Further results on the estimation of dynamic panel logit models with fixed effects.

Hugo Kruiniger* Durham University

This version: 28 January 2023 Previous versions: 27 October 2020, 26 April 2021 and 17 January 2023

JEL classification: C12, C13, C23.

Keywords: dynamic panel logit models, exogenous regressors, fixed effects.

^{*}Address: hugo.kruiniger@durham.ac.uk; Department of Economics, 1 Mill Hill Lane, Durham DH1 3LB, England. I thank N. Peyerimhoff for helpful comments. All remaining errors are mine.

Abstract

Kitazawa (2013, 2016) showed that the common parameters in the panel logit AR(1) model with strictly exogenous covariates and fixed effects are estimable at the root-n rate using the Generalized Method of Moments. Honoré and Weidner (2020) extended his results in various directions: they found additional moment conditions for the logit AR(1) model and also considered estimation of logit AR(p) models with p > 1. In this note we prove a conjecture in their paper and show that for given values of the initial condition, the covariates and the common parameters $2^T - 2T$ of their moment functions for the logit AR(1) model are linearly independent and span the set of valid moment functions, which is a $2^T - 2T$ -dimensional linear subspace of the 2^T -dimensional vector space of real valued functions over the outcomes $y \in \{0,1\}^T$. We also prove that when p = 2 and $T \in \{3,4,5\}$, there are, respectively, $2^T - 4(T-1)$ and $2^T - (3T-2)$ linearly independent moment functions for the panel logit AR(2) models with and without covariates.

1 Proof of a conjecture in Honoré and Weidner (2020)

We adopt the notation of Honoré and Weidner (2020). In p.16 of their paper they define for triples of time periods $t, s, r \in \{1, 2, ..., T\}$ with t < s < r the moment functions $m_{y_0}^{(a/b)(t,s,r)}(y,x,\beta,\gamma)$. Let $z_{t,s}(y_0,y,x,\beta,\gamma) = (x_t - x_s)'\beta + \gamma(y_{t-1} - y_{s-1})$. Then

$$m_{y_0}^{(a)(t,s,r)}(y,x,\beta,\gamma) = \begin{cases} \exp[z_{t,s}(y_0,y,x,\beta,\gamma)] & \text{if } (y_t,y_s,y_r) = (0,1,0), \\ \exp[z_{t,r}(y_0,y,x,\beta,\gamma)] & \text{if } (y_t,y_s,y_r) = (0,1,1), \\ -1 & \text{if } (y_t,y_s) = (1,0), \\ \exp[z_{r,s}(y_0,y,x,\beta,\gamma)] - 1 & \text{if } (y_t,y_s,y_r) = (1,1,0), \\ 0 & \text{otherwise}, \end{cases}$$

$$m_{y_0}^{(b)(t,s,r)}(y,x,\beta,\gamma) = \begin{cases} \exp[z_{s,r}(y_0,y,x,\beta,\gamma)] - 1 & \text{if } (y_t,y_s,y_r) = (0,0,1), \\ -1 & \text{if } (y_t,y_s) = (0,1), \\ \exp[z_{s,t}(y_0,y,x,\beta,\gamma)] & \text{if } (y_t,y_s,y_r) = (1,0,0), \\ \exp[z_{s,t}(y_0,y,x,\beta,\gamma)] & \text{if } (y_t,y_s,y_r) = (1,0,0), \\ 0 & \text{otherwise}. \end{cases}$$

In p.17 of their paper they conjecture that for $\gamma_0 \neq 0$ (and arbitrary y_0 , x and β_0 ; index i is omitted) any moment function $m_{y_0}(y, x, \beta, \gamma) = \overline{w}(y_1, ..., y_{t-1}) m_{y_0}^{(a/b)(t,s,r)}(y, x, \beta, \gamma)$ for the panel logit AR(1) model with strictly exogenous regressors and $T \geq 3$, where $\overline{w}_{y_1,...,y_{t-1}}(y_1, ..., y_{t-1}) : \{0, 1\}^{t-1} \to \mathbb{R}$, can be written as

$$m_{y_0}(y, x, \beta, \gamma) = \sum_{t=1}^{T-2} \sum_{s=t+1}^{T-1} [w_{y_0}^{(a)}(t, s, y_1, ..., y_{t-1}, x, \beta, \gamma) m_{y_0}^{(a)(t, s, T)}(y, x, \beta, \gamma) + w_{y_0}^{(b)}(t, s, y_1, ..., y_{t-1}, x, \beta, \gamma) m_{y_0}^{(b)(t, s, T)}(y, x, \beta, \gamma)]$$

with weights $w_{y_0}^{(a/b)}(t, s, y_1, ..., y_{t-1}, x, \beta, \gamma) \in \mathbb{R}$ that are uniquely determined by the function $m_{y_0}(., x, \beta, \gamma)$.

We will prove this conjecture by showing for given values of y_0 , x, β and γ (i) that the set of valid moment functions is a linear subspace of the 2^T -dimensional vector space of real valued functions over the outcomes $y \in \{0,1\}^T$ that has a dimension of at most $2^T - 2T$, and (ii) that the $2^T - 2T$ moment functions of the form $w_{y_1,...,y_{t-1}}(y_1,...,y_{t-1})m_{y_0}^{(a/b)(t,s,T)}(y,x,\beta,\gamma)$, where $w_{y_1,...,y_{t-1}}(y_1,...,y_{t-1}):\{0,1\}^{t-1} \to \{0,1\}$ are 2^{t-1} linearly independent indicator functions and $1 \le t < s < T$, are linearly independent and span this subspace.

