The 12th International Conference of
International Association for Computer Methods and Advances in Geomechanics (IACMAG)
1-6 October, 2008
Goa, India

The Effects of Geotechnical Material Properties on the Convergence of Iterative Solvers

C.E. Augarde, R.S. Crouch, T. Li
School of Engineering, Durham University, UK

A. Ramage
Dept of Mathematics, Strathclyde University, UK

Keywords: finite element analysis, iterative solvers, preconditioners

ABSTRACT: There is increasing interest in the use of iterative rather than direct solvers for geotechnical finite element analysis. For large 3D problems iterative solvers offer the only possibility of economical solution on a desktop PC. The major stumbling block with iterative solvers is ensuring fast convergence to a suitably accurate solution. The system of equations is always "preconditioned" to improve convergence and the design of preconditioners is a current hot topic in many areas of computational engineering. Effective preconditioning for geotechnical FE is particularly difficult due to (a) the wide range of elasto-plastic constitutive models used and (b) the changing nature of the equations during analysis (due to development of zones of plasticity for instance). In this paper we examine the features of some elasto-plastic material models that affect convergence of iterative solution methods, focussing on analysis of condition numbers of stiffness matrices. It is well-known that frictional material models lead to unsymmetric systems of equations and here we examine the role of the angle of dilation on system condition. The use of the consistent constitutive matrix, instead of the standard constitutive matrix is shown to have an effect on the condition numbers of the systems to be solved.

1 Introduction

Nonlinear finite element modelling in geotechnics is now routine. Engineers appear happy to use complex material models for soils and structures to determine displacements, porewater pressure distributions and the likelihood of instability. As the ambition of modellers increases, to complex multi-stage analyses and three-dimensions, so the time required to reach a solution increases. Serial hardware improvements alone are insufficient to meet the demand for more complex modelling. In an engineering practice it is common to find that the complexity of modelling is limited by the time required for a solution (e.g. maximum one day or overnight).
Confining study to either drained or undrained boundary value problems the linear systems arising take this general form:

$$\mathbf{Ku} = \mathbf{f} \tag{1}$$

where $\mathbf{K}$ is the structure stiffness matrix, $\mathbf{u}$ is the vector of nodal displacements and $\mathbf{f}$ is the vector of nodal loads. For the majority of constitutive models $\mathbf{K} = \mathbf{K}(\boldsymbol{\sigma})$ where $\boldsymbol{\sigma}$ is the vector of stresses from the stress tensor and therefore the systems are a result of linearization within an iterative nonlinear solution procedure, such as Modified Newton-Raphson etc.
Direct methods for solving Equation 1 are now highly optimized often based on frontal solution techniques, and are the methods of choice for small systems (e.g. anything in 2D). For very large systems in 3D direct solvers lose advantage to iterative solvers, which do not require the comparatively larger storage requirements during solution of the former. Iterative solvers are however less well-developed for geotechnical constitutive models. Many of the advances in this area have arisen outside geotechnics and have concentrated on solvers for symmetric linear systems which are unsuitable for many soil models as discussed below. However some advances have been made in recent years in areas such as the modelling of consolidation (e.g. Chan et al. 2001; Bergamaschi et al. 2007) and 3D modelling (Mroueh & Shahrour 1999)
The simplest iterative solution method is conjugate gradients (CG), originally developed by Hestenes & Stiefel (1952). An initial guess is made for $\mathbf{u}$ (often the zero vector) followed by successive updates based on residuals. It can be shown that the number of iterations required for convergence to within a prescribed tolerance is proportional to the square root of the condition number $\kappa = \lambda_{\max}/\lambda_{\min}$ where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $\mathbf{K}$. To accelerate convergence it is necessary to apply a preconditioning matrix $\mathbf{P}$

to change the system solved from Equation 1 to:

$$\mathbf{P}^{-1}\mathbf{K}\mathbf{u} = \mathbf{P}^{-1}\mathbf{f} \tag{2}$$

producing the preconditioned conjugate gradient method (PCG). Successful use of the PCG method thus requires the condition number of $\mathbf{P}^{-1}\mathbf{K}$ to be significantly lower than the unpreconditioned value. Determination of suitable preconditioners is a vibrant research area in numerical analysis.

