On optimal redistributive capital taxation

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Abstract

This paper addresses conflicting results regarding the optimal taxation of capital income. Judd (1985) proves that in steady state there should be no taxation of capital income. Lansing (1999) studies a logarithmic example of one of Judd's models and finds that the optimal steady state tax on capital income is not always zero — it is positive in some specifications, negative in some others. There appears to be a contradiction. However, I show that Lansing derives his result by relaxing the hypotheses of Judd's theorem — with less restrictive hypotheses, a wider range of outcomes is possible. This raises the question of whether yet more outcomes are possible with yet weaker hypotheses. I find that the answer is no: the only possible interior steady states for the model are essentially Judd's zero capital tax and Lansing's unitary elasticity of marginal utility.

JEL Classification: H2

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1 Introduction

Chamley (1986) and Judd (1985) prove that capital income should not be taxed in a steady state. Lansing (1999) provides a counterexample to this result. The example is particularly intriguing since it is a special case of one of Judd's models. There appears to be a contradiction. Lansing offers explanations to reconcile the differences. He also considers extensions of the model that revive the zero tax result. However, one is still left wondering what goes wrong in the counterexample. Lansing states on page 449, "Future research should be directed at developing a solution method that gives the right answer in all cases." Judd's solution method is optimal control theory (as is Lansing's). It would be very troubling indeed if optimal control theory failed to give the right answer. Fortunately, the contradiction can be resolved: Judd and Lansing have proved two different theorems with two different sets of hypotheses. For the special case with logarithmic utility that Lansing considers, his theorem's hypotheses are less restrictive than Judd's so the range of possible outcomes is wider. In particular, Judd's zero capital tax result is one possible outcome, but not the only one.

The hypotheses in question deal with the convergence properties of various co-state variables (Lagrange multipliers). Kemp, Long, and Shimomura (1993) have also observed that the convergence hypotheses of Judd's theorem might not be satisfied. Among the possibilities is that the steady state of the economy could be completely unstable in which case the zero capital tax result may not apply. In Lansing's example it turns out that there is a somewhat different reason why Judd's result does not apply. The issue is not the local dynamics about the steady state, but rather the dynamical system might not even have a steady state. Further work regarding the convergence properties has been done by Straub and Werning (2015). They state, "Reinhorn ... correctly clarified that in the logarithmic case the Lagrange multipliers explode, explaining the difference in results" between Judd (1985) and Lansing (1999). Straub and Werning (2015) also state, "[W]e believe the issue can be framed exactly as Reinhorn ... did, emphasizing the non convergence of multipliers."

Since the co-state variables/multipliers are shadow prices that are not observable, one would rather not make assumptions about their behavior. On the other hand, it is quite reasonable to assume that observable macroeconomic variables have stable long run behavior since this is consistent with most developed economies. (E.g., page 304 of Lucas 1990 for the US.) In the case of Judd's model, which abstracts from demographics and technological change, stability boils down to convergence to an interior steady state. Thus, I study the behavior of the optimal tax on capital income, assuming only that the observable macro variables converge to positive limits, with no assumptions about co-states.[1] I find that there are only two possible

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[1] Throughout the paper, the theorems' hypotheses will be stated in terms of the convergence of endogenous variables. The theorems do not characterize the primitives (utility functions, production function, parameters, initial condition) that satisfy convergence. Some primitives will satisfy Judd's hypotheses, some will satisfy Lansing's, and some neither. However, as discussed above, it seems reasonable to focus on those primitives that lead to stable long run behavior.

outcomes: either the modified golden rule holds in the limit or else savings are insensitive to the after-tax interest rate in the limit. In the former case we get Judd's zero tax result. In the latter case, the income and substitution effects of an interest rate change just cancel, and this is what occurs in Lansing's example with logarithmic utility. If interest does not affect savings, this undermines the benefit from a zero tax on interest/capital income and we can see why Judd's result does not necessarily hold in this case.

Straub and Werning (2015) raise serious concerns about Judd's convergence hypotheses in the case where the capitalist in the model has CES utility. In particular, when the capitalist's intertemporal elasticity of substitution is less than one, the solution to the optimal tax problem cannot converge to an interior steady state. If, in addition, the social welfare function places zero weight on the capitalist and all weight on the worker, then the solution to the optimal tax problem does converge, but to a non-interior steady state with a positive tax rate on capital income. Straub and Werning conclude that Judd's model cannot be used to unequivocally justify a zero long run tax on capital income. I agree with Straub and Werning. But since the CES case with elasticity less than one leads to a non-interior steady state, and since this is inconsistent with stable long run behavior, I prefer to exclude these utility functions from consideration and instead focus on utility functions (and other primitives) that do lead to stable long run behavior.

