)$$

$$m_{2,t}^1 = \frac{\beta^\sigma p_{t+1}^{1-\sigma} [p_t^2 e_t y_1^2 + p_{t+1}^2 e_t y_2^2]}{A_t} \quad (8)$$

$$m_{2,t}^2 = \frac{\beta^\sigma (p_{t+1}^2 e_t)^{1-\sigma} p_t^2 e_t y_1^2 - p_{t+1}^2 e_t y_2^2 [p_t^{1-\sigma} + (p_t^2 e_t)^{1-\sigma} + \beta^\sigma p_{t+1}^{1-\sigma}]}{e_t A_t} \quad (9)$$

where  $A_t \equiv p_t^{1-\sigma} + (p_t^2 e_t)^{1-\sigma} + \beta^\sigma p_{t+1}^{1-\sigma} + \beta^\sigma (e_t p_{t+1}^2)^{1-\sigma}$ .

It can be noticed that agents' demand for the foreign currency is always positive, since they are not endowed with the foreign good. The holdings of the domestic currency are instead positive as long as the numerators of (6) and (9) are strictly positive. In particular, we have that

$$m_{1,t}^1 > 0 \quad \Leftrightarrow \quad \frac{\beta^\sigma p_t^{1-\sigma} y_1^1}{p_{t+1}^{1-\sigma} y_2^1} > 1 + \left(\frac{p_t^2 e_t}{p_t^1}\right)^{1-\sigma} + \beta^\sigma \left(\frac{p_{t+1}^2 e_t}{p_t^1}\right)^{1-\sigma} \quad (10)$$

$$m_{2,t}^2 > 0 \quad \Leftrightarrow \quad \frac{\beta^\sigma p_t^{2-\sigma} y_1^2}{p_{t+1}^{2-\sigma} y_2^2} > 1 + \left(\frac{p_t^1}{p_t^2 e_t}\right)^{1-\sigma} + \beta^\sigma \left(\frac{p_{t+1}^1}{p_t^2 e_t}\right)^{1-\sigma} \quad (11)$$

Since the right hand side of both equations is greater than one, it follows that a necessary (but not sufficient) condition for money holdings of the domestic currency to be positive in each country, is:

$$\beta^\sigma p_t^{\ell\sigma} y_1^\ell > p_{t+1}^{\ell\sigma} y_2^\ell \quad \ell = 1, 2 \quad (12)$$

The value of the endowment when old must be sufficiently small as compared to the value of the endowment when young, i.e. the economy must be Samuelsonian (Gale, 1973). However, equations (10) and (11) are more stringent than the standard Samuelsonian condition. In fact, the demand for the domestic currency will also depend on the demand for the foreign good in the two periods. In particular, a higher price of the foreign good leads to a higher (lower) demand for the domestic good (and hence for the domestic currency) if the elasticity of substitution is higher (lower) than one.

Next, we plug (6), (7), (8) and (9) into the budget constraints and derive

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<sup>11</sup>We show the main steps in the Supplementary Material.

the optimal demands for the goods:

$$c_{1h,t}^1 = \frac{p_t^{1-\sigma}}{A_t} w_{h,t}, \quad c_{1h,t}^2 = \frac{(e_t p_t^2)^{-\sigma}}{A_t} w_{h,t}, \quad (13)$$

$$c_{2h,t+1}^1 = \frac{\beta^\sigma p_{t+1}^{1-\sigma}}{A_t} w_{h,t}, \quad c_{2h,t+1}^2 = \frac{\beta^\sigma (e_t p_{t+1}^2)^{-\sigma}}{A_t} w_{h,t} \quad (14)$$

where  $w_{1,t} \equiv p_t^1 y_1^1 + p_{t+1}^1 y_2^1$  and  $w_{2,t} = p_t^2 e_t y_1^2 + p_{t+1}^2 e_t y_2^2$ .

Finally, we introduce the maximization problem of the initial old. The initial old is endowed with some units of the domestic currency and of the domestic good. To simplify the problem, we assume that the initial old gain utility only from the domestic good.<sup>12</sup> In country 1:

$$\max_{c_{21,0}^1} \frac{c_{21,0}^{1-\frac{1}{\sigma}}}{1 - \frac{1}{\sigma}} \quad (15)$$

subject to:

$$p_0^1 (c_{21,0}^1 - y_{21}^1) = M^1 \quad (16)$$

The solution is straightforward:

$$c_{21,0}^1 = \frac{M^1}{p_0^1} + y_{21}^1 \quad (17)$$

$$(18)$$

The maximisation problem of the initial old in country 2 is similar. Its solution requires that the old agent in country 2 consumes its domestic endowment plus the real money balances:

$$c_{22,0}^2 = \frac{M^2}{p_0^2} + y_{22}^2 \quad (19)$$

## 2.1 Equilibrium

We are now ready to give a definition of monetary equilibrium.

**Definition 1** *A monetary equilibrium is any sequence of strictly positive nominal prices and exchange rates,  $\{p_t^1, p_t^2, e_t\}_{t=0}^\infty$ , a strictly positive consumption allocation,  $\{c_{ah,t}^1, c_{ah,t}^2\}_{t=0}^\infty$  for every  $a, h$ , and a strictly positive portfolio allocation,  $\{m_{h,t}^1, m_{h,t}^2\}_{t=0}^\infty$  for every  $h$ , such that:*

- (i) *Each agent  $h$  maximises her utility subject to her constraints at any  $t$ .*
- (ii) *Goods' markets clear, i.e.,  $\sum_h c_{1h,t}^\ell + \sum_h c_{2h,t}^\ell = y_1^\ell + y_2^\ell \quad \forall \ell, t$ .*
- (iii) *Money markets clear, i.e.,  $\sum_h m_{h,t}^\ell = M^\ell \quad \forall \ell, t$ .*

<sup>12</sup>If we assumed that the initial old gained utility from both goods, we would need to assume that he is endowed with some units of the foreign good (given our utility function). We avoid this to ensure consistency with the pattern of endowments of future generations. Assuming that every generation is endowed with some units of the foreign good would considerably complicate the notation but not change our results.

In the next Proposition, we derive the key equations that any monetary equilibrium must satisfy.

**Proposition 1** *The dynamics of the world economy is fully described by the system:*

$$\beta^\sigma p_{t+1}^{1-\sigma} [p_t^1 y_1^1 - M^1] = p_t^{1-\sigma} [M^1 + p_{t+1}^1 y_2^1] \quad (20)$$

$$\beta^\sigma p_{t+1}^{2-\sigma} [p_t^2 y_1^2 - M^2] = p_t^{2-\sigma} [M^2 + p_{t+1}^2 y_2^2] \quad (21)$$

$$e_t = \left( \frac{M^1 + p_{t+1}^1 y_2^1}{M^2 + p_{t+1}^2 y_2^2} \right)^{\frac{1}{\sigma}} \left( \frac{p_{t+1}^2}{p_{t+1}^1} \right)^{\frac{1-\sigma}{\sigma}} \quad (22)$$

where  $p_t^1 > 0$ ,  $p_t^2 > 0$  for any  $t \geq 0$ , plus the inequality constraints (10) and (11), which guarantee strictly positive money demands.

The first observation is that the dynamics of the prices of the two goods can be studied independently from each other<sup>13</sup>. However, the system does not dichotomizes completely. The dynamic inequality constraints (10) and (11), which have to be respected, link the prices of the two goods together.

In the analysis of the dynamics of the economy, we will proceed by studying first the price sequences emerging from the system (20)-(21) and then verify numerically that the inequality constraints hold for any  $t$ .

Equation (22) shows that the behaviour of the nominal exchange rate is driven by two different channels. Firstly, it depends on the endowment when old of the agents. Suppose that agents born in country 1 have a higher endowment when old. Everything else equal, they would need to acquire less domestic currency when young to achieve their desired level of consumption. A lower demand for currency 1 implies a depreciation ( $e_t$  increases).

The nominal exchange rate also responds to movements in future price levels. The next result summarizes the effects of changes in future price levels on the nominal exchange rate.

**Corollary 1** *If  $\sigma > 1$ , then an increase (decrease) in the expected price of the domestic good leads to a currency depreciation (appreciation). If  $\sigma < 1$ , then the impact of a price change on the exchange rate is ambiguous.*

In Appendix B, we show that price changes can have an effect on the nominal exchange rate through two separate components. Suppose that the price of good 2 goes up. Firstly, this generates a wealth effect: as the value of the endowment of good 2 when old increases, agents need to accumulate less currency 2 and a lower demand for the currency 2 leads to a depreciation ( $e_t$

<sup>13</sup>This result depends on the fact that preferences across the two goods are separable. If we changed the utility function to a CES aggregator of the kind  $c_{ah,t} = \left[ c_{ah,t}^1 \frac{\sigma-1}{\sigma} + c_{ah,t}^2 \frac{\sigma-1}{\sigma} \right]^{\frac{\sigma}{\sigma-1}}$ , the dynamics of the two goods would be interdependent and the system cannot be solved in blocks.

falls). But there is also a price effect: when  $\sigma > 1$ , goods are substitutes which means that an increase in the price of good 2 in the future implies that agents demand less good 2 in favour of good 1. Agents would then demand less currency 2, which also adds to the depreciation of currency 2. Therefore, when  $\sigma > 1$ , the price channel always reinforces the wealth channel. When  $\sigma < 1$ , then the price channel works in the opposite direction as goods are complements. Hence, the overall effect of an increase in the price of good 2 is ambiguous when  $\sigma < 1$ <sup>14</sup>.

### 3 The monetary steady state: long-run determinants of the nominal exchange rate

Firstly, let us define the growth rates of the nominal price levels and the terms of trade of the economy respectively as:  $\pi_t^\ell \equiv \frac{p_{t+1}^\ell}{p_t^\ell} - 1$  and  $\varepsilon_t \equiv \frac{p_t^2 e_t}{p_t^1}$ .

From now onwards, we star steady state variables. At the steady state, the nominal price levels must grow at a constant rate:  $\pi_t^1 = \pi^{1*}$  and  $\pi_t^2 = \pi^{2*}$ . Combining the first-order conditions (37) and (38) and imposing stationarity in consumption, it can also be observed that  $\varepsilon_t = \varepsilon^*$ . It then follows that the nominal exchange rate grows at a constant rate at the steady state of the economy:

$$\frac{e_{t+1}}{e_t} = \frac{1 + \pi^{1*}}{1 + \pi^{2*}} \quad (23)$$

**Definition 2** *A monetary steady state is any monetary equilibrium with strictly positive prices  $(p_t^{1*}, p_t^{2*}, e_t^*)$ , inflation rates  $(\pi^{1*}, \pi^{2*})$ , and strictly positive and constant consumption allocations  $(c_{ah}^{1*}, c_{ah}^{2*})$  for every  $a, h$ .*

The next Proposition shows that, at a stationary monetary equilibrium, inflation rates are equal to zero in both countries. This also implies that the nominal exchange rate is constant. Moreover, it shows that a stationary monetary equilibrium exists and is unique.

**Proposition 2** *At a stationary monetary equilibrium, we have that:*

$$(p_t^{1*}, p_t^{2*}, e_t^*) = \left( \frac{M^1(1 + \beta^\sigma)}{\beta^\sigma y_1^1 - y_2^1}, \frac{M^2(1 + \beta^\sigma)}{\beta^\sigma y_1^2 - y_2^2}, \frac{M^1}{M^2} \frac{\beta^\sigma y_1^2 - y_2^2}{\beta^\sigma y_1^1 - y_2^1} \left( \frac{y_1^1 + y_2^1}{y_1^2 + y_2^2} \right)^{\frac{1}{\sigma}} \right) \quad (24)$$

and the two inequality constraints (10) and (11) become

$$\frac{\beta^\sigma y_1^1}{y_1^2} > 1 + (1 + \beta^\sigma) \varepsilon^{*1-\sigma}, \quad \text{and} \quad \frac{\beta^\sigma y_1^2}{y_2^2} > 1 + (1 + \beta^\sigma) \varepsilon^{*\sigma-1} \quad (25)$$

<sup>14</sup>Notice that when  $\sigma = 1$  (log utility), the nominal exchange rate responds to movements of price levels only through the wealth channel. On the other hand, when the endowments of the old are equal to zero, only the price channel matters.

At the steady state of the economy, the nominal exchange rate is determinate and a function of the fundamentals of the economy. In particular, it depends on three different sets of parameters: (1) relative money supplies; (2) relative savings; (3) relative aggregate endowments or the terms of trade of the economy:

$$e^* = \underbrace{\frac{M^1}{M^2}}_{\text{relative money supply}} \cdot \underbrace{\frac{\beta^\sigma y_1^2 - y_2^2}{\beta^\sigma y_1^1 - y_2^1}}_{\text{relative savings}} \cdot \underbrace{\left(\frac{y_1^1 + y_2^1}{y_1^2 + y_2^2}\right)^{\frac{1}{\sigma}}}_{\text{terms of trade}}$$

For instance, currency 1 depreciates ( $e$  increases) under the following circumstances: (1) an increase in the domestic money supply; (2) a fall in savings in good 1; (3) an increase in the domestic aggregate endowment. The first channel does not probably need an explanation. The second channel involves the relative savings by the young in the two goods. In fact, aggregating the budget constraint of the old across agents, we obtain an equation for the savings of good 1:

$$p^{1*}(c_2^1 - y_2^1) = M^1 \quad \Rightarrow \quad p^{1*}(y_1^1 - c_1^1) = M^1 \quad \Rightarrow \quad y_1^1 - c_1^1 = \frac{\beta^\sigma y_1^1 - y_2^1}{1 + \beta^\sigma}$$

where  $p^{1*}$  is known from Proposition 2. It can be observed that a fall in the supply of savings of good 1 implies a higher price for the good. Since currency 1 in units of the domestic good is worth less in terms of purchasing power, then it depreciates<sup>15</sup>. Finally, an increase in the aggregate endowment of good 1 implies a depreciation of the domestic currency as the relative price of good 1 falls (i.e. the terms of trade worsens):

$$\varepsilon^* = \left(\frac{y_1^1 + y_2^1}{y_1^2 + y_2^2}\right)^{\frac{1}{\sigma}} \quad (26)$$

It is worth observing that the second channel is specific to our OLG model. In cash-in-advance infinite horizon models à la Lucas (1982), the equilibrium exchange rate is instead equal to:

$$e^* = \underbrace{\frac{M^1}{y^1} \frac{y^2}{M^2}}_{\text{quantity theory equations}} \cdot \underbrace{\left(\frac{y^1}{y^2}\right)^{\frac{1}{\sigma}}}_{\text{terms of trade}} = \frac{M^1}{M^2} \left(\frac{y^1}{y^2}\right)^{\frac{1-\sigma}{\sigma}}$$

which can be seen as a specific case of our equation when  $y_2^l = 0$ . In this case, the young hold all the aggregate output of the economy. As in Lucas (1982), an increase in the endowment leads to a currency appreciation (depreciation) whenever the elasticity of substitution is higher (lower) than 1.