The simplest form of preconditioning sets $\mathbf{P} = diag(\mathbf{K})$, hence the name *diagonal* scaling. Geotechnical FE modelling using PCG with diagonal scaling is described in Smith and Griffiths (2004) and in Potts and Zdravković (2001) who present results of 3D linear elastic analyses and show that the efficiency of this iterative approach is affected strongly by Poisson's ratio (a point we will return to below). They also indicate some other reasons why direct solvers are, in many cases, better than iterative solvers, although perhaps not for very large problems.
The structure stiffness matrix $\mathbf{K}$ is formed by assembly of element stiffness matrices $\mathbf{K}_e$ where

$$\mathbf{K}_e = \int_V \mathbf{B}^T \mathbf{D}\mathbf{B}\, dV \tag{3}$$

and $\mathbf{B}$ is the strain-displacement matrix, $\mathbf{D}$ the constitutive matrix and the integration is taken over the element volume V. The nature of the linear system in Equation 1 is therefore related to the constitutive matrices $\mathbf{D}$ of the integration points internal to the elements. For linear elasticity $\mathbf{D}$ is fixed so it is easy to predict the nature of the linear system, and hence determine the preconditioning strategy to adopt. For elasto-plasticity, the nature of the linear system at any stage in the analysis cannot be determined *a priori* since $\mathbf{D} = \mathbf{D}_{ep}(\boldsymbol{\sigma})$, i.e. the constitutive matrix is a function of the current stress state. However by studying the nature of $\mathbf{D}$ it is possible to make some general statements about preconditioning. Probably because of the dominance of direct solvers relatively few researchers have investigated the changing nature of the elasto-plastic constitutive matrix $\mathbf{D}_{ep}$ for geomaterials. In her doctoral thesis and later publications (Van der Veen 1998) describes analysis of the eigenvalue spectra of $\mathbf{D}_{ep}$ for some simple failure criteria under various loading paths. $\mathbf{D}_{ep}$ on its own is always singular (unless there is hardening) so that the spectrum will not necessarily provide a clear measure of the condition number of the stiffness matrix (Equation 3) and hence of the ease of solution using iterative methods.

The aim of this paper is to demonstrate the effects of elastic and plastic parameters on the systems that must be solved in geotechnical FE analyses. While we have indicated the advantages of iterative methods to lie with 3D FE analyses, most discussion and some results in this paper will relate to the 2D case (either plane strain or axisymmetry) for simplicity.

2 Effects of elastic parameters

Despite the necessary concentration on the plastic part of the constitutive model by geotechnical researchers, the majority of a given problem domain during an analysis will be elastic. Therefore it is important to develop robust preconditioning techniques to deal with elasticity as much as plasticity Augarde et al. (2005). Taking Young's modulus E and Poisson's ratio ν as our elastic parameters Augarde et al. (2006), and others, have shown that the effect of the former is to scale all entries in $\mathbf{D}$ which has the effect of scaling all eigenvalues and hence the condition number of the system is unchanged. The effect of Poisson's ratio is however significant. The constitutive matrix for 2D plane strain linear elasticity

$$\mathbf{D}_e = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \tag{4}$$

where the stress vector associated with $\mathbf{D}_e$ is $\boldsymbol{\sigma} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}$, σ_{zz} being obtainable from $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$. At values close to 0.5 (as often employed to model undrained behaviour) the condition number rises considerably and convergence is severely affected. Figure 1 shows the variation in the condition numbers of the structure stiffness matrix and the constitutive matrix for a single six-noded right angled triangular element with max side length unity and $E = 10$ from which it is clear that the latter rises at a greater rate than the former as high values of Poisson's ratio are used and hence the role of the constitutive matrix is significant.

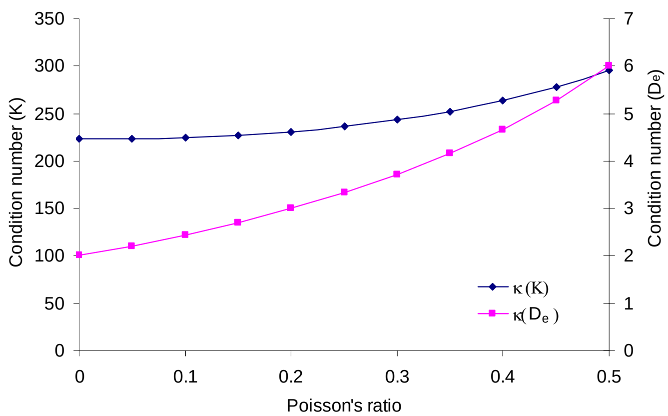

Figure 1. Effect of Poisson's ratio on condition numbers for a single element stiffness matrix.