Section 2 presents the model. Section 3 presents the theorems of Judd and Lansing, explains the relationship between these two theorems, and also provides the general result described above. Section 4 offers a concluding comment.

2   Model

The model has four economic actors: capitalist, worker, firm, government. The capitalist has access to the capital market but does no work. The worker supplies labor inelastically but does not have access to the capital market. The firm is a price taking profit maximizer that uses capital and labor to produce output. The government chooses a time path for the tax rate on capital income and uses the proceeds to provide lump sum transfers to the worker. There is no government debt. Hence the transfers must equal the taxes at each point in time. We now proceed to describe the model in detail.

The capitalist has an infinite horizon and maximizes discounted utility, $\int_0^\infty e^{-\rho t} u(c_t^c) dt$, where $\rho > 0$ is the subjective discount rate and $c_t^c \geq 0$ is instantaneous consumption. The superscript identifies the capitalist; c_t^w will be the worker's consumption. The instantaneous utility function u is smooth, strictly increasing, strictly concave, and satisfies Inada conditions. At the beginning of time the capitalist's wealth consists of the economy's entire stock of capital, $k_0 > 0$. This stock of wealth/capital evolves through time according to the capital accumulation equation: $\dot{k}_t = (1 - \tau_{kt})(r_t - \delta)k_t - c_t^c$ where τ_{kt} is the tax rate on net capital income (subsidy rate if negative), r_t is the pre-tax interest rate gross of depreciation, and δ is the depreciation rate. Note the lack of wage income which reflects the assumption that the capitalist supplies no

labor. For ease of notation, let $\bar{r}_t := (1 - \tau_{kt})(r_t - \delta)$ denote the after tax, net of depreciation, interest rate. Then the capital accumulation equation is $\dot{k}_t = \bar{r}_t k_t - c_t^c$. Let $\bar{R}_t := \int_0^t \bar{r}_s ds$ be the cumulative interest factor. With this definition we can integrate the capital accumulation equation to get $e^{-\bar{R}_T} k_T - k_0 = -\int_0^T e^{-\bar{R}_t} c_t^c dt$. When $T \to \infty$ this equation gives the capitalist's lifetime budget. In order to prevent Ponzi schemes we will require that the present value of wealth be non-negative in the limit: $\lim_{T \to \infty} e^{-\bar{R}_T} k_T \geq 0$. Then the lifetime present value budget constraint is $\int_0^\infty e^{-\bar{R}_t} c_t^c dt \leq k_0$. The capitalist maximizes lifetime utility subject to this budget. At the solution, the intertemporal marginal rate of substitution must equal the ratio of present value prices, and the budget must hold with equality:

$$e^{-\rho t} u'(c_t^c)/u'(c_0^c) = e^{-\bar{R}_t} \text{ for almost all } t \geq 0 \text{ and } \int_0^\infty e^{-\bar{R}_t} c_t^c dt = k_0. \tag{1}$$

Equivalently, the first of these conditions can be log differentiated to give the consumption Euler equation $\dot{c}_t^c u''(c_t^c)/u'(c_t^c) = \rho - \bar{r}_t$. The second equation in (1) can be expressed in its no-Ponzi form as $\lim_{t \to \infty} e^{-\bar{R}_t} k_t = 0$, or, by the first equation in (1), $\lim_{t \to \infty} e^{-\rho t} u'(c_t^c) k_t = 0$.

The worker inelastically supplies a flow of one unit of labor and immediately consumes all wages and transfers due to the lack of access to the capital market. So the worker is a passive actor who makes no decisions. The instantaneous utility function is $v(c_t^w)$. The worker's consumption (and income) is $c_t^w = w_t + TR_t$ where w_t is the wage and TR_t is the transfer. The assumptions that were imposed on the capitalist's utility function u are also imposed on v.

The firm is a price taking profit maximizer with constant returns to scale in labor and capital. The production function in intensive form is $f(k_t)$. The capital to labor ratio coincides with the capital stock since the labor supply is always one unit. We assume that $f(0) = 0$ and that f satisfies the same conditions as the utility functions u and v. At the firm's optimum, $f'(k_t) = r_t$ and $f(k_t) - k_t f'(k_t) = w_t$.

Given the restriction against government debt, tax revenue must equal the transfer at each instant: $\tau_{kt}(r_t - \delta)k_t = TR_t$. Hence, from the definition of $\bar{r}_t$ and the firm's profit maximization condition, $TR_t = -\bar{r}_t k_t + [f'(k_t) - \delta]k_t$. Then the worker's consumption is

$$c_t^w = w_t + TR_t = [f(k_t) - k_t f'(k_t)] - \bar{r}_t k_t + [f'(k_t) - \delta]k_t = f(k_t) - \delta k_t - \bar{r}_t k_t. \tag{2}$$

In equilibrium, consumption plus investment must equal output: $c_t^c + c_t^w + \delta k_t + \dot{k}_t = f(k_t)$. Substitute for c_t^w to get $\dot{k}_t = \bar{r}_t k_t - c_t^c$, which is satisfied by the capitalist's flow budget constraint (Walras' Law).