<sup>15</sup>Notice that savings can fall due to the following reasons: a fall in the endowment when young, an increase in the endowment when old or a fall in the discount factor.

More generally, the nominal exchange rate depends on the distribution of the aggregate endowment across cohorts. Let us calculate the partial derivatives of  $e^*$  with respect to  $y_1^2$  and  $y_2^2$  to gain some more intuition about the effect of an increase in endowments on the exchange rate:

$$\begin{aligned}\frac{\partial e^*}{\partial y_1^2} &= \frac{M^1}{M^2} \frac{1}{\beta^\sigma y_1^1 - y_2^1} \left( \frac{y_1^1 + y_2^1}{y_1^2 + y_2^2} \right)^{\frac{1}{\sigma}} \left[ \beta^\sigma - \frac{\beta^\sigma y_1^2 - y_2^2}{\sigma(y_1^1 + y_2^1)} \right] \\ \frac{\partial e^*}{\partial y_2^2} &= -\frac{M^1}{M^2} \frac{1}{\beta^\sigma y_1^1 - y_2^1} \left( \frac{y_1^1 + y_2^1}{y_1^2 + y_2^2} \right)^{\frac{1}{\sigma}} \left[ y_2^2 + \frac{\beta^\sigma y_1^2 - y_2^2}{\sigma(y_1^1 + y_2^1)} \right]\end{aligned}$$

which implies that

$$\begin{aligned}\frac{\partial e^*}{\partial y_1^2} &> 0 & \text{if} & \quad \beta^\sigma y_1^2(\sigma - 1) + y_2^2(1 + \beta\sigma) > 0 \\ \frac{\partial e^*}{\partial y_2^2} &< 0\end{aligned}$$

Firstly, let us comment on the effect of an increase in the endowment of the young on the exchange rate. It can be seen that there are two effects at play here. On the one hand, an increase in the endowment of the young increases savings. Hence, a higher demand for the domestic currency leads to a currency appreciation. On the other hand, an increase in the endowment of the young causes a deterioration of the terms of trade, which instead leads the currency to depreciate. It can be seen that the first effect always dominates whenever the domestic and the foreign good are substitutes ( $\sigma > 1$ ). However, differently from Lucas (1982), there might be instances in which the sign of the derivative is still positive when the goods are complements (and it is always positive when  $\sigma = 1$ ).

On the other hand, an increase in the endowment of the old unambiguously causes a currency depreciation. In fact, this leads to lower savings (hence to a lower demand for the domestic currency) as well as to a deterioration of the terms of trade.

## 4 Local dynamics

To start with, we study the dynamics of the economy around the monetary steady state. Proposition 1 shows that the nominal exchange rate is pinned down by next period's price levels (equation (22)). Hence, it suffices to study the dynamics of the two prices in order to pin down the equilibrium path of the nominal exchange rate and the quantity variables.

As it emerged clearly from Proposition 1, this means to study the scalar difference equation:

$$F(p_t, p_{t+1}) \equiv p_t^{1-\sigma}(M + p_{t+1}y_2) - \beta^\sigma p_{t+1}^{1-\sigma}(p_t y_1 - M) = 0, \quad (27)$$

and more specifically the local stability of its unique steady state:

$$p^* = \frac{M(1 + \beta^\sigma)}{\beta^\sigma y_1 - y_2} \quad (28)$$

with  $p^* > 0$  as long as  $y_1 > \beta^{-\sigma} y_2 \equiv \bar{y}$ . The superscripts have been dropped to make the notation less cumbersome.<sup>16</sup>

Before proceeding, we observe that the price level is a no-predetermined variable and therefore if  $p^*$  is unstable, then it is locally determinate, while if it is locally stable, then it is locally indeterminate.

**Proposition 3**  *$p^*$  is locally determinate if one of the following conditions is satisfied:*

- i)  $y_1 \in (\bar{y}, \infty)$  and  $\sigma \in [1, \infty)$
- ii)  $y_1 \in (\bar{y}, \tilde{y})$  and  $\sigma \in (0, 1)$ ;
- iii)  $y_1 \in (\tilde{y}, \infty)$  and  $\sigma \in [\frac{1}{2}, 1)$ ;
- iv)  $y_1 \in (\tilde{y}, \infty)$ ,  $\sigma \in (0, \frac{1}{2})$  and  $\beta^\sigma \in [1 - 2\sigma, 1]$  or
- v)  $y_1 \in (\tilde{y}, y^\circ)$ ,  $\sigma \in (0, \frac{1}{2})$  and  $\beta^\sigma \in (0, 1 - 2\sigma)$ .

On the other hand,  $p^*$  is locally indeterminate if the following conditions hold:

$$y_1 \in (y^\circ, \infty), \quad \sigma \in \left(0, \frac{1}{2}\right) \quad \text{and} \quad \beta^\sigma \in (0, 1 - 2\sigma),$$

while  $p^*$  is nonhyperbolic when  $y_1 = y^\circ$ .<sup>17</sup> The thresholds of  $y_1$  are equal to

$$\bar{y} \equiv \beta^{-\sigma} y_2, \quad \tilde{y} \equiv \frac{(1 + \sigma\beta^\sigma)y_2}{(1 - \sigma)\beta^\sigma} \quad \text{and} \quad y^\circ \equiv \frac{\beta^\sigma(1 - 2\sigma) - 1}{(\beta^\sigma + 2\sigma - 1)} y_2 \beta^{-\sigma}.$$

Proposition 3 shows the conditions under which the steady state price of each good is either determinate or indeterminate, since the dynamics of the two prices are independent. However, the determinacy or indeterminacy of the stationary monetary equilibrium will be the consequence of the behavior of both price levels. Following Proposition 3, it is straightforward to identify situations in which the stationary monetary equilibrium is locally indeterminate.

**Result 1** *The monetary steady state is locally indeterminate if the following conditions hold:  $\sigma \in (0, \frac{1}{2})$ ,  $\beta^\sigma \in (0, 1 - 2\sigma)$  and  $y_1 \in (y^\circ, \infty)$  for at least one of the two countries.*

<sup>16</sup>More precisely, we should notice that  $F(p_t, p_{t+1}) = 0$  is an (implicit) function when  $\sigma > 1$  and a correspondence when  $\sigma < 1$  (See the Supplementary Material). This finding is well known in the literature (see Grandmont (1985, p. 1007)). In studying the local dynamics when  $\sigma < 1$ , we will characterise a subset of the existing equilibria.

<sup>17</sup>A steady state  $x^*$  of a scalar difference equation  $x_{t+1} = F(x_t)$  is non-hyperbolic if  $|F'(x^*)| = 1$ . If a steady state is non-hyperbolic then the local stable manifold theorem (e.g. Kuznetsov [14], Theorem 2.3 page 50) does not hold, i.e. the local dynamics around  $x^*$  is no more a “good approximation” of the global dynamics.



First of all, the indeterminacy of the monetary steady state depends on a sufficiently low elasticity of substitution and discount factor. The importance of a strong income effect for indeterminacy to occur is well known in the overlapping-generations literature.<sup>18</sup>

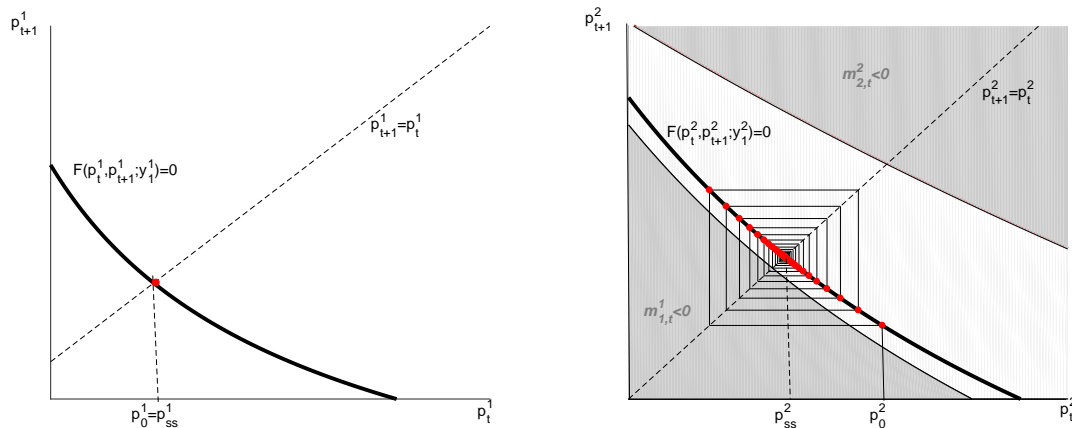


Figure 1: An example of a locally indeterminate monetary steady state

The novelty of this paper is that the indeterminacy of the monetary steady state is also tied to the relative endowments of the two countries. In fact, this is the main source of heterogeneity in the model, as preferences are identical across countries. For indeterminacy to emerge, the endowment of the young must be higher than the bifurcation point  $y^\circ$  for at least one country.

For instance, let us consider the following example where the two agents have the same endowment when old. This implies that the two countries have the same  $y^\circ$  as a threshold. Suppose that country 1 has an endowment when young below the threshold and country 2 above the threshold. We can then construct an equilibrium around the monetary steady state where  $p_0^1 = p^{1*}$  (see left side of Figure 1) while any arbitrary  $p_0^2$  sufficiently close to  $p^{2*}$  will converge to  $p^{2*}$  according to Proposition 3 (see right side of Figure 1). To draw the picture, we have chosen parameter values as follows:  $\sigma = 0.25$ ,  $\beta^\sigma = 0.4729$ ,  $M^1 = M^2 = 1$ ,  $y_2^1 = y_2^2 = 1$ , which implies that  $y^\circ = 14$ . Consistently with what described before, the young of country 1 is endowed with  $y_1^1 = 19 > y^\circ$ , while the young of country 2 with  $y_1^2 = 13 < y^\circ$ . Since there is local indeterminacy in the price  $p_t^2$ , we have also arbitrarily chosen the initial price level so that it is 1% higher than the steady state value. We need also to check that money demands are strictly positive, i.e. the inequality constraints (10)-(11) hold. Observe that these two inequality constraints can be represented as regions in the space  $(p_t^2, p_{t+1}^2)$  given the path of the other price

<sup>18</sup>See e.g. Woodford (1984).

( $p_t^1 = p^{1*}$  for every  $t$ ) and the path of the nominal exchange rate which can be derived from (22) once the paths for the two prices are known. Therefore, we check numerically that the price sequence  $\{p_t^2\}_{t=0}^T$  lies within the region where both money demands are positive, i.e. not in the grey regions of Figure 1.

We can now comment on the dynamics of the exchange rate. As explained before, equation (22) pins down the path of the nominal exchange rate once the paths of the two prices are known:

$$e_t = \left( \frac{M^1 + p^{1*} y_2^1}{M^2 + p_{t+1}^2 y_2^2} \right)^{\frac{1}{\sigma}} \left( \frac{p_{t+1}^2}{p^{1*}} \right)^{\frac{1-\sigma}{\sigma}} \quad (29)$$

For any given  $p_0^2$ , equation (27) will pin down a value for  $p_1^2$ , which will then determine the nominal exchange rate at time 0 through equation (29). Any given  $p_0^2$  will give rise to a different path for the price of good 2 and hence a different path for the nominal exchange rate. Once  $p_t^2$  converges to  $p^{2*}$ , so will the nominal exchange rate converge to its long-run value.

The exchange rate is constant only under very special circumstances.

**Result 2** *In the neighbourhood of the monetary steady state, the nominal exchange rate is constant only when the following conditions hold simultaneously:  $p_0^1 = p_0^2$ ,  $y_a^1 = y_a^2$  for every  $a$  and  $M^1 = M^2$ .*

Equation (22) shows that even if agents coordinate on an equilibrium where  $p_0^1 = p_0^2$ , the exchange rate is not necessarily constant. In fact, equation (27) shows that, given the same  $p_0$ ,  $p_1$  will not be identical across countries if some of the fundamentals of the two economies are different. Only if all dimensions of heterogeneity across countries are completely shut down, the exchange rate is constant and equal to the steady state value (which, in this case, would be equal to 1).

## 5 The “exchange rate disconnect”

The aim of this section is to show that, although the nominal exchange rate is pinned down by the fundamentals of the economy at the monetary steady state, persistent fluctuations of the nominal exchange rate around its long-run value can arise in the absence of shocks to the fundamentals of the economy. Hence, our model can help to explain why there seems to be a disconnect between the exchange rate and the fundamentals, especially at shorter horizons.

In particular, we will examine the existence of endogenous periodic cycles in the nominal exchange rate in section 5.1, while we will provide conditions for the existence of stationary sunspot equilibria in section 5.2.