3 Plasticity

Stability issues preclude FE analyses where very much of the problem domain is yielding and if we restrict study to applications with monotonic loading (for simplicity) an analysis begins with small or no yielded zones. These two points mean that the majority of time is spent in the majority of analyses solving systems which are dominated by elastic effects, as dealt with above. It seems obvious however that including plasticity in a soil model will alter the nature of the linear systems produced in FE calculations, and hence may require a change to the solution strategy and this is the subject of the following sections.

Changes to the constitutive matrix $\mathbf{D}$ at the onset of yielding are affected by the yield criterion $F = F(\boldsymbol{\sigma}, \mathbf{k})$ and the plastic potential $P = P(\boldsymbol{\sigma})$ where $\mathbf{k}$ is a vector of hardening parameters dependent on the type of hardening.

For elasto-plasticity the constitutive matrix can be shown to be

$$\mathbf{D}_{ep} = \mathbf{D}_e - \frac{\mathbf{D}_e P,_{\boldsymbol{\sigma}} F,_{\boldsymbol{\sigma}}{}^T \mathbf{D}_e}{F,_{\boldsymbol{\sigma}}{}^T \mathbf{D}_e P,_{\boldsymbol{\sigma}} + A} \tag{5}$$

where

$$F,_{\boldsymbol{\sigma}} = \left\{ \frac{\partial F}{\partial \sigma_{xx}}, \frac{\partial F}{\partial \sigma_{yy}}, \frac{\partial F}{\partial \sigma_{xy}} \right\}^T, P,_{\boldsymbol{\sigma}} = \left\{ \frac{\partial P}{\partial \sigma_{xx}}, \frac{\partial P}{\partial \sigma_{yy}}, \frac{\partial P}{\partial \sigma_{xy}} \right\}^T \tag{6}$$

and A is related to the hardening parameters in $\mathbf{k}$. For associated flow $F = P$ and for non-associated flow $F \neq P$ and from Equation 5 it can be seen this implies symmetric $\mathbf{D}_{ep}$ for the former and unsymmetric $\mathbf{D}_{ep}$ for the latter. This feature has serious implications for iterative solvers in geotechnical FE, where non-associated plasticity is routinely used (e.g. with the Mohr-Coulomb criterion). The linear system requiring solution (Equation 1) may change from symmetric to unsymmetric during the analysis, requiring a change both to the type of solver used and the preconditioning strategy.

4 Numerical experiments

Augarde et al. (2007) details initial experiments to study the effect of plasticity on the performance of iterative solvers for geotechnical FEA focussing on iteration counts for analyses. In this study further finite element calculations have been carried out to investigate the effect of some geotechnical material models on the likely speed of solution using iterative methods focussing on the stiffness matrix condition numbers. Analysis concentrates on the eigenvalue spectrums of the linear systems produced, since as detailed above, it is this which determines rate of convergence to a solution using an iterative method.

Two very simple yield criteria were used in this study: von Mises and Matsuoka models. The former is suitable for undrained analysis since it can provide an approximation to a Tresca surface which is smooth, hence avoiding the numerical problems met when stress paths hit a vertex. The Matsuoka model is another smooth yield surface (apart from at the cone apex) which has been widely used to model drained frictional soils.

To investigate the effect of plasticity on the ease with which an iterative solver can be used in FE calculations a number of studies at the element and problem level have been carried out. Firstly consider the 2D plane strain smooth footing problem shown in Figure 2. The problem is geometrically symmetric about the vertical centreline and can therefore be analysed with a mesh of dimensions 10B representing one-half of the problem. A sequence of analyses using increasingly refined structured grids of the simplest elements (3-noded triangles) and with varying Poisson's ratios were undertaken. This is clearly an inefficient meshing paradigm in practice but is used here to avoid obscuring the effects of changing material parameters, which are our main concern. Initial studies were carried out using an associated perfect plasticity model (von Mises) where $F = \sqrt{J_2} - \alpha$; J_2 is the second invariant of the deviatoric stress tensor and α is a strength measure related to undrained shear strength (Smith & Griffiths, 2004; Potts & Zdravković, 2001) This criterion is used primarily for undrained analyses setting Poisson's ratio close to 0.49; its main advantage is the surface smoothness removing problems with corners experienced by some other surfaces. Condition numbers for the structure stiffness matrices for this problem preconditioned using diagonal scaling are given in Figure 3. Plots are shown on log-log axes. N equals the number of element sides per side of the square domain (so that N = 256 corresponds to 131072 elements and 66049 nodes). Each analysis is carried out over 100 steps over which the prescribed displacement is applied. The first element Gauss points go plastic around the 30th step.