3   Optimal taxation

The government maximizes social welfare $\int_0^\infty e^{-\rho t}[\gamma v(c_t^w) + u(c_t^c)]dt$ subject to the equilibrium conditions: the capitalist maximizes lifetime utility, the worker consumes all available income, firms maximize profits, the government's budget is in balance at every instant so the worker's income is as described in (2), and

markets clear. Note that the government applies the capitalist's discount factor to both consumers, and the welfare weight γ is time invariant. There is one further constraint: $\bar{r}_t \geq 0$. This is a policy restriction that prevents the government from imposing a tax rate in excess of 100 percent. Substitute for c_t^w from (2) to get the following problem:

$$\text{maximize} \quad \int_0^\infty e^{-\rho t}[\gamma v(f(k_t) - \delta k_t - \bar{r}_t k_t) + u(c_t^c)]dt$$

$$\text{subject to} \quad \dot{k}_t = \bar{r}_t k_t - c_t^c$$

$$\dot{c}_t^c = (\rho - \bar{r}_t)u'(c_t^c)/u''(c_t^c)$$

$$\bar{r}_t \geq 0$$

with $k_0 > 0$ given and $\lim_{t\to\infty} e^{-\rho t}u'(c_t^c)k_t = 0$. The optimal time path for the tax rate can be recovered from the definition of $\bar{r}_t$. The current value Hamiltonian is

$$H(k, c^c, \bar{r}, q_1, q_2, \eta) = \gamma v\big(f(k) - \delta k - \bar{r}k\big) + u(c^c) + q_1(\bar{r}k - c^c) + q_2(\rho - \bar{r})u'(c^c)/u''(c^c) + \eta\bar{r}.$$

The state variables are k_t (with co-state q_{1t}) and c_t^c (with co-state q_{2t}), $\bar{r}_t$ is the control, and η_t is the Lagrange multiplier for the constraint $\bar{r}_t \geq 0$. The optimality conditions are

$$\partial H/\partial k \quad = \quad \gamma v'(c_t^w)[f'(k_t) - \delta - \bar{r}_t] + q_{1t}\bar{r}_t = \rho q_{1t} - \dot{q}_{1t} \tag{3a}$$

$$\partial H/\partial c^c \quad = \quad u'(c_t^c) - q_{1t} + q_{2t}(\rho - \bar{r}_t)\left\{1 - [u''(c_t^c)]^{-2}u'(c_t^c)u'''(c_t^c)\right\} = \rho q_{2t} - \dot{q}_{2t} \tag{3b}$$

$$\partial H/\partial \bar{r} \quad = \quad -\gamma v'(c_t^w)k_t + q_{1t}k_t - q_{2t}u'(c_t^c)/u''(c_t^c) + \eta_t = 0 \tag{3c}$$

$$\partial H/\partial q_1 \quad = \quad \bar{r}_t k_t - c_t^c = \dot{k}_t \tag{3d}$$

$$\partial H/\partial q_2 \quad = \quad (\rho - \bar{r}_t)u'(c_t^c)/u''(c_t^c) = \dot{c}_t^c \tag{3e}$$

$$\eta_t\bar{r}_t = 0, \quad \lim_{t\to\infty} e^{-\rho t}q_{1t}k_t = 0, \quad \lim_{t\to\infty} e^{-\rho t}q_{2t}c_t^c = 0 \tag{3f}$$

together with the problem's two boundary conditions. The last line includes the complementary slackness and transversality conditions.

3.1 Theorem (Judd)[2] *Suppose a solution to (3) has the property that k_t, c_t^c, $\bar{r}_t$, and q_{1t} converge as t tends to infinity, with strictly positive limits for k_t, c_t^c, and c_t^w. Then $\lim_{t\to\infty} \tau_{kt} = 0$.*

Proof Drop the time subscripts to denote limiting values. From (3e),[3] $\bar{r} = \rho$. Therefore (3a) yields $f'(k) - \delta - \bar{r} = 0$. The theorem now follows from the definition $\bar{r}_t = (1 - \tau_{kt})[f'(k_t) - \delta]$. ∎

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[2] See theorem 2 and equations (24) on page 72 of Judd (1985).