## 5.1 Endogenous periodic cycles

In this section, we investigate the dynamics of the economy when the steady state  $p^*$  is nonhyperbolic, i.e. when

$$y_1 = y^\circ, \quad \sigma \in \left(0, \frac{1}{2}\right) \quad \text{and} \quad \beta^\sigma \in (0, 1 - 2\sigma)$$

and therefore  $\left. \frac{dp_{t+1}}{dp_t} \right|_{p_t=p^*} = -1$  as shown in Proposition 3. The bifurcation under investigation is a flip bifurcation which may induce two-period cycles in the price level. In addition, we have chosen  $y_1$  as the bifurcation parameter. For this reason we refer, from now on, to the scalar difference equation (27) as  $F(p_t, p_{t+1}; y_1) = 0$ .

A preliminary step of our analysis consists in using the Implicit Function Theorem and observe that, under the assumption  $F_{p_{t+1}}(p^*; y_1) \neq 0$ , i.e.  $y_1 \neq \tilde{y}$ , there exist two positive constants,  $a$  and  $b$ , and a function  $f : \mathcal{I} \rightarrow \mathcal{I}$  with  $\mathcal{I} \equiv (p^* - a, p^* + b)$  such that

$$F(p_t, p_{t+1}; y_1) = 0 \quad \text{in} \quad \mathcal{I} \times \mathcal{I} \quad \Leftrightarrow \quad p_{t+1} = f(p_t; y_1)$$

The strategy to prove that a period-two cycle emerges through a flip bifurcation consists in showing that the dynamics of the difference equation  $p_{t+1} = f(p_t; y_1)$  is topologically equivalent near  $p_t = p^*$  to the dynamics of one of the scalar difference equations  $x_{t+1} = -(1 + \alpha)x_t \pm x_t^3$  near the steady state  $x^* = 0$ . In fact, both these difference equations have a dynamic behavior characterized by a flip bifurcation and, therefore, a period-two cycle in a left or right neighbourhood of  $\alpha = 0$ . If topologically equivalent to one of these two difference equations,  $p_{t+1} = f(p_t; y_1)$  will have the same dynamic behavior in a left or right neighbourhood of  $y_1 = y^\circ$ .

In the following, we will adapt Theorem 4.3 and 4.4 in Kuznetsov [14] to our framework in order to provide conditions under which the two dynamics are topologically equivalent near their respective steady state.

**Proposition 4** *Consider the one-dimensional dynamical system  $p_{t+1} = f(p_t; y_1)$  having a nonhyperbolic steady state  $p_t = p^*$  at  $y_1 = y^\circ$ . Then the following two conditions are generically satisfied*

- i)  $\frac{1}{2}(f_{p_t p_t}(p^*; y^\circ))^2 + \frac{1}{3}f_{p_t p_t p_t}(p^*; y^\circ) \neq 0$ ;
- ii)  $f_{p_t y_1}(p^*; y^\circ) \neq 0$ .

For this reason,  $p_{t+1} = f(p_t; y_1)$  and one of the difference equations  $x_{t+1} = -(1 + \alpha)x_t \pm x_t^3$  are locally topologically equivalent near their respective steady states.

**Theorem 1** *The one-dimensional dynamical system  $p_{t+1} = f(p_t; y_1)$  has a flip bifurcation if  $y_1 = y^\circ$  and two-period cycles emerge for  $y_1$  in a right or left neighborhood of  $y^\circ$ .*

Theorem 1 shows that each price can exhibit cyclical behaviour for an open set of the endowment. Hence, we can conclude that the conditions for the existence of endogenous business cycles in our framework are the following.

**Result 3** *Two-period cycles around the monetary steady state exist if the following conditions hold:  $\sigma \in (0, \frac{1}{2})$ ,  $\beta^\sigma \in (0, 1 - 2\sigma)$  and  $y_1$  belongs to the open set around  $y^\circ$  as specified by Theorem 1 for at least one of the two countries.*

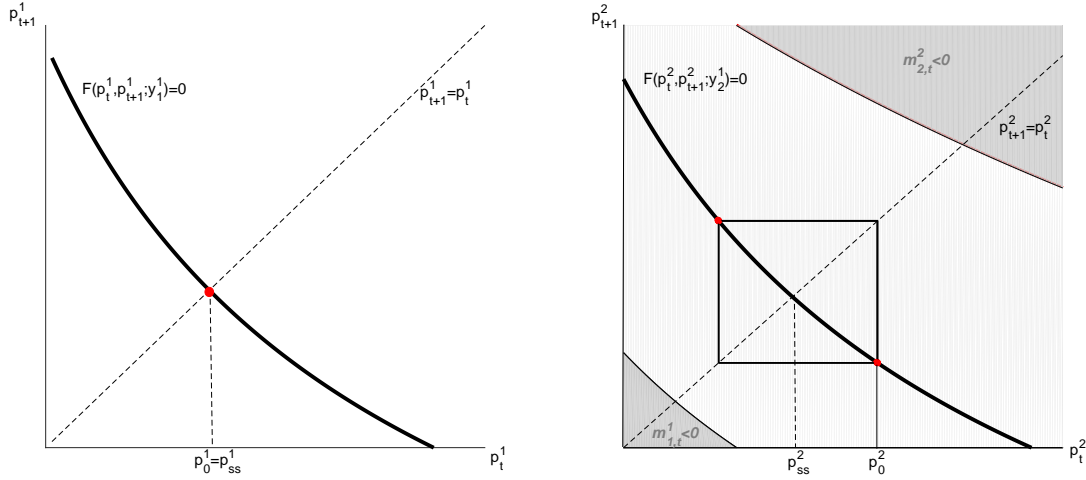


Figure 2: An example of a periodic cycle

Figure 2 shows an example where while  $p^{1*}$  is locally determinate and the initial price level is exactly the steady state,  $y_1^2$  is such that the price of the good produced in country 2 exhibits cyclical behaviour. Crucially, the initial price level in country 2 is not determinate as in country 1 and it could be either of the two values which arise in the cycle. We consider the same parameter values as in the indeterminacy example, except for the value of the endowment of good 2. We check numerically that the money demands remain positive, doing the same procedure described previously for Figure 1. As it can be appreciated by looking at Figure 2, the dynamics of the price of good 2 lies outside the grey regions where the money demands are negative.

Since  $p_0^2 > p^{2*}$ , then we can define  $p_H^2 = p_t^2$  for  $t = 0, 2, 4, \dots$  and  $p_L^2 = p_t^2$  for  $t = 1, 3, 5, \dots$ . The nominal exchange rate also exhibit cyclical behaviour, since

its dynamics is driven by the cycle in the price of good 2. More specifically:

$$\begin{aligned}
 e_L &= \left( \frac{M^1 + p^{1*} y_2^1}{M^2 + p_L^2 y_2^2} \right)^{\frac{1}{\sigma}} \left( \frac{p_L^2}{p^{1*}} \right)^{\frac{1-\sigma}{\sigma}} & t = 0, 2, 4, \dots \\
 e_H &= \left( \frac{M^1 + p^{1*} y_2^1}{M^2 + p_H^2 y_2^2} \right)^{\frac{1}{\sigma}} \left( \frac{p_H^2}{p^{1*}} \right)^{\frac{1-\sigma}{\sigma}} & t = 1, 3, 5, \dots
 \end{aligned}$$

It can be observed that  $e$  is high when the expected price of good 2 is high, which implies that an appreciation of currency 2 is associated with an expected inflation in the domestic good (which means that in the current period there is deflation). However, it is important to stress that this is just an example. As we have observed in Corollary 1, the effect of price changes on the nominal exchange rate is generally ambiguous when  $\sigma < 1$ . In fact, our paper can capture the fact that the exchange rate tends also to be disconnected from other macroeconomic variables such as inflation rates, as well as from the fundamentals of the underlying economies.

When  $p^{1*}$  is locally indeterminate, we can construct instead an equilibrium where  $p_0^1 \neq p^{1*}$ . In this case, the dynamics of the nominal exchange rate is more complex as it depends on whether  $p_0^1$  and  $p_0^2$  start above or below the steady state, as well as on the strength of the wealth channel relatively to the price channel. The interesting thing is that the economy (including the nominal exchange rate) converges to a two-period cycle as  $p^2$  converges to  $p^{2*}$ .

Finally, in the case where both prices can exhibit cyclical behaviour, there also exist periodic equilibria around the monetary steady state.

## 5.2 Stationary sunspot equilibria

In the previous section, we have shown that there exist deterministic fluctuations of the nominal exchange rate around its long-run value which are completely unrelated to fluctuations of the fundamentals of the economy.

In this section, we investigate whether fluctuations of the nominal exchange rate around the monetary steady state can be generated by agents' beliefs that prices are stochastic. In a one-currency one-good OLG economy, Azariadis (1981) has explored the existence of stationary sunspot equilibria, where beliefs (and, since they are self-fulfilling, equilibrium prices) follow a simple two-state Markov process. In particular, he showed that gross complementarity between consumption and leisure and the local indeterminacy of the monetary steady state are sufficient conditions. Our next aim is to explore the existence of stationary sunspot equilibria in this framework.

Let us assume that agents believe that the world economy can be in two states of nature:  $S = \{a, b\}$ . Since we have two agents born in each period, beliefs can potentially be different between them. Let  $\Pi_h$  be a stationary

transition probability matrix, where the element  $\pi_h(ij)$  is the probability that agent  $h$  assigns to state  $j$  tomorrow when today's state is  $i$ :

$$\Pi_h = \begin{pmatrix} \pi_h(aa) & \pi_h(ab) \\ \pi_h(ba) & \pi_h(bb) \end{pmatrix} \quad (30)$$

where  $\sum_{s'} \pi_h(ss') = 1$ .

The maximisation problem that each agent faces is:

$$\max_{\mathbf{c}_{1h}(s), \mathbf{c}_{2h}(ss'), \mathbf{m}_h(s)} \frac{c_{1h}^1(s)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \frac{c_{1h}^2(s)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \beta \sum_{s'} \pi_h(ss') \left[ \frac{c_{2h}^1(ss')^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \frac{c_{2h}^2(ss')^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \right] \quad (31)$$

subject to the following constraints<sup>19</sup>:

$$\begin{aligned} p^1(s)c_{1h}^1(s) + p^2(s)e(s)c_{1h}^2(s) + m_h^1(s) + e(s)m_h^2(s) &= w_h(s) \\ p^1(s')c_{2h}^1(ss') &= m_h^1(s) \\ p^2(s')c_{2h}^2(ss') &= m_h^2(s) \end{aligned}$$

Let us define  $\tilde{m}_h^\ell(s) \equiv \frac{m_h^\ell(s)}{p^\ell(s)}$  and use the definition of the terms of trade of the economy ( $\varepsilon(s) \equiv \frac{p^2(s)e(s)}{p^1(s)}$ ) to rewrite the above budget constraints as follows:

$$c_{1h}^1(s) + \varepsilon(s)c_{1h}^2(s) + \tilde{m}_h^1(s) + \varepsilon(s)\tilde{m}_h^2(s) = \tilde{w}_h(s) \quad (32)$$

$$c_{2h}^1(ss') = \tilde{m}_h^1(s) \frac{p^1(s)}{p^1(s')} \quad (33)$$

$$c_{2h}^2(ss') = \tilde{m}_h^2(s) \frac{p^2(s)}{p^2(s')} \quad (34)$$

where  $\tilde{w}_1(s) = y_1^1$  and  $\tilde{w}_2(s) = \varepsilon(s)y_1^2$ .

In the Appendix, we derive the first-order conditions of the maximisation problem and show that it involves agent  $h$  choosing  $\tilde{m}_h^1(s)$  and  $\tilde{m}_h^2(s)$  which solve the following two equations:

$$\left( \frac{\tilde{w}_h(s) - \tilde{m}_h^1(s) - \varepsilon(s)\tilde{m}_h^2(s)}{1 + \varepsilon(s)^{1-\sigma}} \right)^{-\frac{1}{\sigma}} = \beta \sum_{s'} \pi_h(ss') \left( \tilde{m}_h^1(s) \frac{p^1(s)}{p^1(s')} \right)^{-\frac{1}{\sigma}} \frac{p^1(s)}{p^1(s')} \quad (35)$$

$$\left( \frac{\tilde{w}_h(s) - \tilde{m}_h^1(s) - \varepsilon(s)\tilde{m}_h^2(s)}{1 + \varepsilon(s)^{1-\sigma}} \right)^{-\frac{1}{\sigma}} = \frac{\beta}{\varepsilon(s)} \sum_{s'} \pi_h(ss') \left( \tilde{m}_h^2(s) \frac{p^2(s)}{p^2(s')} \right)^{-\frac{1}{\sigma}} \frac{p^2(s)}{p^2(s')} \quad (36)$$

We now introduce a definition of stationary sunspot equilibrium.

**Definition 3** *Given  $\Pi_h$ , a stationary sunspot equilibrium is a system of prices  $\varepsilon(s) \in \mathbb{R}_{++}$  and  $\mathbf{p}(s) \in \mathbb{R}_{++}^2$ , consumption allocations  $\mathbf{c}_{1h}(s) \in \mathbb{R}_{++}$ ,  $\mathbf{c}_{2h}(ss') \in \mathbb{R}_{++}$  and portfolio allocations  $\tilde{\mathbf{m}}_h(s) \in \mathbb{R}_{++}$  such that:*

- (i) *Agent  $h$  maximizes his utility function (31) subject to the budget constraints (32), (33) and (34) in every  $s$*

<sup>19</sup>For simplicity, we assume that the endowment of the old is zero.

$$(ii) \sum_h (c_{1h}^\ell(s) + c_{2h}^\ell(s's)) = y_1^\ell \quad \forall s, s' \text{ and } \forall \ell$$

$$(iii) \sum_h \tilde{m}_h^\ell(s) = \tilde{m}^\ell(s) \quad \forall s, \ell$$

(iv) at least one of the following holds:  $p^1(a) \neq p^1(b)$ ,  $p^2(a) \neq p^2(b)$ ,  $\varepsilon(a) \neq \varepsilon(b)$ .

$$(v) 0 < \pi_h(aa), \pi_h(bb) < 1$$

As it is standard in the literature, our definition excludes the degenerate cases where the economy either ends up in one state of nature ( $\pi_h(aa) = 1$  or  $\pi_h(bb) = 1$ ) or in a two-period cycle (when  $\pi_h(ab) = 1$  and  $\pi_h(ba) = 1$ )<sup>20</sup>. In our two-country world, three prices can potentially fluctuate: the nominal prices of the two goods  $p^1(s)$  and  $p^2(s)$  and the terms of trade  $\varepsilon(s)$ . Since  $e(s) \equiv \frac{\varepsilon(s)p^1(s)}{p^2(s)}$ , then the nominal exchange rate can also fluctuate as a consequence.