Proof: Recall that $Pr(Y_i = y_i | Y_{i0} = y_{i0}, X_i = x_i, A_i = \alpha_i) \equiv$

$$p_{y_{i0}}(y_i, x_i, \beta_0, \gamma_0, \alpha_i) = \prod_{t=1}^{T} \frac{1}{1 + exp[(1 - 2y_{it})(x'_{it}\beta_0 + y_{i,t-1}\gamma_0 + \alpha_i)]}.$$

We drop the index i. A valid moment function $m_{y_0}(y, x, \beta, \gamma)$ satisfies

$$E[m_{y_0}(Y, X, \beta_0, \gamma_0)|Y_0 = y_0, X = x, A = \alpha] = 0 \text{ for all } \alpha \in \mathbb{R}$$

or equivalently

$$\sum_{y \in \{0,1\}^T} p_{y_0}(y, x, \beta_0, \gamma_0, \alpha) m_{y_0}(y, x, \beta_0, \gamma_0) = 0 \text{ for all } \alpha \in \mathbb{R}.$$

Let $T \geq 2$ and $\alpha_1 < \alpha_2 < ... < \alpha_{2^T}$. Define the $2^T \times 2^T$ matrix \bar{P} with $\bar{P}_{g,h} = p_{y_0}(y,x,\beta_0,\gamma_0,\alpha_g)$ for $g,h=1,2,...,2^T$ with $h=1+2^0y_1+2^1y_2+...+2^{T-1}y_T$. Next let $P_{g,t}=\exp(x_t'\beta_0+\alpha_g)$ and define the $2^T \times 2^T$ matrix \check{P} with $\check{P}_{g,h}=P_{g,T}^{y_T}\prod_{t=1}^{T-1}(P_{g,t}(1+P_{g,t+1})/(1+P_{g,t+1}e^{\gamma_0}))^{y_t}$ for $g,h=1,2,...,2^T$ with $h=1+2^0y_1+2^1y_2+...+2^{T-1}y_T$. Finally, let $\overline{D}=\overline{D}(x,\beta_0,\gamma_0,\alpha)$ and $\check{D}=\check{D}(\gamma_0)$ be nonsingular diagonal matrices with $\overline{D}_{g,g}=((1+P_{g,1}e^{\gamma_0})/(1+P_{g,1}))^{y_0}\prod_{t=1}^T(1+P_{g,t})$ and $\check{D}_{h,h}=\prod_{t=1}^T\exp(-\gamma_0y_{t-1}y_t)$ for $g,h=1,2,...,2^T$ with $h=1+2^0y_1+2^1y_2+...+2^{T-1}y_T$. Then it is easily verified that $\check{P}=\overline{D}\check{P}\check{D}$. Hence $rk(\check{P})=rk(\bar{P})$. We also define $y^S=\sum_{t=1}^Ty_t$ for later use.

We now show (i). If the model does not contain covariates, i.e., if $\beta_0 = 0$, then \check{P} does not depend on x and there exist $2^T - rk(\check{P})$ linearly independent moment functions, which will not depend on x. Furthermore, the number of linearly independent moment functions available for the model without covariates is at least as large as the number of linearly independent moment functions available for the model that does include them, i.e., that allows $\beta_0 \neq 0$. In the appendix we show that $rk(\check{P}) \geq 2T$ irrespective of whether $\beta_0 = 0$ or $\beta_0 \neq 0$, that is, we prove Lemma 1, which states that the 2T columns of \check{P} corresponding to vectors y with either the first k or the last k elements equal to 1 and the remaining elements (if any) equal to 0 for k = 0, 1, 2, ... T are linearly independent. Recall that $rk(\check{P}) = rk(\bar{P})$. It follows that claim (i) is correct. We now show (ii):

It is easily seen that for any t_1 and s_1 with $t_1 < s_1 < T$, the 2^{t_1} moment functions

More generally, any 2T columns of \check{P} will be linearly independent if they correspond to the following 2T y-vectors: the two y-vectors that satisfy $y^S = 0$ or $y^S = T$ and for each $k \in \{1, 2, ..., T-1\}$ two y-vectors that satisfy $y^S = k$, one with $y_T = 0$ and the other with $y_T = 1$.

 $w_{y_1,\dots,y_{t_1-1}}(y_1,\dots,y_{t_1-1})m_{y_0}^{(a/b)(t_1,s_1,T)}$ are linearly independent because the 2^{t_1-1} indicator functions $w_{y_1,\dots,y_{t_1-1}}(y_1,\dots,y_{t_1-1})$ are linearly independent and $m_{y_0}^{(a)(t_1,s_1,T)}$ and $m_{y_0}^{(b)(t_1,s_1,T)}$ are linearly independent. Furthermore, any nontrivial linear combination of the moment functions $w_{y_1,\dots,y_{t_1-1}}(y_1,\dots,y_{t_1-1})m_{y_0}^{(a/b)(t_1,s_1,T)}(y,x,\beta,\gamma)$ with $t_1 < s_1 < T$ is linearly independent of $w_{y_1,\dots,y_{t_1-1}}(y_1,\dots,y_{t_1-1})m_{y_0}^{(a/b)(t_1,s_1,T)}(y,x,\beta,\gamma)$ with t < s < T and $(t,s) \neq (t_1,s_1)$ because only the former depends on $\exp[\pm z_{t_1,s_1}(y_0,y,x,\beta,\gamma)]$, where $z_{t_1,s_1}(y_0,y,x,\beta,\gamma) = (x_{t_1}-x_{s_1})'\beta + \gamma(y_{t_1-1}-y_{s_1-1})$. This is still true when $\beta = 0$. Hence the 2^T-2T functions $w_{y_1,\dots,y_{t-1}}(y_1,\dots,y_{t-1})m_{y_0}^{(a/b)(t,s,T)}(y,x,\beta,\gamma)$ are linearly independent. They are also valid moment functions, see Honoré and Weidner (2020). It follows that they span a 2^T-2T -dimensional linear subspace of the 2^T -dimensional vector space of real valued functions over the outcomes $y \in \{0,1\}^T$ that contains the valid moment functions.