[3] The assumption that $\lim_{t\to\infty} \xi_t$ exists does not always imply $\lim_{t\to\infty} \dot{\xi}_t = 0$ (e.g., $t^{-1}\sin t^2$). However, this is not a problem here. Equations (3a, d, e) are of the form $\dot{\xi}_t = G(k_t, c_t^c, \bar{r}_t, q_{1t})$ with G continuous, where $\dot{\xi}_t$ represents $\dot{q}_{1t}$, $\dot{k}_t$, or $\dot{c}_t^c$. Therefore, under stated assumptions, $\dot{\xi}_t$ has a limit as t tends to infinity. That limit must be zero; otherwise ξ_t (no dot) would fail to converge as t tends to infinity. A similar argument can be applied to Lansing's theorem, and to parts of theorem 3.6, below.

In Lansing's example, $u = \log$. Then (1) simplifies to $c_t^c = c_0^c e^{\bar{R}_t - \rho t}$ and $c_0^c = \rho k_0$. From (3d), $d[e^{-\bar{R}_t} k_t]/dt = -e^{-\bar{R}_t} c_t^c$, so with $c_t^c = \rho k_0 e^{\bar{R}_t - \rho t}$ this yields $d[e^{-\bar{R}_t} k_t]/dt = d[k_0 e^{-\rho t}]/dt$. Integrate, and use $\bar{R}_0 = 0$ to identify the constant of integration. The result is $k_t = k_0 e^{\bar{R}_t - \rho t}$. Hence $c_t^c = \rho k_t$. Substitute this and $u = \log$ into (3) to get:

$$\partial H/\partial k = \gamma v'(c_t^w)[f'(k_t) - \delta - \bar{r}_t] + q_{1t}\bar{r}_t = \rho q_{1t} - \dot{q}_{1t} \tag{4a}$$

$$\partial H/\partial c^c = 1/\rho k_t - q_{1t} - q_{2t}(\rho - \bar{r}_t) = \rho q_{2t} - \dot{q}_{2t} \tag{4b}$$

$$\partial H/\partial \bar{r} = -\gamma v'(c_t^w)k_t + q_{1t}k_t + q_{2t}\rho k_t + \eta_t = 0 \tag{4c}$$

$$\partial H/\partial q_1 = \bar{r}_t k_t - \rho k_t = \dot{k}_t \tag{4d}$$

$$\partial H/\partial q_2 = -(\rho - \bar{r}_t)\rho k_t = \rho \dot{k}_t \tag{4e}$$

$$\eta_t \bar{r}_t = 0, \quad \lim_{t \to \infty} e^{-\rho t} q_{1t} k_t = 0, \quad \lim_{t \to \infty} e^{-\rho t} q_{2t}\rho k_t = 0. \tag{4f}$$

This system characterizes the solution to the optimal tax problem when $u = \log$. One of the properties of (4) is that generically $\lim_{t \to \infty}(k_t, c_t^c, \bar{r}_t, q_{1t})$ does not exist. I.e., it may be that some of these variables converge, but in general they cannot all converge. Thus, for this special utility function the hypotheses of Judd's theorem generically cannot be satisfied. The reason is as follows. If all these variables were to converge, the proof of Judd's theorem would apply so in the limit $\bar{r} = \rho$ (hence $\eta = 0$) and $f'(k) = \delta + \rho$. The latter condition would uniquely determine k (modified golden rule). Then, from (4c), q_{2t} would converge and its limit would satisfy $\gamma v'(c^w) = q_1 + \rho q_2$. Also, in the limit, (4b) would yield $1/\rho k = q_1 + \rho q_2$. Hence $1/\rho k = \gamma v'(c^w) = \gamma v'(f(k) - \delta k - \rho k)$, where the last equality uses (2). This would impose a second condition on k, in addition to $f'(k) = \delta + \rho$. Only in exceptional cases will the same value of k satisfy both these conditions. Generically there will be no k that satisfies both. Nonetheless, (4) is still valid — it still characterizes the solution to the optimal tax problem when $u = \log$. The fact that (generically) its variables do not all converge is neither here nor there.

Given the simplifications associated with $u = \log$, Lansing states directly the optimal tax problem for this special case:

$$\text{maximize} \quad \int_0^\infty e^{-\rho t}[\gamma v(f(k_t) - \delta k_t - \bar{r}_t k_t) + \log(\rho k_t)]dt$$

$$\text{subject to} \quad \dot{k}_t = (\bar{r}_t - \rho)k_t$$

$$\bar{r}_t \geq 0$$

with $k_0 > 0$ given. The $\dot{c}_t^c$ equation is dropped because it is redundant. Thus the $\dot{k}_t$ equation has a dual role. Not only is it the capital accumulation equation; it is also the consumption Euler equation for the capitalist. The current value Hamiltonian for this problem is $H(k, \bar{r}, q_3, \eta) = \gamma v(f(k) - \delta k - \bar{r}k) + \log(\rho k) + q_3(\bar{r} - \rho)k + \eta \bar{r}$.