Since the general case is quite cumbersome to deal with, we make the following assumptions on agents' beliefs:

**Assumption 1** *Agents' beliefs are specified as follows:*

$$\begin{aligned} \tilde{m}^1(a) &= \tilde{m}^{1*} + zn \\ \tilde{m}^1(b) &= \tilde{m}^{1*} + zx & \Rightarrow & \quad p^1(a) \neq p^1(b) \\ \tilde{m}^2(s) &= \tilde{m}^{2*} & \Rightarrow & \quad p^2(s) = p^{2*} \\ \varepsilon(s) &= \varepsilon^* \\ \pi_h(ss') &= \pi(ss') \quad \forall s, s' \end{aligned}$$

where  $z$ ,  $n$  and  $x$  are non-zero numbers.

In other words, agents believe that only the price of good 1 is subject to random fluctuations while the other prices are those prevailing at the monetary steady state. In fact, Assumption 1 implies that the price of good 1 fluctuates as follows:

$$p^1(a) = \frac{M^1}{\tilde{m}^1(a)} = \frac{M^1}{\tilde{m}^{1*} + zn} \quad p^1(b) = \frac{M^1}{\tilde{m}^1(b)} = \frac{M^2}{\tilde{m}^{1*} + zn}$$

It is also easy to see that fluctuations in the price of one good is sufficient to generate fluctuations of the nominal exchange rate:

$$\begin{aligned} e(a) &= \frac{\varepsilon^*}{p^{2*}(\tilde{m}^{1*} + zn)} \\ e(b) &= \frac{\varepsilon^*}{p^{2*}(\tilde{m}^{1*} + zx)} \end{aligned}$$

Finally, we assume that agents share the same beliefs about the uncertainty affecting the world economy.

<sup>20</sup>See Azariadis et al. (1986) for a discussion and for an investigation of the connections between stationary sunspot equilibria and two-period cycles.

**Proposition 5** *Under Assumption 1, stationary sunspot equilibria exist as long as  $p^{1*}$  is locally indeterminate.*

Proposition 5 shows that, in our open economy setting, it is possible to construct stationary sunspot equilibria in the neighborhood of the monetary steady state. If agents believe that the price of one good follows a first-order Markov process, then the nominal exchange rate would also fluctuate accordingly as beliefs are self-fulfilling. When the price of good 1 goes up (down), currency 1's purchasing power falls (increases) hence it depreciates (appreciates). Fluctuations of the nominal exchange rate are purely driven by agents' beliefs that the real interest rate in one country changes over time with no relationship to changes in the fundamentals of the economy<sup>21</sup>.

As in Azariadis (1981), a sufficient condition for stationary sunspot equilibria to arise is that the steady state price of the good under consideration is locally stable or indeterminate. However, the literature has pointed out that neither the indeterminacy of the steady state nor the complementarity between the goods are necessary conditions for the existence of stationary sunspot equilibria<sup>22</sup>. While the objective of this paper is to show that fluctuations of the nominal exchange rate purely driven by agents' beliefs exist, there is no reason to believe that our example is somewhat unique or driven by our assumptions e.g. on preferences.

Let us conclude this section with some welfare considerations. Manuelli et al. (1990) have also looked at the issue of exchange rate volatility in an OLG setting. In a one-good two-currencies model with random shocks to the endowments, they show that for each equilibrium allocation it is possible to construct arbitrary paths of the nominal exchange rate. Hence, the exchange rate volatility generated by their model has no welfare implications. In this paper, there are no shocks to fundamentals and yet the exchange rate can exhibit some degree of volatility as a result of self-fulfilling beliefs. Sunspot equilibria are known to be Pareto inefficient from the point of view of ex-ante expected utility (Peck, 1988). Hence, we can make the argument that the fluctuations of the nominal exchange rate which arise in this setting are inefficient.

## 6 Conclusion

We have provided a theory of exchange rate determination where the value of the nominal exchange rate at any given period is pinned down by the expected purchasing power of the two currencies in units of the respective domestic

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<sup>21</sup>We have constructed sunspot equilibria in the case where the price of good 1 fluctuates, but the same could be done for good 2.

<sup>22</sup>See e.g. Azariadis (1981) and Woodford (1984).



goods. At the monetary steady state, the prices of the two goods are a function of the fundamentals of the respective economies, hence the nominal exchange rate is itself a function of the fundamentals. Empirical evidence indeed suggests that the link between exchange rates and fundamentals is stronger at longer horizons.

Under some conditions on agents' preferences and endowments, the two prices can be locally indeterminate around the monetary steady state. It is enough that one of the two prices is indeterminate for the existence of a continuum of equilibrium paths of the nominal exchange rate which all converge to the monetary steady state. The path that will prevail will depend on the initial prices of the two goods and this gives rise to a different equilibrium allocation. On the other hand, the fundamentals of the economy are assumed to be constant. Our framework can then explain why the econometrician struggles to find a correlation between exchange rates and fundamentals at shorter horizons.

We also show that different types of fluctuations of the nominal exchange rate around its long-run value can emerge. Firstly, we show the existence of deterministic cycles in the nominal exchange rates for an open set of the endowments of the economy. Secondly, we construct stationary sunspot equilibria where random fluctuations of the nominal exchange rate arise as a result of self-fulfilling beliefs. For instance, if we observe that a currency appreciates but the fundamentals of the underlying economy have not changed, it could be because agents believe that the purchasing power of the currency goes up. Therefore, "the exchange rate disconnect" puzzle can be a simple consequence of people's "animal spirits".

Finally, it is worth to spend some words on the conditions under which these types of fluctuations take place. Our paper has established that some of the sufficient conditions for sunspots and cycles in the nominal exchange rate to occur are the same as in one-good, one-currency economies: strong income effects and low values of the discount factor. However, the literature has also shown that these are not necessary conditions for either deterministic cycles or sunspot equilibria to arise in overlapping generations models. Hence, there is reason to believe that other examples of economies displaying cycles or sunspot behaviour in the nominal exchange rate can be constructed. We leave such attempts to future research.

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## 7 Appendix

### 7.1 Appendix A: First-order conditions

#### 7.1.1 Deterministic economy

The first-order conditions are:

$$c_{1h,t}^1 : c_{1h,t}^1{}^{-\frac{1}{\sigma}} = \lambda_{1h,t} p_t^1 \quad (37)$$

$$c_{1h,t}^2 : c_{1h,t}^2{}^{-\frac{1}{\sigma}} = \lambda_{1h,t} p_t^2 e_t \quad (38)$$

$$c_{2h,t+1}^1 : \beta c_{2h,t+1}^1{}^{-\frac{1}{\sigma}} = \lambda_{2h,t+1} p_{t+1}^1 \quad (39)$$

$$c_{2h,t+1}^2 : \beta c_{2h,t+1}^2{}^{-\frac{1}{\sigma}} = \lambda_{2h,t+1} p_{t+1}^2 \quad (40)$$

$$m_{h,t}^1 : -\lambda_{1h,t} + \lambda_{2h,t+1}^1 = 0 \quad (41)$$

$$m_{h,t}^2 : -e_t \lambda_{1h,t} + \lambda_{2h,t+1}^2 = 0 \quad (42)$$

$$\lambda_{1h,t} : w_{1h,t} - m_{h,t}^1 - e_t m_{h,t}^2 - p_t^1 c_{1h,t}^1 - p_t^2 e_t c_{1h,t}^2 = 0 \quad (43)$$

$$\lambda_{2h,t+1}^1 : m_{h,t}^1 - p_{t+1}^1 (c_{2h,t+1}^1 - y_{2h}^1) = 0 \quad (44)$$

$$\lambda_{2h,t+1}^2 : m_{h,t}^2 - p_{t+1}^2 (c_{2h,t+1}^2 - y_{2h}^2) = 0 \quad (45)$$

#### 7.1.2 Stochastic economy

In the sunspot economy, the first-order conditions of the maximisation problem are:

$$c_{1h}^1(s) : c_{1h}^1(s) {}^{-\frac{1}{\sigma}} = \lambda_{1h}(s) \quad (46)$$

$$c_{1h}^2(s) : c_{1h}^2(s) {}^{-\frac{1}{\sigma}} = \lambda_{1h}(s) \varepsilon(s) \quad (47)$$

$$c_{2h}^1(ss') : \beta \pi_h(ss') c_{2h}^1(ss') {}^{-\frac{1}{\sigma}} = \lambda_{2h}^1(ss') \quad (48)$$

$$c_{2h}^2(ss') : \beta \pi_h(ss') c_{2h}^2(ss') {}^{-\frac{1}{\sigma}} = \lambda_{2h}^2(ss') \quad (49)$$

$$\tilde{m}_h^1(s) : -\lambda_{1h}(s) + \sum_{s'} \lambda_{2h}^1(ss') \frac{p^1(s)}{p^1(s')} = 0 \quad (50)$$

$$\tilde{m}_h^2(s) : -\varepsilon(s) \lambda_{1h}(s) + \sum_{s'} \lambda_{2h}^2(ss') \frac{p^2(s)}{p^2(s')} = 0 \quad (51)$$

$$\lambda_{1h}(s) : \tilde{w}_{1h}(s) - \tilde{m}_h^1(s) - \varepsilon(s) \tilde{m}_h^2(s) - c_{1h}^1(s) - \varepsilon(s) c_{1h}^2(s) = 0 \quad (52)$$

$$\lambda_{2h}^1(ss') : \tilde{m}_h^1(s) \frac{p^1(s)}{p^1(s')} - c_{2h}^1(ss') = 0 \quad (53)$$

$$\lambda_{2h}^2(ss') : \tilde{m}_h^2(s) \frac{p^2(s)}{p^2(s')} - c_{2h}^2(ss') = 0 \quad (54)$$

Combine (46) and (47) to obtain:

$$c_{1h}^2(s) = \frac{c_{1h}^1(s)}{\varepsilon(s)^\sigma}$$

Plugging the last equation into (52):

$$c_{1h}^1(s) = \frac{\tilde{w}_{1h}(s) - \tilde{m}_h^1(s) - \varepsilon(s)\tilde{m}_h^2(s)}{1 + \varepsilon(s)^{1-\sigma}} \quad (55)$$

which also implies that:

$$c_{1h}^2(s) = \frac{\tilde{w}_{1h}(s) - \tilde{m}_h^1(s) - \varepsilon(s)\tilde{m}_h^2(s)}{\varepsilon(s)^\sigma(1 + \varepsilon(s)^{1-\sigma})} \quad (56)$$

Using (46)-(49), (53), (54), (55) and (56), the two first-order conditions for the real money balances can be rewritten as:

$$\begin{aligned} \tilde{m}_h^1(s) &: \left( \frac{\tilde{w}_{1h}(s) - \tilde{m}_h^1(s) - \varepsilon(s)\tilde{m}_h^2(s)}{1 + \varepsilon(s)^{1-\sigma}} \right)^{-\frac{1}{\sigma}} = \beta \sum_{s'} \pi_h(ss') \left( \tilde{m}_h^1(s) \frac{p^1(s)}{p^1(s')} \right)^{-\frac{1}{\sigma}} \frac{p^1(s)}{p^1(s')} \\ \tilde{m}_h^2(s) &: \left( \frac{\tilde{w}_{1h}(s) - \tilde{m}_h^1(s) - \varepsilon(s)\tilde{m}_h^2(s)}{1 + \varepsilon(s)^{1-\sigma}} \right)^{-\frac{1}{\sigma}} = \frac{\beta}{\varepsilon(s)} \sum_{s'} \pi_h(ss') \left( \tilde{m}_h^2(s) \frac{p^2(s)}{p^2(s')} \right)^{-\frac{1}{\sigma}} \frac{p^2(s)}{p^2(s')} \end{aligned}$$

## 7.2 Appendix B: Proofs

**Proof of Proposition 1.** Using the demand functions (13) and (14), we can write the market clearing equations for good 1 at time  $t$  as follows:

$$\frac{1}{p_t^{1\sigma}} \sum_h \frac{w_{h,t}}{A_{h,t}} + \frac{\beta^\sigma}{p_t^{1\sigma}} \sum_h \frac{w_{h,t-1}}{A_{h,t-1}} = y_1^1 + y_2^1 \quad (57)$$

Using (3) and (14), the demand for currency 1 at time  $t$  and  $t - 1$  can be written as:

$$m_{h,t}^1 = \beta^\sigma p_{t+1}^{1-\sigma} \frac{w_{h,t}}{A_{h,t}} - p_{t+1}^1 y_{2h}^1 \quad (58)$$

$$m_{h,t-1}^1 = \beta^\sigma p_t^{1-\sigma} \frac{w_{h,t-1}}{A_{h,t-1}} - p_t^1 y_{2h}^1 \quad (59)$$

Summing across  $h$  and assuming that the market for currency 1 clear at any  $t$ , we get upon rearranging:

$$\sum_h \frac{w_{h,t}}{A_{h,t}} = \frac{M^1 + p_{t+1}^1 y_2^1}{\beta^\sigma p_{t+1}^{1-\sigma}} \quad (60)$$

$$\sum_h \frac{w_{h,t-1}}{A_{h,t-1}} = \frac{M^1 + p_t^1 y_2^1}{\beta^\sigma p_t^{1-\sigma}} \quad (61)$$

Finally, plug equations (60) and (61) into (57) and rearrange to obtain (20). Notice that  $p_t^1 y_1^1 > M^1$  holds since the aggregate consumption of the young of good 1 is positive.