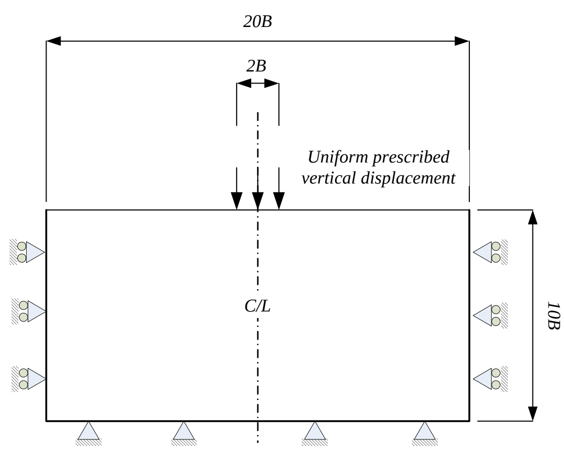

Figure 2. Smooth footing problem.

All plots in Figure 3 are shown with identical axis scaling to allow easy comparisons to be made. The first feature to note is the increase in condition numbers with Poisson's ratio for all analyses, matching the conclusions to be drawn from analysis of the elastic constitutive matrix. Secondly the scatter in condition numbers between different steps reduces with increasing Poisson's ratio, again a consequence of the influence of the elastic rather than the plastic parameters. At low Poisson's ratios the influence of plasticity is more significant as seen in the difference between steps (i.e. the results at steps 50 and 100 contain significant zones of yield as compared to the results at steps 1 and 25). Overall however, the magnitudes of the maximum condition numbers reached tend to the same value regardless of elastic or plastic features.

For drained analysis the failure surface due to Matsuoka (1976) is often used. It too is smooth (apart from the apex) and can be written as

$$F = I_1 I_2 - \varsigma I_3 \tag{7}$$

where I_1, I_2, I_3 are the invariants of the stress tensor,

$$\varsigma = 9 + 8\tan^2 \phi_{tc} \tag{8}$$

and ϕ_{tc} is the triaxial compression friction angle (Burd et al. 1989).