The optimality conditions are

$$\partial H/\partial k \;=\; \gamma v'(c_t^w)[f'(k_t) - \delta - \bar{r}_t] + 1/k_t + q_{3t}(\bar{r}_t - \rho) = \rho q_{3t} - \dot{q}_{3t} \tag{5a}$$

$$\partial H/\partial \bar{r} \;=\; -\gamma v'(c_t^w)k_t + q_{3t}k_t + \eta_t = 0 \tag{5b}$$

$$\partial H/\partial q_3 \;=\; (\bar{r}_t - \rho)k_t = \dot{k}_t \tag{5c}$$

$$\eta_t \bar{r}_t = 0, \quad \lim_{t\to\infty} e^{-\rho t} q_{3t} k_t = 0 \tag{5d}$$

with k_0 given. In Lansing (1999), this appears as (21) on page 435. Note the new notation q_{3t} for the co-state here in (5). Since the $\dot{k}_t$ equation has a dual role here so does its co-state.[4] Indeed q_{3t} is distinct from both of the co-states in (4), q_{1t} (for capital) and q_{2t} (for the capitalist's consumption). However, they are related to one another.

3.2 Lemma *Equations (4) and (5) are equivalent, with*

$$q_{3t} \;=\; q_{1t} + \rho q_{2t} \tag{6}$$

$$k_t q_{1t} - k_0 q_{10} \;=\; t + k_t q_{3t} - k_0 q_{30} - \rho \int_0^t k_s q_{3s} ds \tag{7}$$

$$k_t q_{2t} - k_0 q_{20} \;=\; -t/\rho + \int_0^t k_s q_{3s} ds \tag{8}$$

where the initial conditions in (7) and (8) satisfy $q_{10} + \rho q_{20} = q_{30}$.

Proof First, given a solution to (4), verify that (5) is satisfied when q_{3t} is defined by (6). Clearly (5b) follows from (4c), (5c) follows from (4d), and (5d) follows from (4f). Given (4a), (5a) will be satisfied if

$$1/k_t + \dot{q}_{3t} + q_{3t}(\bar{r}_t - \rho) - \rho q_{3t} = \dot{q}_{1t} + q_{1t}(\bar{r}_t - \rho). \tag{9}$$

Use (6) to substitute for q_{3t} and $\dot{q}_{3t}$:

$$1/k_t + \dot{q}_{1t} + \rho \dot{q}_{2t} + (q_{1t} + \rho q_{2t})(\bar{r}_t - 2\rho) = \dot{q}_{1t} + q_{1t}(\bar{r}_t - \rho).$$

This is satisfied by (4b).

Next, given a solution to (5), verify that (4) is satisfied when q_{1t} and q_{2t} are defined by (7) and (8) with $q_{10} + \rho q_{20} = q_{30}$. Take (7) and add to it (8) multiplied by ρ to get (6). Hence (4c) follows from (5b). Clearly, (4d) and (4e) follow from (5c), while (4a) will follow from (5a) if (9) holds. To verify (9), first take the time derivative of (7):

$$\dot{k}_t q_{1t} + k_t \dot{q}_{1t} = 1 + \dot{k}_t q_{3t} + k_t \dot{q}_{3t} - \rho k_t q_{3t}.$$

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[4]Cf Lansing (1999) where the same notation q_{1t} is used for the dual role co-state in (21) on page 435 and also for capital's co-state in (17) on page 432 where utility is not restricted to be logarithmic.

Substitute for $\dot{k}_t$ from (5c):

$$(\bar{r}_t - \rho)k_t q_{1t} + k_t \dot{q}_{1t} = 1 + (\bar{r}_t - \rho)k_t q_{3t} + k_t \dot{q}_{3t} - \rho k_t q_{3t}.$$

Now divide through by k_t to get (9). To verify (4b), first take the time derivative of (8):

$$\dot{k}_t q_{2t} + k_t \dot{q}_{2t} = -1/\rho + k_t q_{3t}.$$

Substitute for $\dot{k}_t$ from (5c), and, since (7) and (8) yield (6), also substitute for q_{3t} from (6). The result is (4b) multiplied through by k_t. Finally, (4f) follows from (7), (8), (5d), and, if necessary, application of l'Hopital's rule to $\lim_{t\to\infty} \int_0^t k_s q_{3s} ds / e^{\rho t}$. ∎

3.3 Theorem (Lansing)[5] *Suppose a solution to* (5) *has the property that* k_t, $\bar{r}_t$, *and* q_{3t} *converge as* t *tends to infinity, with strictly positive limits for* k_t, c_t^w, *and* $f'(k_t) - \delta$. *Then, dropping the time subscripts to denote limiting values,* $\mathrm{sgn}(\tau_k) = \mathrm{sgn}\big(\rho\gamma v'(c^w)k - 1\big)$.