Since the maximization problem of the initial old is different, we need to check that this difference equation also holds at  $t = 0$ . The market clearing

condition for good 1 at  $t = 0$  is:

$$\frac{1}{p_0^{1-\sigma}} \sum_h \frac{w_{h,0}}{A_{h,0}} + \frac{M^1}{p_0^1} + y_2^1 = y_1^1 + y_2^1$$

Substituting equation (58) at  $t = 0$ , we get:

$$\frac{1}{p_0^{1-\sigma}} \frac{M^1 + p_1^1 y_2^1}{\beta^\sigma p_1^{1-\sigma}} + \frac{M^1}{p_0^1} = y_1^1$$

which, upon rearranging, satisfies equation (20) at  $t = 0$ .

In a similar way, we can derive the equation describing the dynamics of good 2. Following the same steps as for good 1, the money market clearing equation for currency 2 at  $t$  can be written as:

$$\sum_h \frac{w_{h,t}}{A_{h,t}} = \frac{M^2 + p_{t+1}^2 y_2^2}{\beta^\sigma p_{t+1}^{2-\sigma} e_t^{-\sigma}} \quad (62)$$

Plugging the latter equation at  $t$  and  $t - 1$  into the market clearing equations for good 2, we can derive equation (21).

Finally, the expression for the nominal exchange rate (22) can be found by combining (60) and (62). Once the paths of the nominal price levels are known, equation (22) will pin down the path of nominal exchange rate. Hence, the consumption and the portfolio allocations can be calculated using (93)-(??) and (6)-(9). ■

**Proof of Corollary 1.** Let us calculate the derivative of  $e_t$  with respect to  $p_{t+1}^2$ . After a few steps, it can be shown that:

$$\frac{\partial e_t}{\partial p_{t+1}^2} = \frac{1}{\sigma} \left( \frac{M^1 + p_{t+1}^1 y_2^1}{M^2 + p_{t+1}^2 y_2^2} \right)^{\frac{1}{\sigma}} \left[ \underbrace{\frac{-y_2^2}{M^2 + p_{t+1}^2 y_2^2}}_{\text{wealth channel}} - \underbrace{\frac{\sigma - 1}{p_{t+1}^2}}_{\text{price channel}} \right]$$

It can be immediately observed that  $\frac{\partial e_t}{\partial p_{t+1}^2} < 0$  is always negative for  $\sigma > 1$ . This means that currency 2 depreciates when the expected price of good 2 increases. When  $\sigma < 1$ , it is easy to see that the overall effect is ambiguous:

$$\frac{\partial e_t}{\partial p_{t+1}^2} > 0 \quad \left( \frac{\partial e_t}{\partial p_{t+1}^2} < 0 \right) \quad \text{price effect} > (<) \text{ wealth effect}$$

■

**Proof of Proposition 2.**

From (3), stationarity in consumption implies that

$$p_{t+1}^1 (c_{2h}^1 - y_{2h}^1) = m_{h,t}^1$$

Aggregating and using the market clearing conditions, we have that

$$p_{t+1}^1 \sum_h (c_{2h}^1 - y_{2h}^1) = \sum_h m_{h,t}^1 = M^1$$

Therefore,  $p^1$  must be constant and therefore  $\pi^1 = 0$ . Looking at (4), we can similarly show that  $p^2$  must be constant and therefore  $\pi^2 = 0$ . At constant prices, the system plus the two inequality constraints in Proposition 1 rewrite as (24) and the two inequality constraints in Proposition 2. ■

**Proof of Proposition 3.** We investigate the local dynamics around  $p^*$  by looking at the slope of  $F(p_t, p_{t+1})$  at the steady state  $p^*$ :

$$m \equiv \left. \frac{dp_{t+1}}{dp_t} \right|_{p_t=p^*} = - \frac{(1-\sigma)\beta^\sigma y_2 - (\sigma + \beta^\sigma)\beta^\sigma y_1}{(1 + \sigma\beta^\sigma)y_2 - (1-\sigma)\beta^\sigma y_1} \quad (63)$$

Under the restriction  $y_1 > \bar{y}$  it is easy to show that the numerator is always negative. On the other hand, the denominator is positive if one of the following conditions is satisfied:  $\sigma \geq 1$  or  $\sigma < 1$  and  $y_1 < \tilde{y}$ . On the other hand, if  $\sigma < 1$  and  $y_1 > \tilde{y}$  the denominator is positive. Observe also that  $\tilde{y} > \bar{y}$  when  $\sigma < 1$ . Taking into account this information we are now ready to investigate whether  $|m|$  is lower, equal or greater than one.

We need to distinguish two cases:

**Case 1:**  $\sigma \geq 1$  **OR**  $\sigma < 1$  **and**  $y_1 \in (\tilde{y}, \bar{y})$ : in this case the steady state is locally unstable because  $m > 1$  always. In fact

$$m > 1 \quad \Leftrightarrow \quad y_1 > \bar{y}$$

which is always satisfied from what said above.

**Case 2:**  $\sigma < 1$  **and**  $y_2 \in (\tilde{y}, \infty)$ : in this case  $m < 0$  since both the denominator and numerator of (63) are negative. Therefore, to establish the local determinacy of  $p^*$ , we need to check whether  $m < -1$ . Doing that leads to the following:

$$m < -1 \quad \Leftrightarrow \quad \underbrace{(\beta^\sigma - 2\sigma\beta^\sigma - 1)}_{\equiv \Gamma_1} y_2 < \underbrace{(2\sigma + \beta^\sigma - 1)}_{\equiv \Gamma_2} \beta^\sigma y_1. \quad (64)$$

Now,  $\Gamma_1$  is always negative. It is straightforward to see that it is so when  $1 - 2\sigma < 0$ . It continues to be the case when  $1 - 2\sigma > 0$  because a positive sign would require  $\beta^\sigma > \frac{1}{1-2\sigma} > 1$ .

Looking now at  $\Gamma_2$ , it is clear that  $\Gamma_2 > 0$  always if  $2\sigma - 1 \geq 0$ . In addition,  $\Gamma_2 > 0$  if  $2\sigma - 1 < 0$  and  $\beta^\sigma > 1 - 2\sigma$  while  $\Gamma_2 < 0$  if  $2\sigma - 1 < 0$  and  $\beta^\sigma < 1 - 2\sigma$ .

Summing up the sign conditions just found, we have that:

- a)  $\Gamma_1 < 0$  and  $\Gamma_2 > 0$  if  $\sigma \geq \frac{1}{2}$  or if  $\sigma < \frac{1}{2}$  and  $\beta^\sigma \in (1 - 2\sigma, 1]$ ;
- b)  $\Gamma_1 < 0$  and  $\Gamma_2 < 0$  if  $\sigma < \frac{1}{2}$  and  $\beta^\sigma \in (0, 1 - 2\sigma)$

Clearly, under condition a) and looking at (64) we conclude that  $p^*$  is locally determinate. On the other hand, condition b) joint with (64) lead to conclude that  $p^*$  is locally determinate if

$$y_1 < \frac{\Gamma_1}{\Gamma_2} \beta^{-\sigma} y_2 \equiv y^\circ$$

and indeterminate otherwise. Finally, we need to verify whether  $y^\circ$  is greater or lower than  $\tilde{y}$ . After some computations, it emerges that  $y^\circ > \tilde{y}$  because otherwise  $\beta^{2\sigma} + 2 + 2\beta^\sigma < 0$ . Therefore, we conclude that under condition b)  $p^*$  is locally determinate when  $y_1 \in (\tilde{y}, y^\circ)$  while indeterminate if  $y_2 \in (y^\circ, \infty)$ .

■

**Proof of Proposition 4.** Conditions i) and ii) are the conditions to check for the two difference equations to be topologically equivalent according to Theorem 4.3 in Kuznetsov [14]. More precisely, the conditions in Kuznetsov [14] are adapted to our framework to take into account that  $y^\circ$  is generically different from zero and that the nonhyperbolic steady state of  $p_{t+1} = f(p_t; y^\circ)$  is not zero but rather  $p^*$ .<sup>23</sup>

To check these two conditions we need to compute the derivatives  $f_{p_t y_1}(p^*; y^\circ)$ ,  $f_{p_t p_t}(p^*; y^\circ)$  and  $f_{p_t p_t p_t}(p^*; y^\circ)$ . This can be done by applying several times the implicit function theorem to  $F(p_t, p_{t+1}; y_1) = 0$  in  $\mathcal{I} \times \mathcal{I}$ .

We start with:

$$f_{p_t}(p_t, y_1) = -\frac{F_{p_t}}{F_{p_{t+1}}} = -\frac{(1-\sigma)p_t^{-\sigma}(M + p_{t+1}y_2) - \beta^\sigma p_{t+1}^{1-\sigma}y_1}{p_t^{1-\sigma}y_2 - \beta^\sigma(1-\sigma)p_{t+1}^{-\sigma}(p_t y_1 - M)} \quad (65)$$

from which we have that  $f_{p_t}(p^*, y^\circ) = -1$  as shown in Proposition 3.

To verify **condition ii)** we need to differentiate again with respect to  $y_1$  remembering that  $p_{t+1}$  is also a function of  $y_1$ , i.e.  $p_{t+1} = f(p_t; y_1)$ . Observe that the last term of (65) can be seen as a function  $G(p_t, p_{t+1}; y_1)$  and then differentiating both sides of (65) with respect to  $y_1$  leads to

$$f_{p_t y_1}(p_t, y_1) = G_{y_1}(p_t, p_{t+1}; y_1) + G_{p_{t+1}}(p_t, p_{t+1}; y_1)f_{y_1}(p_t, y_1) \quad (66)$$

where the last component has been obtained using the chain rule. In particular, we find that:

$$\begin{aligned} f_{y_1}(p_t, y_1) &= -\frac{F_{y_1}}{F_{p_{t+1}}} = \frac{\beta^\sigma p_{t+1}^{1-\sigma} p_t}{p_t^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} (p_t y_1 - M)} \\ G_{y_1}(p_t, p_{t+1}; y_1) &= \frac{\beta^\sigma p_{t+1}^{1-\sigma} + G(p_t, p_{t+1}; y_1) \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} p_t}{p_t^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} (p_t y_1 - M)} \\ G_{p_{t+1}}(p_t, p_{t+1}; y_1) &= \frac{(1-\sigma) [p_{t+1}^{-\sigma} \beta^\sigma y_1 - p_t^{-\sigma} y_2 - G(p_t, p_{t+1}; y_1) \sigma \beta^\sigma p_{t+1}^{-1-\sigma} (p_t y_1 - M)]}{p_t^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} (p_t y_1 - M)} \end{aligned}$$

substituting these expressions into (100) and evaluating the result at  $(p^*, y^\circ)$  leads to

$$f_{p_t y_1}(p^*, y^\circ) = \frac{\beta^\sigma p^{*2(1-\sigma)} [(2\sigma - 1)y_2 + (1-\sigma)\beta^\sigma y^\circ]}{[p^{*1-\sigma} y_2 - \beta^\sigma (1-\sigma) p^{*- \sigma} (p^* y^\circ - M)]^2}. \quad (67)$$

<sup>23</sup>The same kind of modification of the conditions in Theorem 4.3 has been implicitly done in Example 4.1 at page 123 by Kuznetsov [14].



Details of this derivation can be found in the Supplementary material. Now we need to show that this cross derivative is generically different from zero. We observe that this is equivalent to show that generically  $(2\sigma-1)y_2+(1-\sigma)\beta^\sigma y^\circ \neq 0$ . To show that, we use the definition of  $y^\circ \equiv \frac{\beta^\sigma(1-2\sigma)-1}{[\beta^\sigma+2\sigma-1]}\beta^\sigma y_2$  and notice that

$$(2\sigma-1)y_2+(1-\sigma)\beta^\sigma y^\circ \neq 0 \quad \Leftrightarrow \quad y_2 \left\{ \frac{(2\sigma-1)(\sigma\beta^\sigma+2\sigma-1)-(1-\sigma)}{[\beta^\sigma+2\sigma-1]} \right\} \neq 0$$

which is equal to zero only for a zero measure subset of  $\{(\sigma, \beta) : \sigma \in (0, \frac{1}{2}) \text{ and } \beta^\sigma \in (0, 1-2\sigma)\}$ . Therefore condition ii) is proven.