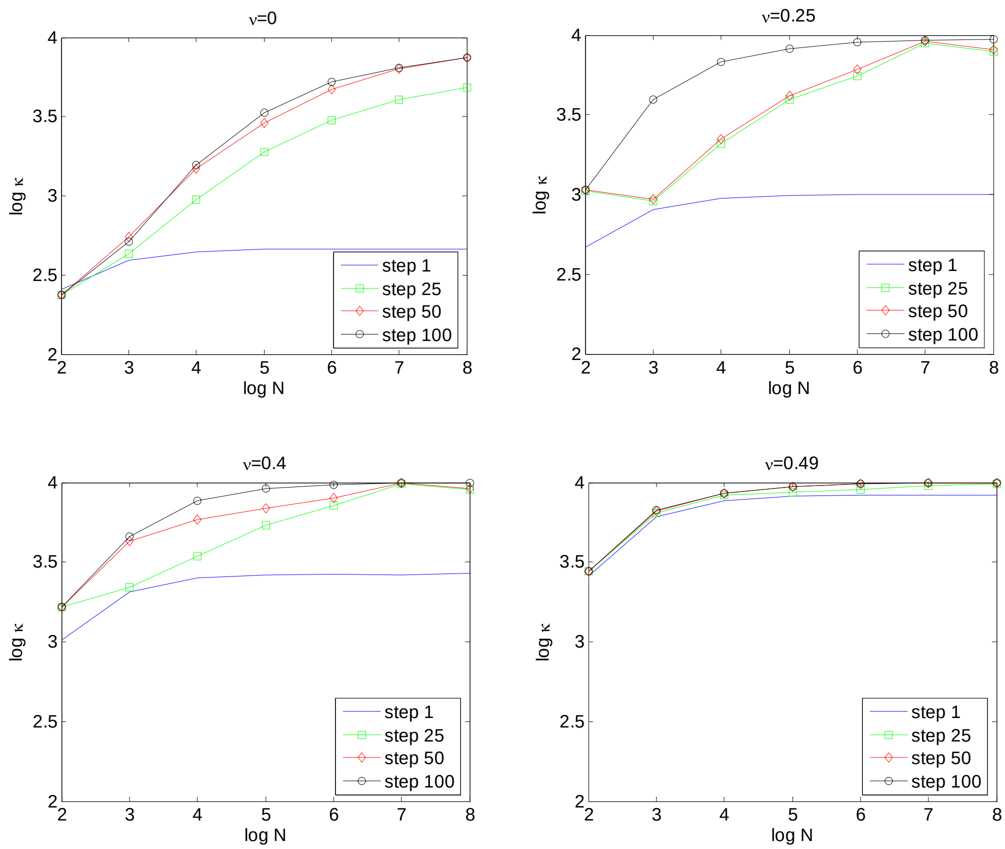

Figure 3. Log-log plots of condition number against N for varying Poisson's ratios for the footing problem.

A number of analyses of the same footing problem as used for the associated case have been carried out using the Matsuoka surface and a non-associated flow rule. The plastic potential employed is in the form of Equation 7 but with angle of dilation ψ replacing ϕ_{tc} (= 26.5° throughout). All analyses use the same Poisson's ratio of 0.35 to make the effect of plasticity clear, and to match typical values used in a drained analysis. The results are presented in Figure 4 for three values of degree of association γ_a (Burd et al. 1989) where $\gamma_a = 1.0$ is fully-associated and $\gamma_a = 0$ is fully non-associated. While the maximum refinements used in these analyses are not as great as the associated case the condition numbers are in all cases higher than those found for the associated case indicating the additional difficulties one would meet obtaining convergence with an iterative solver and this material model. Another interesting feature is there does not appear to be any levelling out of κ with increasing N as witnessed in the associated case.

5 Use of the consistent tangent constitutive matrix

One of the most important (and computationally intensive) steps in elasto-plastic FE analysis is stress updating.

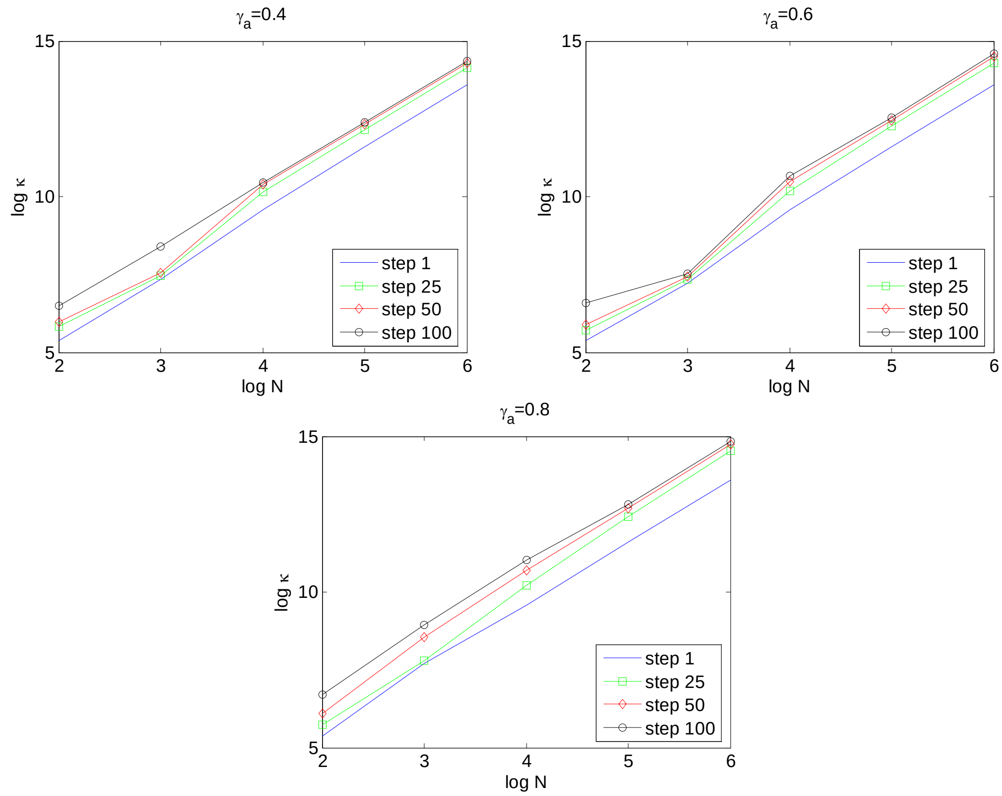

Figure 4: Log-log plots of condition number against N for the footing problem with the Matsuoka failure criterion.