Proof From (5c), $\bar{r} = \rho$. From (5b), $q_3 = \gamma v'(c^w)$ since $\eta = 0$ ($\bar{r} > 0$) and $k > 0$. Therefore, (5a) yields $\gamma v'(c^w)[f'(k) - \delta - \bar{r}] = \rho\gamma v'(c^w) - 1/k$. The theorem now follows from $\bar{r}_t = (1 - \tau_{kt})[f'(k_t) - \delta]$. ∎

Judd's hypotheses are more restrictive than Lansing's. That is, in (4) Judd's hypotheses are that k_t, c_t^c, $\bar{r}_t$, and q_{1t} all converge. Recall that generically this does not happen, but when it does, $1/\rho k = \gamma v'(c^w)$. So in this special case Lansing's theorem yields $\tau_k = 0$ in the limit, just like Judd's theorem: When $u = \log$, Judd's theorem is a special (and exceptional) case of Lansing's.

Furthermore, when Judd's hypotheses are satisfied, q_{2t} also converges by (4c). Hence, by (6), q_{3t} converges in (5). So Lansing's hypotheses are satisfied. The converse does not necessarily hold. It is possible for q_{3t} to converge while q_{1t} and q_{2t} diverge. The following corollary states this formally.

3.4 Corollary *Suppose a solution to* (5) *has the property that* k_t, $\bar{r}_t$, *and* q_{3t} *converge as* t *tends to infinity, with strictly positive limits for* k_t, c_t^w, *and* $f'(k_t) - \delta$. *Then, in* (4),

$$\lim_{t\to\infty} q_{1t}/t \;=\; (1 - \rho k q_3)/k \;=\; 1/k - \rho\gamma v'(c^w)$$

$$\lim_{t\to\infty} q_{2t}/t \;=\; (-1/\rho + k q_3)/k \;=\; -1/\rho k + \gamma v'(c^w)$$

where $k = \lim_{t\to\infty} k_t$, *etc. So if* $\rho\gamma v'(c^w)k \neq 1$ *then both* q_{1t} *and* q_{2t} *fail to converge. Since* $\mathrm{sgn}(\tau_k) = \mathrm{sgn}\big(\rho\gamma v'(c^w)k - 1\big)$ *from theorem 3.3, it follows that if* $\tau_k \neq 0$ *then* q_{1t} *fails to converge so Judd's hypotheses are not satisfied.*

Proof In (7) and (8), apply l'Hopital's rule to the integrals divided by t, and use $q_3 = \gamma v'(c^w)$ from the proof of theorem 3.3. ∎

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[5] See proposition 2 on page 435 of Lansing (1999).

The following example rigs the initial conditions and parameter values to illustrate the corollary.

3.5 Example Suppose

$$\gamma v'(c_0^w)[f'(k_0) - \delta - 2\rho] + 1/k_0 = 0, \quad y_0 = f(k_0) - \delta k_0 - \rho k_0 > 0, \quad 0 < f'(k_0) - \delta \neq \rho. \tag{10}$$

Then $k_t \equiv k_0$, $\bar{r}_t \equiv \rho$ (hence $\eta_t \equiv 0$), and $q_{3t} \equiv \gamma v'(c_0^w)$ solves (5). So Lansing's hypotheses are satisfied. From (7) and (8), $q_{1t} = q_{10} + [1/k_0 - \gamma \rho v'(c_0^w)]t$ and $q_{2t} = q_{20} - [1/k_0 - \gamma \rho v'(c_0^w)]t/\rho$, with $q_{10} + \rho q_{20} = q_{30} = \gamma v'(c_0^w)$ by (6). So, from (10), q_{1t} and q_{2t} do not converge; but they do satisfy transversality. The tax rate on capital income is not zero: $\tau_{kt}[f'(k_0) - \delta] = f'(k_0) - \delta - \rho \neq 0$. ∎

Return now to the general case (3) when the capitalist's utility is not necessarily $u = \log$. As stated in the introduction, the focus of attention is time paths for which the observables k_t, c_t^c, $\bar{r}_t$ converge to positive limits as t tends to infinity. As a first step, suppose the initial condition k_0 is such that these observables are time invariant: $(k_t, c_t^c, \bar{r}_t) \equiv (k, c^c, \bar{r})$. The behavior of the capital tax in this case will shed light on the time varying case.