We need now to verify **condition i)**. Apply again the chain rule of differentiation on (65) we find that

$$f_{p_t p_t}(p_t, y_1) = G_{p_t}(p_t, p_{t+1}; y_1) + G_{p_{t+1}}(p_t, p_{t+1}; y_1) f_{p_t}(p_t, y_1) \quad (68)$$

where  $G_{p_{t+1}}(p_t, p_{t+1}; y_1)$  and  $f_{p_t}(p_t, y_1)$  were previously found and

$$G_{p_t}(p_t, p_{t+1}; y_1) = \frac{(1-\sigma)[\sigma p_t^{-1-\sigma}(M+p_{t+1}y_2) - G(p_t, p_{t+1}; y_1)(p_t^{-\sigma}y_2 - \beta^\sigma p_{t+1}^{-\sigma}y_1)]}{p_t^{1-\sigma}y_2 - \beta^\sigma(1-\sigma)p_{t+1}^{-\sigma}(p_t y_1 - M)}. \quad (69)$$

Therefore:

$$f_{p_t p_t}(p_t, y_1) = -(1-\sigma)G(p_t, p_{t+1}; y_1) \cdot \frac{2[p_t^{-\sigma}y_2 - \beta^\sigma p_{t+1}^{-\sigma}y_1] + \sigma[\beta^\sigma p_{t+1}^{-1-\sigma}(p_t y_1 - M) + p_t^{-\sigma-1}(M+p_{t+1}y_2)]}{p_t^{1-\sigma}y_2 - \beta^\sigma(1-\sigma)p_{t+1}^{-\sigma}(p_t y_1 - M)} \quad (70)$$

Let us call the last term on the right hand side  $\Gamma(p_t, p_{t+1}; y_1)$ . Then

$$f_{p_t p_t p_t}(p_t, y_1) = -(1-\sigma) \left[ (G_{p_t} + G_{p_{t+1}} f_{p_t}) \Gamma + (\Gamma_{p_t} + \Gamma_{p_{t+1}} f_{p_t}) G \right] \quad (71)$$

Observe that (71) can be rewritten as

$$f_{p_t p_t p_t}(p_t, y_1) = (1-\sigma)^2 G \Gamma^2 - (1-\sigma) (\Gamma_{p_t} + \Gamma_{p_{t+1}} f_{p_t}) G \quad (72)$$

since  $(G_{p_t} + G_{p_{t+1}} f_{p_t}) = f_{p_t p_t} = -(1-\sigma)G\Gamma$ . Therefore condition i) is equivalent to check that:

$$\frac{1}{6}(1-\sigma)^2 \Gamma^{*2} + (1-\sigma)(\Gamma_{p_t}^* - \Gamma_{p_{t+1}}^*) \neq 0 \quad (73)$$

where  $\Gamma^* \equiv \Gamma(p^*; y^0)$ . To show that this relation generically hold, we need to find  $\Gamma^*$  and  $\Gamma_{p_t}^* - \Gamma_{p_{t+1}}^*$ . Substituting  $p_t = p^* = \frac{1+\beta^\sigma}{\beta^\sigma y^\circ - y_2} M$  and  $y_1 = y^\circ$  into  $\Gamma$  leads to:

$$\Gamma^* = \frac{c_0 + c_1 M^2}{p^* y_2 - \beta^\sigma(1-\sigma)(p^* y^\circ - M)} \quad (74)$$

where  $c_0 = 2(y_2 - \beta^\sigma y^\circ) \neq 0$  and  $c_1 = 2 \frac{\beta^\sigma(1+\beta^\sigma)(y^\circ+y_2)}{(\beta^\sigma y^\circ - y_2)^2} \neq 0$ . On the other hand it can be shown after long derivations reported in the Supplementary material, that:

$$\Gamma_{p_t}^* - \Gamma_{p_{t+1}}^* = \frac{p^*}{p^* y_2 - \beta^\sigma(1-\sigma)(p^* y^\circ - M)} \left\{ \frac{2\sigma(1-2\sigma^2 - \beta^\sigma)}{\beta^\sigma + 2\sigma - 1} + \Gamma^*(1-\sigma) \left[ 1 + \beta^\sigma - \beta^\sigma \sigma M \left( \frac{y_0 + y_2}{\beta^\sigma y^\circ - y_2} \right) \right] \right\} \quad (75)$$

Substituting (74) and (75) into (73) allows us to rewrite condition i) as it follows:

$$\frac{(1-\sigma)^2(c_0 + c_1M^2)^2}{6} + \frac{(1-\sigma)(1+\beta^\sigma)M}{\beta^\sigma y^\circ - y_2} \left\{ \frac{2\sigma(1-2\sigma^2-\beta^\sigma)}{\beta^\sigma + 2\sigma - 1} \left[ \frac{1+\beta^\sigma}{\beta^\sigma y^\circ - y_2} y_2 - \beta^\sigma(1-\sigma) \left( \frac{y^\circ + y_2}{\beta^\sigma y^\circ - y_2} \right) \right] M \right. \\ \left. + (c_0 + c_1M^2) \left[ 1 + \beta^\sigma - \beta^\sigma \sigma M \left( \frac{y^\circ + y_2}{\beta^\sigma y^\circ - y_2} \right) \right] \right\} \neq 0$$

A quick inspection reveals that the left hand side is a 4th degree polynomial in  $M$  and therefore it may have, at most, 4 distinctive zeros unless all the coefficients of the polynomial are zero. But this is not true since at least the coefficient of  $M^3$  is different from zero. In particular it is equal to  $\sigma(1-\sigma)\beta^\sigma \frac{(1+\beta^\sigma)^3(y^\circ+y_2)}{(\beta^\sigma y^\circ - y_2)^3}$  which is generically different from zero. Therefore condition i) holds. ■

**Proof of Theorem 1.** The result is a direct consequence of Proposition 4. See also Theorem 4.3 and 4.4 in Kuznetsov [14]. ■

**Proof of Proposition 5.**

We proceed in a number of steps, but the main idea is to adapt the strategy followed by Woodford (1984) to analyse a one-currency, one-good and one-agent economy to our framework.

*Step 1* - Firstly, we explore the implications of Assumption 1.

To start with, consider that preferences are homothetic (see equation (31)). This implies that consumption is a constant fraction of wealth. Therefore, we can write the consumption of the old as follows:

$$c_{2h}^\ell(ss') = f_h^\ell(ss') \tilde{w}_h(s)$$

where  $f_h^\ell(ss')$  is a function of current and future prices. Since preferences, as specified by the discount factor, the elasticity of substitution and the beliefs (i.e. the transition probabilities) are the same across individuals, this must imply that:

$$f_1^\ell(ss') = f_2^\ell(ss')$$

Hence, the aggregate consumption of the old can be rewritten as follows:

$$c_2^\ell(ss') = f^\ell(ss') \sum_h \tilde{w}_h(s)$$

Let us now define agent  $h$ 's share of aggregate consumption as:

$$\theta_h^\ell(ss') \equiv \frac{c_{2h}^\ell(ss')}{c_2^\ell(ss')}$$

Using the above reasoning, it is easy to show that the share of consumption only depends on an agent's share of aggregate wealth:

$$\theta_h^\ell(ss') = \frac{\tilde{w}_h(s)}{\sum_h \tilde{w}_h(s)} \quad \Rightarrow \quad \theta_h^\ell(ss') = \theta_h(s)$$

As a consequence, it is independent of the future state and it is also the same across goods (hence the superscript can be dropped).

We have also assumed that agents believe that the terms of trade are constant. Therefore, agents' wealth does not fluctuate across states which means that each agent's share of aggregate consumption is constant and equal to the steady state value:

$$\varepsilon(s) = \varepsilon^* \quad \Rightarrow \quad \tilde{w}_h(s) = \tilde{w}_h \quad \Rightarrow \quad \theta_h(s) = \theta_h^*$$

Using the budget constraint of the old, we can rewrite  $\theta_h$  as follows:

$$\theta_h^* = \frac{c_{2h}^\ell(ss')}{c_2^\ell(ss')} = \frac{\tilde{m}_h^\ell(s)}{\tilde{m}^\ell(s)} \quad (76)$$

The additional requirement that  $p^2(a) = p^2(b)$  has another implication:

$$\tilde{m}^2(s) = \tilde{m}^{2*} \quad \Rightarrow \quad c_2^2(ss') = c_2^{2*} \quad \Rightarrow \quad c_1^2(s) = c_1^{2*}$$

This is a direct consequence of the aggregated budget constraint of the old for good 2: constant aggregate money balances imply constant aggregate consumption of the old, hence constant aggregate consumption of the young. Given that the share of aggregate consumption for each good is also constant, the individual consumption of good 2 is also constant:

$$\theta_h(s) = \theta_h^* \quad \Rightarrow \quad c_{1h}^2(s) = c_{1h}^{2*} \quad \& \quad c_{2h}^2(ss') = c_{2h}^{2*}$$

Taking into account all of the above plus the assumption of equal beliefs across agents, the first-order conditions (35) and (36) can be simplified as follows:

$$\left( \frac{\tilde{w}_1^* - \tilde{m}_h^1(s) - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}} = \beta \sum_{s'} \pi(ss') \left( \frac{\tilde{m}_h^1(s) p^1(s)}{p^1(s')} \right)^{-\frac{1}{\sigma}} \frac{p^1(s)}{p^1(s')} \quad (77)$$

$$\left( \frac{\tilde{w}_1^* - \tilde{m}_h^1(s) - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}} = \frac{\beta}{\varepsilon^*} (\tilde{m}_h^{2*})^{-\frac{1}{\sigma}} \quad (78)$$

where

$$\tilde{m}_h^\ell(s) = \theta_h^* \tilde{m}^\ell(s) \quad (79)$$

given equation (76).

*Step 2* - Since we have two agents and two states, we have four first-order conditions for the real money balances of currency 1 (equation (77)), which can be rewritten as follows:

$$\frac{\pi(aa)}{1 - \pi(aa)} = \frac{\beta \left( \tilde{m}_h^1(a) \frac{p^1(a)}{p^1(b)} \right)^{-\frac{1}{\sigma}} \frac{p^1(a)}{p^1(b)} - \left( \frac{\tilde{w}_h^* - \tilde{m}_h^1(a) - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}}}{\left( \frac{\tilde{w}_h^* - \tilde{m}_h^1(a) - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}} - \beta (\tilde{m}_h^1(a))^{-\frac{1}{\sigma}}} \quad (80)$$

$$\frac{\pi(bb)}{1 - \pi(bb)} = \frac{\beta \left( \tilde{m}_h^1(b) \frac{p^1(b)}{p^1(a)} \right)^{-\frac{1}{\sigma}} \frac{p^1(b)}{p^1(a)} - \left( \frac{\tilde{w}_h^* - \tilde{m}_h^1(b) - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}}}{\left( \frac{\tilde{w}_h^* - \tilde{m}_h^1(b) - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}} - \beta (\tilde{m}_h^1(b))^{-\frac{1}{\sigma}}} \quad (81)$$

Since equation (79) establishes a relationship between individual and aggregate real money balances, and the stochastic process for  $\tilde{m}^1$  is specified by Assumption 1, the transition probabilities can then be pinned down by the following equations:

$$\frac{\pi(aa)}{1 - \pi(aa)} = \frac{\beta(\theta_h^* \tilde{m}^1(b))^{-\frac{1}{\sigma}} \tilde{m}^1(b) - \left( \frac{\tilde{w}_h^* - \theta_h^* \tilde{m}^1(a) - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}} \tilde{m}^1(a)}{\left( \frac{\tilde{w}_h^* - \theta_h^* \tilde{m}^1(a) - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}} \tilde{m}^1(a) - \beta(\theta_h^* \tilde{m}^1(a))^{-\frac{1}{\sigma}} \tilde{m}^1(a)} \quad (82)$$

$$\frac{\pi(bb)}{1 - \pi(bb)} = \frac{\beta(\theta_h^* \tilde{m}^1(a))^{-\frac{1}{\sigma}} \tilde{m}^1(a) - \left( \frac{\tilde{w}_h^* - \theta_h^* \tilde{m}^1(b) - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}} \tilde{m}^1(b)}{\left( \frac{\tilde{w}_h^* - \theta_h^* \tilde{m}^1(b) - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}} \tilde{m}^1(b) - \beta(\theta_h^* \tilde{m}^1(b))^{-\frac{1}{\sigma}} \tilde{m}^1(b)} \quad (83)$$

where  $\tilde{m}^1(a) = \tilde{m}^{1*} + zw$  and  $\tilde{m}^1(b) = \tilde{m}^{1*} + zx$ .

As Woodford (1984), we take the limit for  $z \rightarrow 0$  of the first-order conditions. If the right-hand sides are positive, then a stationary sunspot equilibrium exists as it can be supported by positive probabilities. But since both the numerators and the denominators tend to zero when the economy approaches the monetary steady state, we apply Hopital's rule.

Let us start with the agents born in state  $a$  and define:

$$\frac{\pi(aa)}{1 - \pi(aa)} \equiv \frac{f_h(z)}{g_h(z)}$$

After a few steps, it can be checked that:

$$\frac{\pi(aa)}{1 - \pi(aa)} \rightarrow \frac{\lim_{z \rightarrow 0} f'_h(z)}{\lim_{z \rightarrow 0} g'_h(z)} = \frac{\frac{x}{n} - S_h^*}{S_h^* - 1} \quad (84)$$

where

$$S_h^* = \frac{\left( \frac{\tilde{w}_h^* - \tilde{m}_h^{1*} - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}} + \frac{\theta_h^* \tilde{m}^{1*}}{\sigma(1 + \varepsilon^{*1-\sigma})} \left( \frac{\tilde{w}_h^* - \tilde{m}_h^{1*} - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\left(\frac{1}{\sigma} + 1\right)}}{\beta(\theta_h^* \tilde{m}^{1*})^{-\frac{1}{\sigma}} - \frac{\beta}{\sigma} (\theta_h^* \tilde{m}^{1*})^{-\frac{1}{\sigma}}}$$

Notice that, at the monetary steady state, the following first-order condition holds (see equation (77) and (79) for the right-hand side):

$$\frac{\tilde{w}_h^* - \tilde{m}_h^{1*} - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} = \frac{\tilde{m}_h^{1*}}{\beta^\sigma} = \frac{\theta_h^* \tilde{m}^{1*}}{\beta^\sigma}$$

We now use some results from section 3, although assuming that  $y_2^\ell = 0$ . Firstly, the aggregate real money balances of good 1 are:

$$\tilde{m}^{1*} = \frac{\beta^\sigma y_1^1}{1 + \beta^\sigma} \quad (85)$$

Hence, we can rewrite the above equation as follows:

$$\frac{\tilde{w}_h^* - \tilde{m}_h^{1*} - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} = \frac{\theta_h^* y_1^1}{1 + \beta^\sigma} = \theta_h^* (y_1^1 - \tilde{m}^{1*}) \quad (86)$$

which shows that the consumption of good 1 when young of agent  $h$  is nothing but a share of the savings of good 1.