Strains at each element integration point are determined from the current displacement solution. Stresses are then derived from these strains consistent with the plasticity model used. The FE modelling described above makes use of an explicit (or Forward Euler) scheme of stress updating in which the constitutive matrix is $\mathbf{D}_{ep}$, as specified in Equation 5. Greater accuracy can be achieved using a Backward Euler or implicit approach providing that the considerably more complicated expressions required for this algorithm can be obtained for the material model (Jeremic & Sture, 1997). An implicit approach replaces $\mathbf{D}_{ep}$ with the *consistent tangent,* or *algorithmic* constitutive matrix $\mathbf{D}_{al}$ (Simo & Taylor, 1985). It is the algorithmic constitutive matrix and not the elasto-plastic constitutive matrix which can lead to quadratic convergence in a global Newton-Raphson scheme. In this work we are interested in the effect on the condition of the linear system when $\mathbf{D}_{al}$ is used instead of $\mathbf{D}_{ep}$.

It can be shown (e.g. Simo & Hughes, 1997) that for perfect plasticity,

$$\mathbf{D}_{al} = \mathbf{A}_{mm} - \frac{\mathbf{A}_{mn} P,_{\boldsymbol{\sigma}} F,_{\boldsymbol{\sigma}}{}^{T} \mathbf{A}_{mm}}{F,_{\boldsymbol{\sigma}}{}^{T} \mathbf{A}_{mn} P,_{\boldsymbol{\sigma}}} \tag{9}$$

where $\mathbf{A}_{mm}$ etc are blocks of the matrix $\mathbf{A}$ and

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{mm} & \mathbf{A}_{mn} \\ \mathbf{A}_{nm} & \mathbf{A}_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{e} & \mathbf{I} \\ \lambda P^{T}_{,\sigma\sigma} & -\mathbf{I} \end{bmatrix}^{-1} . \tag{10}$$

In Equation 10 $\mathbf{C}_e$ is the elastic compliance matrix (i.e. the inverse of the matrix in Equation 4), $\mathbf{I}$ is the unit matrix of appropriate size and λ is the conventional incremental plastic multiplier. The most widely used implicit stress integration algorithm is the Backward-Euler Closest Point Projection method (CPPM). For an isotropic model, the principal directions of the final (converged) stresses will be identical to those of the trial stress state. Thus the constitutive model algorithm can be expressed in terms of the principal values (with appropriate transformations to and from the global coordinate system prior to entering and after leaving the routine). The split-operation CPP approach first treats the total strain increment as producing a purely elastic response, thereby arriving at an initial predictor trial stress state. If this lies inside the yield surface then no plastic deformation will have occurred during that strain increment; otherwise, an elasto-plastic response must have taken place. Given the stress state and plastic strains at step *n* plus the new calculated elastic predictor stresses, the central tasks of the CPP algorithm are to find at step $n+1$ (i) the corrective up-dated (return) stresses (ii) the converged plastic strains that simultaneously result in zero residual plastic strains and satisfy the consistency condition. Finally the *algorithmic* constitutive matrix $\mathbf{D}_{al}$ corresponding to the final converged state is calculated.

The effect of the choice of constitutive matrix has been studied by examining the condition number of a single element stiffness matrix as carried out for the elastic case in Section 2 above. Here however an eight-noded brick element is used. Table 1 shows the ratios of $\mathbf{D}_{al}$ to $\mathbf{D}_{ep}$ for a range of material parameters using the Matsuoka model with the same starting stress state (zero stress) and the same predictor stress state $\sigma_1 = 1.0, \sigma_2 = 0.4, \sigma_3 = 0.3$ (in MPa). (Note that these values are not meant to represent any realistic stress state found in geotechnics, rather their relative ratios are important). In Table 1 $\sigma_1^*, \sigma_2^*, \sigma_3^*$ are the converged principal stresses returned by the CPPM which differ due to the varying plastic parameters used. In all cases shown I the Table (and in a large number of other trials not reported here for brevity) the condition number of the stiffness matrix using the consistent constitutive matrix is higher than that using the standard matrix although the ratio is generally found to lie between 1.1 and 2.5.