If $(k_t, c_t^c, \bar{r}_t) \equiv (k, c^c, \bar{r})$ then $\bar{r} = \rho$ from (3e) and $c^c = \rho k$ from (3d). Also, $c_t^w \equiv c^w = f(k) - \delta k - \rho k$ from (2); assume this is positive. From (3a), $q_{1t} = q_{10} - \gamma v'(c^w)[f'(k) - \delta - \rho]t$. Then from (3c) (with $\eta_t \equiv 0$ since $\bar{r}_t \equiv \rho$), $q_{2t} = ku''(\rho k)[u'(\rho k)]^{-1}\{q_{10} - \gamma v'(c^w) - \gamma v'(c^w)[f'(k) - \delta - \rho]t\}$. Note that transversality is satisfied in (3f). All that remains is (3b), which reduces to $u'(\rho k) - q_{1t} = \rho q_{2t} - \dot{q}_{2t}$. With the above solutions for q_{1t} and q_{2t}, this requires that the coefficients of t match up:

$$\gamma v'(c^w)[f'(k) - \delta - \rho] = -\rho k u''(\rho k)[u'(\rho k)]^{-1}\gamma v'(c^w)[f'(k) - \delta - \rho]$$

hence

$$\gamma v'(c^w)[f'(k) - \delta - \rho][1 + \rho k u''(\rho k)/u'(\rho k)] = 0. \tag{11a}$$

It also requires $u'(\rho k) - q_{10} = \rho q_{20} - \dot{q}_{2t}$:

$$u'(\rho k) - q_{10} = ku''(\rho k)[u'(\rho k)]^{-1}\{\rho[q_{10} - \gamma v'(c^w)] + \gamma v'(c^w)[f'(k) - \delta - \rho]\}$$

hence

$$q_{10}[1 + \rho k u''(\rho k)/u'(\rho k)] = u'(\rho k) - k u''(\rho k)[u'(\rho k)]^{-1}\gamma v'(c^w)[f'(k) - \delta - 2\rho]. \tag{11b}$$

The solution to (11a) and (11b) requires one of the following alternatives:

(i) $f'(k) = \rho + \delta$ and $\rho k u''(\rho k)/u'(\rho k) \neq -1$;

(ii) $\rho k u''(\rho k)/u'(\rho k) = -1$ and $\rho u'(\rho k) = -\gamma v'(c^w)[f'(k) - \delta - 2\rho]$.

In each of these, the first condition ensures that (11a) is satisfied, while the second ensures that (11b) is satisfied. In particular, in (i) the second condition allows us to find a unique value for q_{10} that satisfies (11b). In (i), the capital tax is zero whereas in (ii), the capital tax is not restricted to be zero. Lansing's example with $u = \log$ is an instance of alternative (ii): the first condition in (ii) is satisfied identically and the second condition determines the value of k. (I.e., it determines the value of k_0 that would lead to a time invariant path.) When $u \neq \log$, alternative (ii) would impose two distinct restrictions on k making it unlikely to have any solution. Thus, other than $u = \log$, alternative (ii) can be effectively dismissed and this leaves us with alternative (i) — zero tax on capital income.

These results for the time invariant case suggest the following for the time varying case.

3.6 Theorem *Suppose a solution to* (3) *has the property that* k_t, c_t^c, *and* $\bar{r}_t$ *converge as* t *tends to infinity, with strictly positive limits for* k_t, c_t^c, *and* c_t^w. *Then* $\lim_{t\to\infty} \tau_{kt} = 0$ *or* $\lim_{t\to\infty}[c_t^c + u'(c_t^c)/u''(c_t^c)] = 0$ *or both.*

Proof From (3b) and (3e),

$$\frac{d}{dt}[e^{-\rho t}q_{2t}u'(c_t^c)/u''(c_t^c)] = e^{-\rho t}\frac{[u'(c_t^c)]^2}{u''(c_t^c)}[q_{1t}/u'(c_t^c) - 1].$$

Since $u'(c_t^c)/[c_t^c u''(c_t^c)]$ has a finite limit, transversality yields

$$-e^{-\rho t}q_{2t}u'(c_t^c)/u''(c_t^c) = \int_t^\infty e^{-\rho s}\frac{[u'(c_s^c)]^2}{u''(c_s^c)}[q_{1s}/u'(c_s^c) - 1]ds. \tag{12}$$

In preparation for integration by parts, let $z_t := \int_t^\infty e^{-\rho s}[u'(c_s^c)]^2[u''(c_s^c)]^{-1}ds$. From l'Hopital's rule, $\lim_{t\to\infty}[z_t/e^{-\rho t}] = \lim_{t\to\infty}[u'(c_t^c)]^2[\rho u''(c_t^c)]^{-1}$. This will be useful later. From (3a) and (3e) respectively, $e^{\bar{R}_t - \rho t}q_{1t} = q_{10} - \gamma\int_0^t e^{\bar{R}_s - \rho s}v'(c_s^w)[f'(k_s) - \delta - \bar{r}_s]ds$ and $e^{\bar{R}_t - \rho t} = u'(c_0^c)/u'(c_t^c)$. Recall, $\bar{R}_t := \int_0^t \bar{r}_s ds$ is cumulative interest. Therefore,

$$\frac{d}{dt}[q_{1t}/u'(c_t^c) - 1] = [u'(c_0^c)]^{-1}\frac{d}{dt}[e^{\bar{R}_t - \rho t}q_{1t}] = -\gamma[u'(c_t^c)]^{-1}v'(c_t^w)[f'(k_t) - \delta - \bar{r}_t].$$