Secondly, the share of consumption of agent 1 can be rewritten as:

$$\theta_1^* = \frac{w_1^*}{w_1^* + w_2^*} = \frac{y_1^1}{y_1^1 + \varepsilon^* y_1^2} = \frac{y_1^1}{y_1^1 + \left(\frac{y_1^1}{y_1^2}\right)^{\frac{1}{\sigma}} y_1^2} = \frac{1}{1 + \varepsilon^{*1-\sigma}} \quad (87)$$

Plugging (86) and (87) into  $S_h^*$ , we get that:

$$S_h^* = S^* = \frac{(y_1^1 - \tilde{m}^{1*})^{-\frac{1}{\sigma}} + \frac{\theta_1^* \tilde{m}^{1*}}{\sigma} (y_1^1 - \tilde{m}^{1*})^{-\left(\frac{1}{\sigma}+1\right)}}{\beta(\tilde{m}^{1*})^{-\frac{1}{\sigma}} - \frac{\beta}{\sigma} (\tilde{m}^{1*})^{-\frac{1}{\sigma}}} \quad (88)$$

This confirms that beliefs across the two agents born in state  $a$  must be the same for stationary sunspot equilibria to exist:

$$\frac{\pi(aa)}{1 - \pi(aa)} \rightarrow \frac{\lim_{z \rightarrow 0} f'(z)}{\lim_{z \rightarrow 0} g'(z)} = \frac{\frac{x}{n} - S^*}{S^* - 1} \quad (89)$$

Next, substituting (85) into (88) we obtain:

$$S^* = \frac{\sigma + \beta^\sigma \theta_1^*}{\sigma - 1} \quad (90)$$

Following the same procedure for the agents born in state  $b$ , we obtain:

$$\frac{\pi(bb)}{1 - \pi(bb)} \rightarrow \frac{\lim_{z \rightarrow 0} f'(z)}{\lim_{z \rightarrow 0} g'(z)} = \frac{\frac{n}{x} - S^*}{S^* - 1} \quad (91)$$

Notice that, when  $\theta \rightarrow 1$ , we have that  $S^* \rightarrow m$ , which is the slope of the difference equation for good 1 at the monetary steady state<sup>24</sup>. In fact, when  $y_2 = 0$ ,  $m$  becomes:

$$m = \frac{\sigma + \beta^\sigma}{\sigma - 1}$$

We can now link the existence of sunspot equilibria to the local stability of  $p^{1*}$  and hence the monetary steady state.

Firstly, we should note that the conditions for the indeterminacy of  $p^{1*}$  are slightly different when  $y_2 = 0$ . To start with, let us check the conditions under which  $0 < m < 1$ .  $m > 0$  only if  $\sigma > 1$ . However, when  $\sigma > 1$ ,  $m < 1$  is impossible since that would require that  $1 + \beta^\sigma < 0$ . Let us now consider the case  $-1 < m < 0$ . If  $\sigma < 1$ , then  $m < 0$  always holds. It is easy to check that  $m > -1$  when  $\beta^\sigma < 1 - 2\sigma$ . Since  $\beta > 0$ , we would also require that  $\sigma < \frac{1}{2}$ . Therefore, for indeterminacy to occur the conditions on agents' preferences remain the same when  $y_2 = 0$ . However, we do not have any condition on  $y_1$ .

Finally, it can be verified that for  $\theta_1 \rightarrow 0$ , then  $-1 < S^* < 0$  when  $\sigma < \frac{1}{2}$ . Since  $S^*$  is monotonically decreasing in  $\theta_1^*$ , then  $-1 < S^* < 0$  for any  $\theta_1^*$ . It

<sup>24</sup>See the proof of Proposition 3 when  $y_2 = 0$ .

is then possible to find a continuum of  $n$  and  $x$  such that the two ratios of probabilities are positive.

*Step 3* - To conclude, note that as  $z \rightarrow 0$  the first-order condition for currency 2 (78) becomes:

$$\left( \frac{\tilde{w}_h^* - \tilde{m}_h^{1*} - \varepsilon^* \tilde{m}_h^{2*}}{1 + \varepsilon^{*1-\sigma}} \right)^{-\frac{1}{\sigma}} = \frac{\beta}{\varepsilon^*} (\tilde{m}_h^{2*})^{-\frac{1}{\sigma}}$$

which is the first-order condition for currency 2 at the monetary steady state.

■

## Supplementary material

### The imperfect substitutability of currencies

In this section, we show that the assumption that the old can use each currency only to buy the country-specific good is not enough to guarantee that the two currencies are not perfect substitutes as stores of value. Suppose that the old were allowed to readjust their portfolio. The budget constraints of the old would then be written as follows:

$$\begin{aligned} m_{2h,t+1}^1 + e_{t+1}m_{2h,t+1}^2 &= m_{1h,t}^1 + e_{t+1}m_{1h,t}^2 \\ p_{t+1}^1(c_{2h,t+1}^1 - y_{2h}^1) &= m_{2h,t+1}^1 \\ p_{t+1}^2(c_{2h,t+1}^2 - y_{2h}^2) &= m_{2h,t+1}^2 \end{aligned}$$

But then, the latter constraints can be substituted back into the first constraint:

$$p_{t+1}^1(c_{2h,t+1}^1 - y_{2h}^1) + p_{t+1}^2e_{t+1}(c_{2h,t+1}^2 - y_{2h}^2) = m_{1h,t}^1 + e_{t+1}m_{1h,t}^2$$

We now show that the consolidated budget constraint of the old implies that the two currencies are perfect substitutes as stores of value. Let us define  $\lambda_{1h,t}$  as the Lagrange multiplier associated with the budget constraint of the young (2) and  $\lambda_{2h,t+1}$  as the Lagrange multiplier associated with the above constraint for the old. As we are interested in equilibria where both currencies are demanded in positive quantities, the first-order conditions for the two currencies would be:

$$\begin{aligned} m_{1h,t}^1 &: -\lambda_{1h,t} + \lambda_{2h,t+1} = 0 \\ m_{1h,t}^2 &: -\lambda_{1h,t}e_t + \lambda_{2h,t+1}e_{t+1} = 0 \end{aligned}$$

It is easy to see that the first-order conditions for the two currencies are only compatible with a constant exchange rate ( $e_{t+1} = e_t = e$ ) as in Kareken et al. (1981). As the two first-order conditions for the currencies are identical, the portfolio composition cannot be pinned down.

Hence, it is not enough to assume that each currency can only buy the country-specific good to avoid that the currencies are perfect substitutes as stores of value. The inability of the old to readjust their portfolio is crucial to ensure that the two currencies have different rates of return as stores of value.

### Derivation of the agents' portfolios

Combining (37) and (38), we get that:

$$c_{1h,t}^2 = c_{1h,t}^1 \left( \frac{p_t^1}{p_t^2 e_t} \right)^\sigma \quad (92)$$

Plugging the latter equation into the budget constraint of the young (43), we obtain:

$$c_{1h,t}^1 = \frac{1}{p_t^{1-\sigma}} \frac{w_{1h,t} - m_{h,t}^1 - e_t m_{h,t}^2}{p_t^{1-\sigma} + p_t^2 e_t^{1-\sigma}} \quad (93)$$

Substituting (37) and (39) into (41):

$$\frac{p_t^{1-\frac{1-\sigma}{\sigma}}}{(p_t^1 c_{1h,t}^1)^{\frac{1}{\sigma}}} = \frac{\beta p_{t+1}^{1-\frac{1-\sigma}{\sigma}}}{(p_{t+1}^1 c_{2h,t+1}^1)^{\frac{1}{\sigma}}} \quad (94)$$

Plugging (93) and (44) into (94) and rearranging, we obtain:

$$m_{h,t}^1 = \frac{\beta^\sigma p_{t+1}^{1-\sigma} [w_{1h,t} - e_t m_{h,t}^2] - p_{t+1}^1 y_2^1 (p_t^{1-\sigma} + p_t^2 e_t^{1-\sigma})}{p_t^{1-\sigma} + p_t^2 e_t^{1-\sigma} + \beta^\sigma p_{t+1}^{1-\sigma}} \quad (95)$$

From (41) and (42), notice that:

$$e_t \lambda_{2h,t+1}^1 = \lambda_{2h,t+1}^2$$

Substituting (44) and (45) into the latter equation and then (39) and (40), we get:

$$\frac{e_t^\sigma (m_{h,t}^2 + p_{t+1}^2 y_2^2)}{p_{t+1}^{2-\sigma}} = \frac{m_{h,t}^1 + p_{t+1}^1 y_2^1}{p_{t+1}^{1-\sigma}} \quad (96)$$

Notice that (95) and (96) is a system of two equations where the two endogenous variables are  $m_{h,t}^1$  and  $m_{h,t}^2$ . Solving them simultaneously, we obtain agent  $h$ 's demand functions for the two currencies:

$$m_{h,t}^1 = \frac{\beta^\sigma p_{t+1}^{1-\sigma} [w_{1h,t} + p_{t+1}^2 e_t y_2^2] - p_{t+1}^1 y_2^1 [p_t^{1-\sigma} + p_t^2 e_t^{1-\sigma} + \beta^\sigma (p_{t+1}^2 e_t)^{1-\sigma}]}{p_t^{1-\sigma} + (p_t^2 e_t)^{1-\sigma} + \beta^\sigma p_{t+1}^{1-\sigma} + \beta^\sigma (e_t p_{t+1}^2)^{1-\sigma}} \quad (97)$$

$$m_{h,t}^2 = \frac{\beta^\sigma p_{t+1}^{1-\sigma} e_t^{-\sigma} [w_{1h,t} + p_{t+1}^1 y_2^1] - p_{t+1}^2 y_2^2 [p_t^{1-\sigma} + p_t^2 e_t^{1-\sigma} + \beta^\sigma p_{t+1}^{1-\sigma}]}{p_t^{1-\sigma} + (p_t^2 e_t)^{1-\sigma} + \beta^\sigma p_{t+1}^{1-\sigma} + \beta^\sigma (e_t p_{t+1}^2)^{1-\sigma}} \quad (98)$$

**Conditions under which  $F(p_t, p_{t+1}) = 0$  is a function or a correspondence**

$F(p_t, p_{t+1}) = 0$  is an (implicit) function when  $\sigma > 1$  and a correspondence when  $\sigma < 1$ . To prove this proposition, we begin computing

$$\frac{dp_{t+1}}{dp_t} = - \frac{(1-\sigma) p_t^{-\sigma} (M + p_{t+1} y_2) - \beta^\sigma p_{t+1}^{1-\sigma} y_1}{p_t^{1-\sigma} y_2 - (1-\sigma) \beta^\sigma p_{t+1}^{-\sigma} (p_t y_1 - M)}.$$

Observe that  $\frac{dp_{t+1}}{dp_t}$  is strictly positive for  $\sigma > 1$  since the numerator is negative and the denominator is positive. Therefore,  $F(p_t, p_{t+1}) = 0$  is a monotonically increasing function.



On the other hand,  $F(p_t, p_{t+1}) = 0$  is a correspondence when  $\sigma < 1$ . To see this, we prove that  $p_{t+1} = p^*$  is not the only strictly positive zero of  $F(p^*, p_{t+1}) = 0$ . In particular, we have that

$$F(p^*, p_{t+1}) \equiv p^{*1-\sigma} y_2 p_{t+1} - \beta^\sigma \left( \frac{y_1 + y_2}{\beta^\sigma y_1 - y_2} \right) M p_{t+1}^{1-\sigma} + p^{*1-\sigma} M$$

with  $F(p^*, 0) = p^{*1-\sigma} M > 0$  and  $\lim_{p_{t+1} \rightarrow \infty} F(p^*, p_{t+1}) = +\infty$ . In addition,  $F(p^*, p_{t+1})$  has a unique critical point which is the global minimum. Therefore,  $F(p^*, p_{t+1}) = 0$  has two zeros,  $p_{t+1} = p^*$  and another one to the right of it. We conclude the proof showing that if the steady state respects the necessary condition for holding money

$$y_2 p_{t+1}^\sigma < \beta^\sigma y_1 p^{*\sigma} \quad \Leftrightarrow \quad p_{t+1} < \frac{\beta^\sigma y_1}{y_2} p^* \equiv \tilde{p}$$

then such a condition is also respected by the other zero. In fact, the second zero would not be admissible if  $F(p^*, \tilde{p}) < 0$  but boring calculation shows that

$$F(p^*, \tilde{p}) = \frac{\beta^\sigma y_1}{y_2} \left[ \frac{y_2}{y_1} + \left( \frac{\beta^\sigma y_1}{y_2} \right)^{-\frac{1}{\sigma}} \right] > 0$$

#### Further details on the derivations in Proposition 4

Regarding Condition ii), we show the steps to arrive to

$$f_{p_t y_1}(\hat{p}, y^\circ) = \frac{\beta^\sigma \hat{p}^{2(1-\sigma)} [y_2 - (1-\sigma)y^\circ]}{[\hat{p}^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) \hat{p}^{-\sigma} (\hat{p} y^\circ - M)]^2}. \quad (99)$$

from

$$f_{p_t y_1}(p_t, y_1) = G_{y_1}(p_t, p_{t+1}; y_1) + G_{p_{t+1}}(p_t, p_{t+1}; y_1) f_{y_1}(p_t, y_1) \quad (100)$$

where as previously observed  $f_{p_t} \equiv G(p_t, p_{t+1}; y_1)$  while the derivatives are

$$\begin{aligned} f_{y_1}(p_t, y_1) &= \frac{\beta^\sigma p_{t+1}^{1-\sigma} p_t}{p_t^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} (p_t y_1 - M)} \\ G_{y_1}(p_t, p_{t+1}; y_1) &= \frac{\beta^\sigma p_{t+1}^{1-\sigma} + G(p_t, p_{t+1}; y_1) \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} p_t}{p_t^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} (p_t y_1 - M)} \\ G_{p_{t+1}}(p_t, p_{t+1}; y_1) &= \frac{(1-\sigma) [p_{t+1}^{-\sigma} \beta^\sigma y_1 - p_t^{-\sigma} y_2 - G(p_t, p_{t+1}; y_1) \sigma \beta^\sigma p_{t+1}^{-1-\sigma} (p_t y_1 - M)]}{p_t^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} (p_t y_1 - M)} \end{aligned}$$