Extrapolating these findings to the modelling of a boundary value problem is the subject of our current research, although it seems reasonable to presume that these increased condition numbers at the single element level will also be evident for a mesh of elements in which some have yielding integration points.

Table 1: Ratios of condition nos. for different constitutive matrices.

σ_1^*	σ_2^*	σ_3^*	ν	ϕ	φ	Ratio of κ values
0.9626	0.43	0.3635	0	25	25	1.183673
0.9926	0.4457	0.3739	0.25	25	25	1.594488
0.9455	0.4206	0.3579	0	25	20	1.24186
0.9618	0.429	0.3635	0.25	25	20	2.33758
0.9773	0.4381	0.3683	0	25	30	1.157068
1.014	0.4576	0.3808	0.25	25	30	1.358974

6 Conclusions

The strong desire of geotechnical modellers to run 3D analyses is likely to raise the profile of iterative, rather than direct, solution methods for geotechnical FEA. Unlike direct solvers, the nature of the linear system has a significant affect on the speed or indeed the possibility of solution. The linear systems solved in geotechnical FEA are strongly influenced by the nature of the material model and the particular stresses developed in the problem domain during the analysis. The role of elastic parameters is clear and surprisingly important given that most of a problem domain in a typical geotechnical FE model will remain elastic throughout an analysis. The effects of plasticity are also important particularly when dealing with constitutive models that lead to unsymmetric linear systems. The use of implicit rather than explicit stress updating procedures changes the way that the constitutive matrix is derived and worsens the situation for an iterative solver somewhat.

7 References

Augarde C.E., Ramage A., Staudacher J. 2007. Element-based preconditioners for elasto-plastic problems in geotechnical engineering, *International Journal for Numerical Methods in Engineering*, **71**, 757-779.

Augarde C.E., Ramage A., Staudacher J. 2006. An element-based displacement preconditioner for linear elasticity problems. *Computers and Structures*, **84**, 2306–2315.

Augarde C.E. Staudacher J., Ramage A. 2005. On preconditioning strategies for geotechnics. Proc. 11th International

Conference of IACMAG, Turin (Italy), 3-11.

Bergamaschi L., Ferronato M., Gambolati G. 2007. Novel preconditioners for the iterative solution to FE-discretized coupled consolidation equations, *Computer Methods in Applied Mechanics and Engineering*, **196**, 2647-2656.

Burd H.J., Yu H.S., Houlsby G.T. 1989. Finite Element Implementation of Frictional Plasticity Models with Dilation,Proceedings of the International Conference on Constitutive Laws for Engineering Materials, Chongqing, China, 783-787.

Chan S.H. Phoon K.K., Lee F.H. 2001. A modified Jacobi preconditioner for solving ill-conditioned Biot's consolidation equations using symmetric quasi-minimal residual method, *International Journal for Numerical and Analytical Methods in Geomechanics*, **25**, 1001-1025.

Hestenes M.R., Stiefel E. 1952. Methods of Conjugate Gradients for Solving Linear Systems, *J. Res. Nat. Bur. Stand.* **49**, 409-436.

Jeremic B., Sture S. 1997. Implicit integrations in elastoplastic geomechanics, *Mechanics of Cohesive-frictional Materials*, **2**, 165-183.

Matsuoka H. 1976. On the significance of the spatial mobilised plane, *Soils and Foundations*, **16**(1):91-100

Mroueh H., Shahrour I. 1999. Use of sparse iterative methods for the resolution of three-dimensional soil/structure interaction problems, *International Journal for Numerical and Analytical Methods in Geomechanics*. **23**, 1961-1975.

Potts D.M., Zdraković L. 1999. *Finite element analysis in geotechnical engineering: theory*. Thomas Telford, London (UK).

Simo J.C., Hughes T.J.R. 1997. *Computational Inelasticity*. Springer, New York (USA).

Simo J.C., Taylor R.L. 1985. Consistent tangent operators for rate independent plasticity. *Computer Methods in Applied Mechancis and Engineering*, **48**, 101-118.

Smith I.M., Griffiths, D.V. 2004. *Programming the finite element method,* John Wiley and Sons, Chichester (UK).

Van der Veen H. 1998. The significance and use of eigenvalues and eigenvectors in the numerical analysis of elastoplastic soils, Ph.D. Thesis, Delft University of Technology.