Apply integration by parts to (12):

$$\begin{aligned}-e^{-\rho t}q_{2t}u'(c_t^c)/u''(c_t^c) &= \left[-z_s[q_{1s}/u'(c_s^c) - 1]\right]_t^\infty - \gamma\int_t^\infty z_s[u'(c_s^c)]^{-1}v'(c_s^w)[f'(k_s) - \delta - \bar{r}_s]ds \\ &= z_t[q_{1t}/u'(c_t^c) - 1] - \gamma\int_t^\infty z_s[u'(c_s^c)]^{-1}v'(c_s^w)[f'(k_s) - \delta - \bar{r}_s]ds.\end{aligned}$$

The second line follows from transversality and the limiting behavior of z_t. Use this equation to substitute for $q_{2t}u'(c_t^c)/u''(c_t^c)$ in (3c):

$$q_{1t}[k_t + e^{\rho t}z_t/u'(c_t^c)] = \gamma v'(c_t^w)k_t + e^{\rho t}z_t + \gamma e^{\rho t}\int_t^\infty z_s[u'(c_s^c)]^{-1}v'(c_s^w)[f'(k_s) - \delta - \bar{r}_s]ds - \eta_t. \tag{13}$$

All terms on the right side of this equation converge as t tends to infinity. In particular, l'Hopital's rule can be applied to the integral divided by $e^{-\rho t}$, while $\lim_{t\to\infty}\eta_t = 0$ since $\lim_{t\to\infty}\bar{r}_t = \rho$ from (3e). Therefore,

11

the left side must converge. The convergence of q_{1t} would be sufficient for this to occur in which case Judd's theorem would apply so $\lim_{t\to\infty} \tau_{kt} = 0$. But if q_{1t} fails to converge, the required convergence of the left side of (13) implies

$$0 = \lim_{t\to\infty} [k_t + e^{\rho t} z_t / u'(c_t^c)] = \rho^{-1} \lim_{t\to\infty} [c_t^c + u'(c_t^c)/u''(c_t^c)]$$

where the second equality uses $\lim_{t\to\infty} c_t^c = \rho \lim_{t\to\infty} k_t$ from (3d, e), and also the earlier result regarding the limiting behavior of z_t. ∎

3.7 Remark Consider the following cases of the theorem. If $u = \log$, then in the proof $z_t = -e^{-\rho t}/\rho$ and $c_t^c = \rho k_t$, so, dropping time subscripts to denote limiting values, in the limit (13) yields: $0 = \gamma v'(c^w)k[f'(k) - \delta - 2\rho] + 1$ (apply l'Hopital's rule to the integral divided by $e^{-\rho t}$) with $c^w = f(k) - \delta k - \rho k$. This determines the steady state value(s) of k, and hence, by Lansing's theorem, τ_k. If u is any other CES function, $c^c + u'(c^c)/u''(c^c) \neq 0$, so $\tau_k = 0$. In this case, (13) determines q_1, and k solves $f'(k) = \delta + \rho$. For general u, if τ_{kt} fails to converge to zero, k must satisfy $\rho k + u'(\rho k)/u''(\rho k) = 0$, and (13) determines the limiting behavior of the indeterminate form $\lim_{t\to\infty} q_{1t}[k_t + e^{\rho t} z_t / u'(c_t^c)]$. ∎

4   Conclusion

This paper has clarified the relationship between the results of Judd (1985) and Lansing (1999). Judd's theorem states that in steady state the optimal tax rate on capital income is zero.[6] Lansing identifies a logarithmic example of one of Judd's models in which this tax rate can converge to any number, zero or otherwise — the value depends on the model's primitives (the worker's utility function, the production function, etc). It seems odd that the same model can generate two different results. The apparent contradiction is resolved by observing that Lansing has relaxed the hypotheses of Judd's theorem. With less restrictive hypotheses, there are more possible outcomes. One would like to know if yet more outcomes are possible with yet less restrictive hypotheses. Theorem 3.6 addresses this issue and characterizes all possible steady state outcomes for this particular model. There are two, and only two, possibilities: the zero capital tax result is one, while in any other steady state the capitalist's marginal utility must have unitary elasticity. The latter possibility is satisfied identically with logarithmic utility, which was the case considered by Lansing.

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[6] Judd (1999, 2002) has returned to this issue, but not with the worker-capitalist model. The range of views on capital income taxation can be exemplified by Atkeson, Chari, and Kehoe (1999) on the one hand, and Conesa, Kitao, and Krueger (2009) on the other.

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