Remark 1: The analysis above is also valid when there are no covariates, i.e., $\beta_0=0$. Remark 2: When $\beta_0\neq 0$, then \check{P} depends on x and part (i) of the proof implies that $rk(\check{P})\geq 2T$. However, part (ii) of the proof shows that there exist at least 2^T-2T linearly independent moment functions, which in turn implies that $rk(\check{P})\leq 2T$. We conclude that $rk(\check{P})=2T$. When $\beta_0=0$, the proof of the conjecture still implies that $rk(\check{P})=2T$.

Remark 3: It follows from the result under (i) that there are no valid moment functions when T=2. In other words, GMM estimation of the panel logit AR(1) model with fixed effects and possibly strictly exogenous covariates is not possible for T=2. Our proof is more general than that of Honoré and Weidner (2020) for this claim because we also cover the case where the values of α can only be finite. In their proof, Honoré and Weidner (2020) chose two of the four different values of α equal to $\pm \infty$, which leads to probabilities that are equal to 1 for the events where all elements of y are either zero or one. This unnecessarily restricts the moment functions a priori. In contrast, we also allow all the probabilities of observing a y-vector with only zeros or only ones to be less than 1.

Remark 4: The analysis above can also be extended to panel logit AR(p) models with fixed effects and p > 1.

Remark 5: The analysis above can also be used for the static panel logit model, i.e., when $\gamma_0 = 0$. In that case $\check{P}_{g,h} = \prod_{t=1}^T P_{g,t}^{y_t}$. When also $\beta_0 = 0$, \check{P} is equal to a matrix with columns from a Vandermonde matrix of rank T+1. It follows that when $\gamma_0 = 0$, the set of valid moment functions is a linear subspace of the 2^T -dimensional vector space of real valued functions over the outcomes $y \in \{0,1\}^T$ that has at most dimension $2^T - (T+1)$ and in particular that when T=2, there exists at most one valid moment condition.

2 Some results for the panel logit AR(2) model

When p = 2, we have

$$Pr(Y_{i} = y_{i}|Y_{i0} = y_{i0}, Y_{i,-1} = y_{i,-1}, X_{i} = x_{i}, A_{i} = \alpha_{i}) \equiv p_{y_{i}^{(0)}}(y_{i}, x_{i}, \beta_{0}, \gamma_{0}, \alpha_{i}) = \prod_{t=1}^{T} \frac{exp(x'_{it}\beta_{0} + \sum_{l=1}^{2} y_{i,t-l}\gamma_{l,0} + \alpha_{i})}{1 + exp(x'_{it}\beta_{0} + \sum_{l=1}^{2} y_{i,t-l}\gamma_{l,0} + \alpha_{i})},$$

where $y_i^{(0)} = (y_{i0}, y_{i,-1})'$ and $\gamma_0 = (\gamma_{1,0}, \gamma_{2,0})'$. We drop the index i. Let us redefine \bar{P} as a $2^T \times 2^T$ matrix with $\bar{P}_{g,h} = p_{y^{(0)}}(y, x, \beta_0, \gamma_0, \alpha_g)$ for $g, h = 1, 2, ..., 2^T$ with $h = 1 + 2^0 y_1 + 2^1 y_2 + ... + 2^{T-1} y_T$, and let us redefine \bar{P} as a $2^T \times 2^T$ matrix with $\bar{P}_{g,h} = P_{g,T}^{y_T} \prod_{t=2}^{T-1} \left(P_{g,t-1}(\frac{(1+P_{g,t})(1+P_{g,t+1})}{(1+P_{g,t}e^{\gamma_1})(1+P_{g,t+1}e^{\gamma_2})})^{1-y_{t-2}}(\frac{1+P_{g,t}}{1+P_{g,t}e^{\gamma_1+\gamma_2}})^{y_{t-2}}\right)^{y_{t-1}} \times \left(P_{g,T-1}(\frac{1+P_{g,T}}{1+P_{g,T}e^{\gamma_1}})^{1-y_{T-2}}(\frac{1+P_{g,T}}{1+P_{g,T}e^{\gamma_1+\gamma_2}})^{y_{T-2}}\right)^{y_{T-1}}$ for $g, h = 1, 2, ..., 2^T$ with $h = 1 + 2^0 y_1 + 2^1 y_2 + ... + 2^{T-1} y_T$. Note that with these new definitions of \bar{P} and \bar{P} , we still have $\bar{P} = \bar{D}\bar{P}\bar{D}$ for some nonsingular diagonal matrices $\bar{D} = \bar{D}(x, \beta_0, \gamma_0, \alpha)$ and $\bar{D} = \bar{D}(\gamma_0)$.