The first step is to evaluate these derivatives at the nonhyperbolic steady state  $(\hat{p}, y^\circ)$ ; observe that this means that  $G(\hat{p}; y^\circ) = f_{p_t}(\hat{p}; y^\circ) = -1$ . Then we have that

$$\begin{aligned}
f_{y_1}(\hat{p}, y^\circ) &= -\frac{F_{y_1}}{F_{p_{t+1}}} = \frac{\beta^\sigma \hat{p}^{2-\sigma}}{\hat{p}^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) \hat{p}^{-\sigma} (\hat{p} y^\circ - M)} \\
G_{y_1}(\hat{p}; y^\circ) &= \frac{\beta^\sigma \sigma \hat{p}^{1-\sigma}}{\hat{p}^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) \hat{p}^{-\sigma} (\hat{p} y^\circ - M)} = \sigma f_{y_1}(\hat{p}, y^\circ) \hat{p}^{-1} \\
G_{p_{t+1}}(\hat{p}; y^\circ) &= \frac{(1-\sigma) \hat{p}^{-\sigma} [\beta^\sigma y^\circ - y_2 + \sigma \beta^\sigma \hat{p}^{-1} (\hat{p} y^\circ - M)]}{\hat{p}^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) \hat{p}^{-\sigma} (\hat{p} y^\circ - M)}
\end{aligned}$$

and then

$$\begin{aligned}
f_{p_t y_1}(\hat{p}, y^\circ) &= f_{y_1}(\hat{p}, y^\circ) \left\{ \sigma \hat{p}^{-1} + \frac{(1-\sigma) \hat{p}^{-\sigma} [\beta^\sigma y^\circ - y_2 + \sigma \beta^\sigma \hat{p}^{-1} (\hat{p} y^\circ - M)]}{\hat{p}^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) \hat{p}^{-\sigma} (\hat{p} y^\circ - M)} \right\} \\
&= f_{y_1}(\hat{p}, y^\circ) \left\{ \frac{\hat{p}^{-\sigma} [(1-\sigma) \beta^\sigma y^\circ + (2\sigma - 1) y_2]}{\hat{p}^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) \hat{p}^{-\sigma} (\hat{p} y^\circ - M)} \right\}
\end{aligned}$$

Regarding Condition i), we begin with

$$f_{p_t p_t}(p_t, y_1) = G_{p_t}(p_t, p_{t+1}; y_1) + G_{p_{t+1}}(p_t, p_{t+1}; y_1) f_{p_t}(p_t, y_1) \quad (101)$$

where  $G_{p_{t+1}}(p_t, p_{t+1}; y_1)$  and  $f_{p_t}(p_t, y_1)$  were previously found and

$$G_{p_t}(p_t, p_{t+1}; y_1) = \frac{(1-\sigma) [\sigma p_t^{-1-\sigma} (M + p_{t+1} y_2) - G(p_t, p_{t+1}; y_1) (p_t^{-\sigma} y_2 - \beta^\sigma p_{t+1}^{-\sigma} y_1)]}{p_t^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} (p_t y_1 - M)}. \quad (102)$$

Combining these derivatives lead to

$$\begin{aligned}
f_{p_t p_t}(p_t, y_1) &= \frac{(1-\sigma) [\sigma p_t^{-1-\sigma} (M + p_{t+1} y_2) - G(p_t, p_{t+1}; y_1) (p_t^{-\sigma} y_2 - \beta^\sigma p_{t+1}^{-\sigma} y_1)]}{p_t^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} (p_t y_1 - M)} + \\
&\quad \frac{(1-\sigma) G(p_t, p_{t+1}; y_1) [p_{t+1}^{-\sigma} \beta^\sigma y_1 - p_t^{-\sigma} y_2 - G(p_t, p_{t+1}; y_1) \sigma \beta^\sigma p_{t+1}^{-1-\sigma} (p_t y_1 - M)]}{p_t^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} (p_t y_1 - M)} \\
&= \frac{(1-\sigma) \{ \sigma p_t^{-1-\sigma} (M + p_{t+1} y_2) + G(p_t, p_{t+1}; y_1) [2(p_{t+1}^{-\sigma} \beta^\sigma y_1 - p_t^{-\sigma} y_2) - G(p_t, p_{t+1}; y_1) \sigma \beta^\sigma p_{t+1}^{-1-\sigma} (p_t y_1 - M)] \}}{p_t^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) p_{t+1}^{-\sigma} (p_t y_1 - M)}
\end{aligned}$$

where we observe that  $f_{p_t p_t} \equiv H(p_t, p_{t+1}; y_1)$ . We also find that, at the nonhyperbolic steady state, we have that

$$\begin{aligned}
f_{p_t p_t}(\hat{p}, y^\circ) &= \frac{(1-\sigma) \{ \sigma \hat{p}^{-1-\sigma} (M + \hat{p} y_2) - [2(\hat{p}^{-\sigma} \beta^\sigma y^\circ - \hat{p}^{-\sigma} y_2) + \sigma \beta^\sigma \hat{p}^{-1-\sigma} (\hat{p} y^\circ - M)] \}}{\hat{p}^{1-\sigma} y_2 - \beta^\sigma (1-\sigma) \hat{p}^{-\sigma} (\hat{p} y^\circ - M)} \\
&= \frac{(1-\sigma) \{ \sigma \hat{p}^{-1} (M + \hat{p} y_2) - [2(\beta^\sigma y^\circ - \hat{y}_2) + \sigma \beta^\sigma \hat{p}^{-1} (\hat{p} y^\circ - M)] \}}{\hat{p} y_2 - \beta^\sigma (1-\sigma) (\hat{p} y^\circ - M)} \\
&= \frac{(1-\sigma) \{ (2+\sigma) (y_2 - y^\circ \beta^\sigma) + \sigma \hat{p}^{-1} M (1+\beta^\sigma) \}}{\hat{p} y_2 - \beta^\sigma (1-\sigma) (\hat{p} y^\circ - M)}
\end{aligned}$$

and considering that the nonhyperbolic steady state is  $\hat{p} = \frac{(1+\beta^\sigma)M}{\beta^\sigma y^\circ - y_2}$  the last expression simplifies to

$$f_{p_t p_t}(\hat{p}, y^\circ) = \frac{2(1-\sigma)(y_2 - y^\circ \beta^\sigma)}{\hat{p}y_2 - \beta^\sigma(1-\sigma)(\hat{p}y^\circ - M)} \neq 0 \quad (103)$$

which is always different from zero since  $y_2 - y^\circ \beta^\sigma < 0$  to guarantee the existence of a strictly positive steady state.

We now need to compute

$$f_{p_t p_t p_t}(p_t, y_1) = H_{p_t}(p_t, p_{t+1}; y_1) + H_{p_{t+1}}(p_t, p_{t+1}; y_1) f_{p_t}(p_t, y_1). \quad (104)$$

We find that

$$H_{p_t} = \frac{(1-\sigma)\{-\sigma(1+\sigma)p_t^{-2-\sigma}(M + p_{t+1}y_2) + G_{p_t}[2(p_{t+1}^{-\sigma}\beta^\sigma y_1 - p_t^{-\sigma}y_2) - G\sigma\beta^\sigma p_{t+1}^{-1-\sigma}(p_t y_1 - M)]\}}{p_t^{1-\sigma}y_2 - \beta^\sigma(1-\sigma)p_{t+1}^{-\sigma}(p_t y_1 - M)} + \frac{(1-\sigma)\{G[2\sigma p_t^{-1-\sigma}y_2 - G_{p_t}\sigma\beta^\sigma p_{t+1}^{-1-\sigma}(p_t y_1 - M) - G\sigma\beta^\sigma p_{t+1}^{-1-\sigma}y_1] + H[p_t^{-\sigma}y_2 - \beta^\sigma p_{t+1}^{-\sigma}y_1]\}}{p_t^{1-\sigma}y_2 - \beta^\sigma(1-\sigma)p_{t+1}^{-\sigma}(p_t y_1 - M)}$$

and

$$\begin{aligned} H_{p_{t+1}} &= \frac{(1-\sigma)\{\sigma p_t^{-1-\sigma}y_2 + G_{p_{t+1}}[2(p_{t+1}^{-\sigma}\beta^\sigma y_1 - p_t^{-\sigma}y_2) - G\sigma\beta^\sigma p_{t+1}^{-1-\sigma}(p_t y_1 - M)]\}}{p_t^{1-\sigma}y_2 - \beta^\sigma(1-\sigma)p_{t+1}^{-\sigma}(p_t y_1 - M)} + \\ &\frac{(1-\sigma)\{G[-2\sigma p_{t+1}^{-1-\sigma}\beta^\sigma y_1 - G_{p_{t+1}}\sigma\beta^\sigma p_{t+1}^{-1-\sigma}(p_t y_1 - M) + G\sigma(1+\sigma)\beta^\sigma p_{t+1}^{-2-\sigma}(p_t y_1 - M)] + H\sigma\beta^\sigma p_{t+1}^{-\sigma-1}(p_t y_1 - M)\}}{p_t^{1-\sigma}y_2 - \beta^\sigma(1-\sigma)p_{t+1}^{-\sigma}(p_t y_1 - M)} \\ &= \frac{(1-\sigma)\{\sigma p_t^{-1-\sigma}y_2 + G_{p_{t+1}}[2(p_{t+1}^{-\sigma}\beta^\sigma y_1 - p_t^{-\sigma}y_2) - G\sigma\beta^\sigma p_{t+1}^{-1-\sigma}(p_t y_1 - M)]\}}{p_t^{1-\sigma}y_2 - \beta^\sigma(1-\sigma)p_{t+1}^{-\sigma}(p_t y_1 - M)} + \\ &\frac{(1-\sigma)\{G[-2\sigma p_{t+1}^{-1-\sigma}\beta^\sigma y_1 + \sigma\beta^\sigma p_{t+1}^{-1-\sigma}(p_t y_1 - M)(-G_{p_{t+1}} + G(1+\sigma)p_{t+1}^{-1} + \frac{H}{G})]\}}{p_t^{1-\sigma}y_2 - \beta^\sigma(1-\sigma)p_{t+1}^{-\sigma}(p_t y_1 - M)} \end{aligned}$$

Let us now evaluate this two derivatives at the nonhyperbolic steady state:

$$\begin{aligned} H_{p_t} &= \frac{(1-\sigma)\{-\sigma(1+\sigma)\hat{p}^{-2}(M + \hat{p}y_2) + G_{p_t}[2(\beta^\sigma y^\circ - y_2) + \sigma\beta^\sigma \hat{p}^{-1}(\hat{p}y^\circ - M)]\}}{\hat{p}y_2 - \beta^\sigma(1-\sigma)(\hat{p}y^\circ - M)} + \\ &\frac{(1-\sigma)\{-[2\sigma\hat{p}^{-1}y_2 - G_{p_t}\sigma\beta^\sigma \hat{p}^{-1}(\hat{p}y^\circ - M) + \sigma\beta^\sigma \hat{p}^{-1}y^\circ] + H[y_2 - \beta^\sigma y^\circ]\}}{\hat{p}y_2 - \beta^\sigma(1-\sigma)(\hat{p}y^\circ - M)} \\ &= \frac{(1-\sigma)\{-\sigma(1+\sigma)\hat{p}^{-2}(M + \hat{p}y_2) + G_{p_t}[2(\beta^\sigma y^\circ - y_2) + 2\sigma\beta^\sigma \hat{p}^{-1}(\hat{p}y^\circ - M)]\}}{\hat{p}y_2 - \beta^\sigma(1-\sigma)(\hat{p}y^\circ - M)} + \\ &\frac{(1-\sigma)\{-2\sigma\hat{p}^{-1}y_2 - \sigma\beta^\sigma \hat{p}^{-1}y^\circ + H[y_2 - \beta^\sigma y^\circ]\}}{\hat{p}y_2 - \beta^\sigma(1-\sigma)(\hat{p}y^\circ - M)} \end{aligned}$$

and

$$\begin{aligned} H_{p_{t+1}} &= \frac{(1-\sigma)\{\sigma\hat{p}^{-1}y_2 + G_{p_{t+1}}[2(\beta^\sigma y^\circ - y_2) + \sigma\beta^\sigma \hat{p}^{-1}(\hat{p}y^\circ - M)]\}}{\hat{p}y_2 - \beta^\sigma(1-\sigma)(\hat{p}y^\circ - M)} + \\ &\frac{(1-\sigma)\{-[-2\sigma\hat{p}^{-1}\beta^\sigma y^\circ + \sigma\beta^\sigma \hat{p}^{-1}(\hat{p}y^\circ - M)(-G_{p_{t+1}} - (1+\sigma)\hat{p}^{-1} - H)]\}}{\hat{p}y_2 - \beta^\sigma(1-\sigma)(\hat{p}y^\circ - M)} \\ &= \frac{(1-\sigma)\{\sigma\hat{p}^{-1}y_2 + G_{p_{t+1}}[2(\beta^\sigma y^\circ - y_2) + 2\sigma\beta^\sigma \hat{p}^{-1}(\hat{p}y^\circ - M)]\}}{\hat{p}y_2 - \beta^\sigma(1-\sigma)(\hat{p}y^\circ - M)} + \\ &\frac{(1-\sigma)\{2\sigma\hat{p}^{-1}\beta^\sigma y^\circ + \sigma\beta^\sigma \hat{p}^{-1}(\hat{p}y^\circ - M)[(1+\sigma)\hat{p}^{-1} + H]\}}{\hat{p}y_2 - \beta^\sigma(1-\sigma)(\hat{p}y^\circ - M)} \end{aligned}$$