The formula for $P_{g,h}$ suggests that a second conjecture of Honoré and Weidner (2020), henceforth H&W, namely that the number of linearly independent moment functions for the general panel logit AR(p) models with covariates is given by $l = 2^{T} - (T (p+1)2^p$, is plausible: when p increases by one, the number of possible values for a ptuple $(y_{t-p}, y_{t-p+1}, \dots, y_{t-1})$, namely 2^p , doubles, while the number of different sets of p consecutive elements of $\{y_1, y_2, ..., y_{T-1}\}$ that appear in the products of powers in $\check{P}_{g,h}$ decreases by one (this number equals T-2 when p=2) and the factors in $\check{P}_{g,h}$ whose power depends on either y_T or y_0 account for 2^p more possibilities, which explains the (T-p+1) part of the formula. To prove H&W's second conjecture for p>1, one can in principle follow a similar proof strategy as for the case where p=1. However, when p > 1, things are a bit more complicated. As H&W demonstrate, when p > 1, the number of linearly independent moment functions for the general panel logit AR(p) model is smaller than the number of linearly independent moment functions for the panel logit AR(p) model without covariates (i.e., with $\beta_0 = 0$). One can relatively easily establish the latter number for different values of T by using a proof strategy similar to that for the case p=1. The difference between the two numbers of moment functions is equal to the number of linearly independent "special" moment functions that are only valid for "special" versions of the model, e.g. the model with $\beta_0 = 0$, but not for the general model. Thus by subtracting the number of these special moment functions from the total number of linearly independent moment functions for the model with $\beta_0 = 0$, one obtains the number of linearly independent moment functions for the general model.

H&W claim that they have found all moment functions for the general model when $T \leq 5$. However, their claim is premature as they have not shown that there cannot be more than l moment functions for the general model when $T \leq 5$. We have shown this above for p = 1 (and any T) and we will show this in the appendix for p = 2 and $T \leq 5$.

For the panel logit AR(2) model without covariates (i.e., with $\beta_0 = 0$), one can show that $rk(\check{P}) = 4(T-1) - (T-2) = 3T-2$, so that there are $2^T - (3T-2)$ linearly independent moment functions available for this model.³ One can easily obtain these by solving the system $\bar{P}_{[3T-2]}\bar{M}_{3T-2} = 0$, where $\bar{P}_{[3T-2]} = \bar{P}_{[3T-2]}(e^{\gamma_{1,0}}, e^{\gamma_{2,0}})$ is a $(3T-2) \times 2^T$ matrix that consists of (any) 3T-2 rows of the matrix \bar{P} , each evaluated at/corresponding to different values for the α_g , and \bar{M}_{3T-2} is a $2^T \times (2^T - (3T-2))$ matrix with $rk(\bar{M}_{3T-2}) = 2^T - (3T-2)$. The $2^T - (3T-2)$ columns of \bar{M}_{3T-2} span the nullspace of \bar{P} , which is the space of valid moment functions for the panel logit AR(2) model without covariates.

References

- [1] Honoré, B. E., and M. Weidner, 2020, Moment Conditions for Dynamic Panel Logit Models with Fixed Effects, arXiv:2005.05942v3 [econ.EM] 21 Jun 2020.
- [2] Kitazawa, Y., 2013, Exploration of dynamic fixed effects logit models from a traditional angle, Discussion paper No. 60, Kyushu Sangyo University, Faculty of Economics.
- [3] Kitazawa, Y., 2016, Root-N consistent estimations of time dummies for the dynamic fixed effects logit models: Monte Carlo illustrations, Discussion paper No. 72, Kyushu Sangyo University, Faculty of Economics.

²H&W have found one moment function for the panel logit AR(2) model with $\beta_0 = 0$ (given the value of $y^{(0)}$) when T = 3, which is a "special" moment function that is only valid when $x_2 = x_3$. However, they have not shown that when T = 3, there is *only* one moment function for this model.

 $^{^3}$ A proof strategy for the claim that $rk(\check{P}) = 3T - 2$ is discussed in the appendix.

A Appendix

Lemma 1 The 2T columns of \check{P} corresponding to vectors y with either the first k or the last k elements equal to 1 and the remaining elements (if any) equal to 0 for k = 0, 1, 2, ... T are linearly independent a.s. (almost surely) for any $T \geq 2$:

Proof: We will prove this Lemma by showing that the square matrix \widetilde{P}_{2T} (sometimes denoted by \widetilde{P} for short for some value of T) that contains the first 2T rows of these 2T columns of \check{P} has full rank for any $T \geq 2$. We will omit the subscript 0 from β_0 and γ_0 .

We will first consider the special (and most challenging) case where $\beta = 0$.

We define the elements of the matrix \widetilde{P}_{2T} as follows:

If y = (1, ..., 1, 0, ..., 0)' with the first k entries equal to 1 and $0 \le k \le T - 1$:

 $\widetilde{P}_{2T,g,h}$ (or simply $\widetilde{P}_{g,h}$ for some value of T) = $(e^{\alpha_g} \frac{1 + e^{\alpha_g}}{1 + e^{\alpha_g + \gamma}})^k$ for any $g \in \{1, 2, \dots, 2T\}$ and for h = 2k + 1;

if $y = (0, \dots, 0, 1, \dots, 1)'$ with the last k + 1 entries equal to 1 and $0 \le k \le T - 1$:

$$\widetilde{P}_{2T,g,h} = (\widetilde{P}_{g,h} =) e^{\alpha_g} (e^{\alpha_g} \frac{1+e^{\alpha_g}}{1+e^{\alpha_g+\gamma}})^k$$
 for any $g \in \{1, 2, \dots, 2T\}$ and for $h = 2(k+1)$.

Let
$$D_{2T} = diag(1 + e^{\alpha_1 + \gamma}, 1 + e^{\alpha_2 + \gamma}, \dots, 1 + e^{\alpha_{2T} + \gamma})$$
. Note that $det(D_{2T}) \neq 0$.

We will prove the Lemma by induction. When T=2, we consider the 4×4 matrix

$$D_4 \widetilde{P}_4 = \begin{bmatrix} 1 + e^{\alpha_1 + \gamma} & e^{\alpha_1} (1 + e^{\alpha_1 + \gamma}) & e^{\alpha_1} (1 + e^{\alpha_1}) & e^{2\alpha_1} (1 + e^{\alpha_1}) \\ 1 + e^{\alpha_2 + \gamma} & e^{\alpha_2} (1 + e^{\alpha_2 + \gamma}) & e^{\alpha_2} (1 + e^{\alpha_2}) & e^{2\alpha_2} (1 + e^{\alpha_2}) \\ 1 + e^{\alpha_3 + \gamma} & e^{\alpha_3} (1 + e^{\alpha_3 + \gamma}) & e^{\alpha_3} (1 + e^{\alpha_3}) & e^{2\alpha_3} (1 + e^{\alpha_3}) \\ 1 + e^{\alpha_4 + \gamma} & e^{\alpha_4} (1 + e^{\alpha_4 + \gamma}) & e^{\alpha_4} (1 + e^{\alpha_4}) & e^{2\alpha_4} (1 + e^{\alpha_4}) \end{bmatrix}, \text{ and it is easily verified}$$

that $rank(D_4\widetilde{P}_4)=4$ a.s. (Recall that $\gamma \neq 0$, note that any linear combination of the first two columns of $D_4\widetilde{P}_4$ depends on γ and conclude that the k-th column of $D_4\widetilde{P}_4$ cannot be written as a linear combination of the k-1 columns on its LHS for $k=2,\ldots,4$). As $\det(D_4)\neq 0$, it follows that $rank(\widetilde{P}_4)=4$ a.s.

Assuming that the Lemma is correct for T=S+2 for some $S\in\mathbb{N},$ we will now prove that it is also correct for T=S+3:

The $2(S+3)\times 2(S+3)$ matrix $\widetilde{P}=\widetilde{P}_{2(S+3)}$ contains the $2(S+2)\times 2(S+2)$ matrix $\widetilde{P}_{2(S+2)}$ (in the north-west corner) and two more rows and columns:

$$\widetilde{P}_{2(S+3)} = \begin{bmatrix} \widetilde{P}_{2(S+2)} & [\widecheck{P}_{g,2^{S+2}}]_{1 \leq g \leq 2(S+2)} & [\widecheck{P}_{g,2^{S+3}}]_{1 \leq g \leq 2(S+2)} \\ [\widetilde{P}_{2S+5,h}]_{1 \leq h \leq 2(S+2)} & \widecheck{P}_{2S+5,2^{S+2}} & \widecheck{P}_{2S+5,2^{S+3}} \\ [\widetilde{P}_{2S+6,h}]_{1 \leq h \leq 2(S+2)} & \widecheck{P}_{2S+6,2^{S+2}} & \widecheck{P}_{2S+6,2^{S+3}} \end{bmatrix}, \text{ where } \widecheck{P} \text{ is a } 2(S+3) \times 2(S+3) \text{ matrix.}$$

$$(S+3) \times 2(S+3)$$
 matrix. We can partition $D_{2(S+3)}^{S+2}\widetilde{P} = D_{2(S+3)}^{S+2}\widetilde{P}_{2(S+3)}$ as
$$\begin{bmatrix} D_{2(S+2)}^{S+2}\widetilde{P}_{2(S+2)} & B \\ C & F \end{bmatrix}.$$

Let $M \equiv D_{2(S+2)}^{S+2} \widetilde{P}_{2(S+2)} - BF^{-1}C$. Then it follows from a standard result regarding the determinants of partitioned matrices that $\det(D_{2(S+3)}^{S+2}\widetilde{P}) = \det(F)\det(M)$.

It is easily checked that F has full rank, i.e., rank(F) = 2: $F = \begin{bmatrix} (e^{\alpha_{2S+5}}(1+e^{\alpha_{2S+5}}))^{S+2} & e^{\alpha_{2S+5}}(e^{\alpha_{2S+5}}(1+e^{\alpha_{2S+5}}))^{S+2} \\ (e^{\alpha_{2S+6}}(1+e^{\alpha_{2S+6}}))^{S+2} & e^{\alpha_{2S+6}}(e^{\alpha_{2S+6}}(1+e^{\alpha_{2S+6}}))^{S+2} \end{bmatrix} \text{ so } \det(F) \neq 0 \text{ because } e^{\alpha_{2S+6}} - e^{\alpha_{2S+5}} \neq 0.$

It is also easily shown that $M=D_{2(S+2)}^{S+2}\widetilde{P}_{2(S+2)}-BF^{-1}C$ is invertible because it follows from Leibniz's formula for determinants (or from Laplace's expansion of the determinant, which uses cofactors and minors) that $\det(M)$ is equal to a polynomial in the elements of M, because this polynomial can be rewritten as a sum of terms that includes the term $\det(D_{2(S+2)}^{S+2}\widetilde{P}_{2(S+2)})$, because $\det(D_{2(S+2)}^{S+2}\widetilde{P}_{2(S+2)}) \neq 0$ a.s., and because (the sum of) all the other terms in this sum is/are a.s. incapable of canceling out $\det(D_{2(S+2)}^{S+2}\widetilde{P}_{2(S+2)})$:

Let
$$Q = BF^{-1}C \equiv \widetilde{Q}/\det(F)$$
. Then $Q_{g,h} = B_{g,}F^{-1}C_{.,h} = \widetilde{Q}_{g,h}/\det(F)$ with $\widetilde{Q}_{g,h} = (1+e^{\alpha_g+\gamma})^{S+2} \left[\widecheck{P}_{g,2^{S+2}} \quad \widecheck{P}_{g,2^{S+3}} \right] \left[\begin{array}{c} e^{\alpha_{2S+6}}(e^{\alpha_{2S+6}}(1+e^{\alpha_{2S+6}}))^{S+2} & -e^{\alpha_{2S+5}}(e^{\alpha_{2S+5}}(1+e^{\alpha_{2S+5}}))^{S+2} \\ -(e^{\alpha_{2S+6}}(1+e^{\alpha_{2S+6}}))^{S+2} & (e^{\alpha_{2S+5}}(1+e^{\alpha_{2S+5}}))^{S+2} \end{array} \right] \times \left[\begin{array}{c} \widetilde{P}_{2S+5,h}(1+e^{\alpha_{2S+5}})^{S+2} \\ \widetilde{P}_{2S+6,h}(1+e^{\alpha_{2S+6}})^{S+2} \end{array} \right] = \left[(e^{\alpha_g}(1+e^{\alpha_g}))^{S+2} \quad e^{\alpha_g}(e^{\alpha_g}(1+e^{\alpha_g}))^{S+2} \right] \times \left[\begin{array}{c} e^{\alpha_{2S+6}}(e^{\alpha_{2S+6}}(1+e^{\alpha_{2S+6}}))^{S+2} & -e^{\alpha_{2S+5}}(e^{\alpha_{2S+5}}(1+e^{\alpha_{2S+5}}))^{S+2} \\ -(e^{\alpha_{2S+6}}(1+e^{\alpha_{2S+6}}))^{S+2} & (e^{\alpha_{2S+5}}(1+e^{\alpha_{2S+5}}))^{S+2} \end{array} \right] \times \left[\begin{array}{c} (e^{\alpha_{2S+6}})^{\delta}(e^{\alpha_{2S+5}} & \frac{1+e^{\alpha_{2S+5}}}{1+e^{\alpha_{2S+5}+7}})^{\delta}(1+e^{\alpha_{2S+5}})^{S+2} \\ (e^{\alpha_{2S+6}})^{\delta}(e^{\alpha_{2S+6}} & \frac{1+e^{\alpha_{2S+6}}}{1+e^{\alpha_{2S+6}+7}})^{\delta}(1+e^{\alpha_{2S+6}})^{S+2} \end{array} \right] \text{ for some } k \in \{0,1,\ldots,S+1\} \text{ and some } \delta \in \{0,1\}. \text{ Omitting the factor } (e^{\alpha_g}(1+e^{\alpha_g})(1+e^{\alpha_{2S+5}})(1+e^{\alpha_{2S+5}}))^{S+2},$

$$\begin{split} \widetilde{Q}_{g,h} &\propto \left[1 - e^{\alpha_g}\right] \begin{bmatrix} e^{\alpha_{2S+6}} (e^{\alpha_{2S+6}})^{S+2} & -e^{\alpha_{2S+5}} (e^{\alpha_{2S+5}})^{S+2} \\ -(e^{\alpha_{2S+6}})^{S+2} & (e^{\alpha_{2S+5}})^{S+2} \end{bmatrix} \begin{bmatrix} (e^{\alpha_{2S+5}})^{\delta} (e^{\alpha_{2S+5}} \frac{1+e^{\alpha_{2S+5}}}{1+e^{\alpha_{2S+5}}})^k \\ (e^{\alpha_{2S+6}})^{\delta} (e^{\alpha_{2S+6}} \frac{1+e^{\alpha_{2S+5}}}{1+e^{\alpha_{2S+6}}})^k \end{bmatrix} = \\ e^{\delta\alpha_{2S+6}} \left(e^{\alpha_g} e^{(S+2)\alpha_{2S+5}} - e^{\alpha_{2S+5}} e^{(S+2)\alpha_{2S+5}} \right) \left(\frac{e^{\alpha_{2S+6}}}{e^{\gamma+\alpha_{2S+6}} + 1} (e^{\alpha_{2S+6}} + 1) \right)^k - \\ e^{\delta\alpha_{2S+5}} \left(e^{\alpha_g} e^{(S+2)\alpha_{2S+6}} - e^{\alpha_{2S+6}} e^{(S+2)\alpha_{2S+6}} \right) \left(\frac{e^{\alpha_{2S+5}}}{e^{\gamma+\alpha_{2S+5}} + 1} (e^{\alpha_{2S+5}} + 1) \right)^k = \\ e^{\delta\alpha_{2S+6}} e^{(S+2)\alpha_{2S+5}} \left(e^{\alpha_g} - e^{\alpha_{2S+5}} \right) \left(\frac{e^{\alpha_{2S+6}}}{e^{\gamma+\alpha_{2S+6}} + 1} (e^{\alpha_{2S+6}} + 1) \right)^k - \\ e^{\delta\alpha_{2S+5}} e^{(S+2)\alpha_{2S+6}} \left(e^{\alpha_g} - e^{\alpha_{2S+6}} \right) \left(\frac{e^{\alpha_{2S+5}}}{e^{\gamma+\alpha_{2S+6}} + 1} (e^{\alpha_{2S+5}} + 1) \right)^k. \end{split}$$

Note that the expression for $\widetilde{Q}_{g,h}$ cannot be rewritten as an expression that is divisible by the expression $e^{\alpha_{2S+6}} - e^{\alpha_{2S+5}}$ and hence that the expressions for all elements of Q are

ratios with the factor $e^{\alpha_{2S+6}} - e^{\alpha_{2S+5}}$ in the denominator. We conclude that $\det(M)$ can be written as the sum of $\det(D_{2(S+2)}^{S+2}\widetilde{P}_{2(S+2)})$ and one other term, (which itself is the result of summing almost all terms that appear in the aforementioned expansion of det(M) except for $\det(D_{2(S+2)}^{S+2}\widetilde{P}_{2(S+2)})$, and) which is an expression that is given by a ratio with the factor $e^{\alpha_{2S+6}} - e^{\alpha_{2S+5}}$ raised to some positive power appearing in the denominator (as a common factor) and with the same factor also appearing in the numerator but raised to lower positive powers than its power in the denominator so that its presence in the numerator does not completely cancel out this factor in the denominator. 4 However, none of the elements of $D_{2(S+2)}^{S+2}\widetilde{P}_{2(S+2)}$ depend on $e^{\alpha_{2S+5}}$ or $e^{\alpha_{2S+6}}$. It follows that $\det(M) \neq 0$ a.s. and that $\widetilde{P} = \widetilde{P}_{2(S+3)}$ is invertible a.s. (as we have already seen that $\det(F) \neq 0$), i.e., $rank(\widetilde{P}_{2(S+3)}) = 2(S+3)$ a.s. Another way of seeing this is that det(M) can be expressed as a ratio with a numerator that is a polynomial in e^{α_g} for $g=1,2,\ldots,2(S+2)$, in e^{γ} and, unless the second term ("the other term") in the aforementioned sum of two terms is zero (in which case $\det(M) = \det(D_{2(S+2)}^{S+2} \widetilde{P}_{2(S+2)}) \neq 0$ a.s.), also in $e^{\alpha_{2S+5}}$ and $e^{\alpha_{2S+6}}$. Hence det(M) = 0 if and only if this numerator equals zero. Given values of e^{α_g} for $g=1,2,\ldots,2(S+2)$ and e^{γ} , the numerator is a polynomial in $e^{\alpha_{2S+5}}$ and $e^{\alpha_{2S+6}}$ with a finite number of roots. As the values of α_g , $g=1,2,\ldots,2(S+3)$, and $\gamma\neq 0$ can be assumed to be randomly drawn from some continuous distribution(s), the probability that the values of $e^{\alpha_{2S+5}}$ and $e^{\alpha_{2S+6}}$ coincide with these roots is negligible. It follows that $\Pr(\det(M) \neq 0) = 1$ and hence that $\Pr(\det(\widetilde{P}_{2(S+3)}) \neq 0) = 1$.

The arguments generalize to the case where $\beta \neq 0$. Q.E.D.

An alternative proof of the claim that $rk(\bar{P}) = 2T$ for the panel logit AR(1) model without covariates (i.e., with $\beta_0 = 0$):

Consider the $2^T \times 2^T$ matrix \ddot{P} with typical element $\ddot{P}_{g,h} = (1 + P_{g,1}e^{\gamma})^{T-1}P_{g,1}^{y^S} \times \prod_{t=1}^{T-1}((1+P_{g,1})/(1+P_{g,1}e^{\gamma}))^{y_t}$ for $g,h=1,2,...,2^T$ with $h=1+2^0y_1+2^1y_2+...+2^{T-1}y_T$. Note that $\ddot{P}=\ddot{D}\breve{P}$ for some nonsingular diagonal matrix $\ddot{D}=\ddot{D}(\gamma_0,\alpha)$ and that the columns of \ddot{P} correspond to different polynomials in $P_{g,1}$ up to order 2T-1 with all intermediate powers occurring somewhere inside \ddot{P} . It follows that $rk(\bar{P})=rk(\ddot{P})$ is equal to the rank

⁴We have not investigated whether this second term (expression) in the sum is zero. If the latter were the case, we would have $\det(M) = \det(D_{2(S+2)}^{S+2} \tilde{P}_{2(S+2)}) \neq 0$ a.s., i.e., $\det(M) \neq 0$ a.s., which is what we want to show.

of a matrix that consists of linear combinations of the columns of a Vandermonde matrix that is based on powers of $P_{g,1}$ and has rank 2T. Hence $rk(\bar{P}) = rk(\bar{P}) \leq 2T$. To prove that $rk(\bar{P}) = rk(\bar{P}) = 2T$, it suffices to show that $rk(\bar{P}) \geq 2T$. This can be done by selecting the same 2T columns of \bar{P} as those of \bar{P} that underlie the definition of the matrix \bar{P} that is used in the proof of Lemma 1. Of course, it follows from Lemma 1, $rk(\bar{D}) = 2^T$ and $\bar{P} = \bar{D}\bar{P}$ that $rk(\bar{P}) \geq 2T$.

Analysis for the panel logit AR(2) model:

Proof strategy for the claim that $rk(\bar{P}) = 3T - 2$ for the panel logit AR(2) model without covariates (i.e., with $\beta_0 = 0$):

When $y_0=1$, we consider \ddot{P} with typical element $\ddot{P}_{g,h}=(1+P_{g,1}e^{\gamma_1})^{\lfloor 0.5(T-1)\rfloor}(1+P_{g,1}e^{\gamma_1}+\gamma_2)^{T-1}P_{g,1}^{y^S}\left((\frac{1+P_{g,1}}{1+P_{g,1}e^{\gamma_1}})^{1-y_{T-2}}(\frac{1+P_{g,1}}{1+P_{g,1}e^{\gamma_1+\gamma_2}})^{y_{T-2}}\right)^{Y_{T-1}}\times \prod_{t=2}^{T-1}\left((\frac{(1+P_{g,1})^2}{(1+P_{g,1}e^{\gamma_1})(1+P_{g,1}e^{\gamma_2})})^{1-y_{t-2}}(\frac{1+P_{g,1}}{1+P_{g,1}e^{\gamma_1+\gamma_2}})^{y_{t-2}}\right)^{Y_{t-1}}.$ Note that $\ddot{P}=\ddot{D}\breve{P}$ for some nonsingular diagonal matrix $\ddot{D}=\ddot{D}(\gamma_0,\alpha)$ and that the columns of \ddot{P} correspond to different polynomials in $P_{g,1}$ up to order 3(T-1) with all intermediate powers occurring somewhere inside \ddot{P} . It follows that $rk(\bar{P})=rk(\ddot{P})$ is equal to the rank of a matrix that consists of linear combinations of the columns of a Vandermonde matrix that is based on powers of $P_{g,1}$ and has rank 3T-2. Hence $rk(\bar{P})=rk(\ddot{P})\leq 3T-2$. To prove that $rk(\bar{P})=rk(\ddot{P})=3T-2$, it suffices to show that $rk(\ddot{P})\geq 3T-2$. This can be done by selecting 3T-2 suitable columns of \ddot{P} and showing that they are linearly independent similarly to the proof of Lemma 1.

When $y_0=0$, we consider \ddot{P} with typical element $\ddot{P}_{g,h}=(1+P_{g,1}e^{\gamma_1})^{\lfloor 0.5T\rfloor}(1+P_{g,1}e^{\gamma_2})^{\lfloor 0.5(T-1)\rfloor}(1+P_{g,1}e^{\gamma_1+\gamma_2})^{T-2}P_{g,1}^{y^S}\left((\frac{1+P_{g,1}}{1+P_{g,1}e^{\gamma_1}})^{1-y_{T-2}}(\frac{1+P_{g,1}}{1+P_{g,1}e^{\gamma_1+\gamma_2}})^{y_{T-2}}\right)^{y_{T-1}}\times \prod_{t=2}^{T-1}\left((\frac{(1+P_{g,1})^2}{(1+P_{g,1}e^{\gamma_1})(1+P_{g,1}e^{\gamma_2})})^{1-y_{t-2}}(\frac{1+P_{g,1}}{1+P_{g,1}e^{\gamma_1+\gamma_2}})^{y_{t-2}}\right)^{y_{t-1}}$. Note that $\ddot{P}=\ddot{D}\breve{P}$ for some nonsingular diagonal matrix $\ddot{D}=\ddot{D}(\gamma_0,\alpha)$ and that the columns of \ddot{P} correspond to different polynomials in $P_{g,1}$ up to order 3(T-1) with all intermediate powers occurring somewhere inside \ddot{P} . It follows that $rk(\bar{P})=rk(\ddot{P})$ is equal to the rank of a matrix that consists of linear combinations of the columns of a Vandermonde matrix that is based on powers of $P_{g,1}$ and has rank 3T-2. Hence $rk(\bar{P})=rk(\ddot{P})\leq 3T-2$. To prove that $rk(\bar{P})=rk(\ddot{P})=3T-2$, it suffices to show that $rk(\ddot{P})\geq 3T-2$. This can be done by selecting 3T-2 suitable columns of \ddot{P} and showing that they are linearly independent similarly to the proof of Lemma 1.

Proof of the second conjecture of H&W (2020) for p=2 and $T\in\{3,4,5\}$:

We have followed the proof strategy discussed above to show that when p=2 and $\beta_0 = 0$, then $rk(\bar{P}) = 3T - 2$ for all $T \in \{3, 4, 5\}$ and any $y_0 \in \{0, 1\}$. In particular, we have used Mathematica to verify that when p=2 and $\beta_0=0$, then $rk(\ddot{P})=3T-2$ for all $T \in \{3,4,5\}$ and any $y_0 \in \{0,1\}$. We note that when p=2, T=3 and $x_2=x_3$, there is (at least) one extra moment function relative to the number of linearly independent moment functions for the general model (given the value of $y^{(0)}$), cf. H&W (2020) who found one extra moment function for this case; by analogy, when p = 2, T = 4 and $x_2 = x_3 = x_4$, there will be (at least) two extra moment functions relative to the number of linearly independent moment functions for the general model (given the value of $y^{(0)}$); and when p = 2, T = 5 and $x_2 = x_3 = x_4 = x_5$, there will be (at least) three extra moment functions relative to the number of linearly independent moment functions for the general model (given the value of $y^{(0)}$). H&W (2020) also found $l=2^T-4(T-1)$ linearly independent moment functions for the general model (given the value of $y^{(0)}$) when p=2 and $T\in\{3,4,5\}$, so there are at least l of them in these cases. Hence the number of linearly independent "general" and "special" moment functions is at least $2^{T}-4(T-1)+(T-2)=2^{T}-(3T-2)$. However, this number cannot be larger than the number of linearly independent moment functions for the model without covariates (i.e., with $\beta_0 = 0$, which is $2^T - (3T - 2)$. We conclude that when p = 2 and $T \in \{3, 4, 5\}$, there are $2^T - (3T - 2) - (T - 2) = 2^T - 4(T - 1) = l$ linearly independent moment functions for the general model (given the value of $y^{(0)}$).

Q.E.D.