Pancyclicity and panconnectivity in augmented k-ary n-cubes

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***Abstract*—The augmented k-ary n-cube $AQ_{n,k}$ is a recently proposed interconnection network that incorporates an extension of a k-ary n-cube Q_n^k inspired by the extension of a hypercube Q_n to the augmented hypercube AQ_n (as developed by Choudom and Sunita). We extend a recent topological investigation of augmented k-ary n-cubes by proving that any augmented k-ary n-cube $AQ_{n,k}$ is edge-pancyclic and that $AQ_{2,k}$ is panconnected.**

***Keywords*-interconnection networks; pancyclicity; panconnectivity; augmented k-ary n-cube;**

I. Introduction

Hypercubes are perhaps the most well known of all interconnection networks for parallel computing, given their basic simplicity, their generally desirable topological and algorithmic properties, and the extensive investigation they have undergone (not just in the context of parallel computing but also in discrete mathematics in general; see, for example, [13] for some essential properties of hypercubes). However, a multitude of different interconnection networks have been devised and developed in a continuing search for improved performance, with many of these networks having hypercubes at their roots. Amongst these generalisations of hypercubes are k-ary n-cubes [4], augmented cubes [2], cube-connected cycles [12], twisted cubes [8], twisted n-cubes [7], crossed cubes [5], folded hypercubes [6], Möbius cubes [3], generalised twisted cubes [1], shuffle cubes [11], k-skip enhanced cubes [15], twisted hypercubes [10], and Fibonacci cubes [9]. Perhaps the most popular of these generalisations are the k-ary n-cubes. Having the two parameters k and n available allows us to regulate the degree of the nodes yet still incorporate large numbers of processors, although usually at a cost to some other property such as the diameter or the connectivity.

However, recently an interconnection network has been proposed that can be viewed as incorporating not just one but two of the above generalisation techniques. In [16], generalisations of k-ary n-cubes, namely *augmented k-ary n-cubes*, have been proposed as interconnection networks for parallel computing, inspired by Choudum and Sunitha's generalisation of hypercubes as augmented cubes [2]. A k-ary n-cube Q_n^k is extended to an augmented k-ary n-cube $AQ_{n,k}$ in a manner analogous to the extension of an n-dimensional hypercube Q_n to an n-dimensional augmented cube AQ_n; however, the latter extension is more involved than the former, as we now explain. The hypercube Q_n and the k-ary n-cube Q_n^k are spanning subgraphs of the augmented hypercube AQ_n and the augmented k-ary n-cube $AQ_{n,k}$, respectively. In order to build the augmented hypercube AQ_n, one takes two copies of an $(n-1)$-dimensional augmented cube AQ_{n-1} and as well as joining corresponding pairs of vertices, as one does in the hypercube construction, one also joins pairs of vertices of Hamming distance $n-1$ (that is, vertices that are different in every component). In order to build the augmented k-ary n-cube $AQ_{n,k}$, one takes k copies of an augmented k-ary $(n-1)$-cube $AQ_{n-1,k}$ and as well as joining these copies as one would in order to form a k-ary n-cube, one also includes two other edges for every vertex v: one edge going to the vertex whose every component is 1 less that of v (modulo k); and one edge going to the vertex whose every component is 1 plus that of v (modulo k) (precise algebraic definitions are given in the next section). In consequence, the augmented k-ary n-cube $AQ_{n,k}$ is a k-ary n-cube with additional edges. The augmented 5-ary 2-cube is depicted in Fig. 1 (in two different ways, showing different embeddings of Q_2^5 within $AQ_{2,5}$).

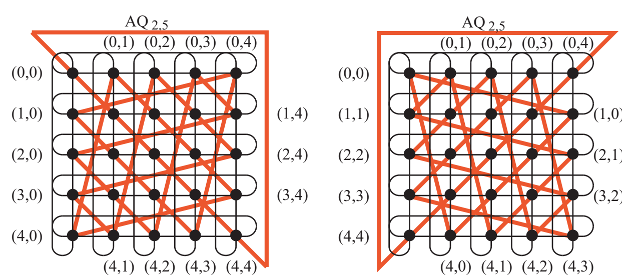


Figure 1. Two views of an augmented 5-ary 2-cube.

Some essential properties of the augmented k-ary n-cube $AQ_{n,k}$ in comparison with the k-ary n-cube Q_n^k are given in Fig. 2 (see [16] for more details). As can be seen, the augmented k-ary n-cube $AQ_{n,k}$ compares very favourably with the k-ary n-cube Q_n^k. Furthermore, and importantly, the augmented k-ary n-cube $AQ_{n,k}$ is 'built on top' of the k-ary n-cube AQ_n^k; that is, Q_n^k is a spanning subgraph of $AQ_{n,k}$. Thus, all routing and broadcasting algorithms which work for Q_n^k also work for $AQ_{n,k}$.

	Q_n^k	$AQ_{n,k}$
vertices/edges	k^n/nk^n	$k^n/(2n-1)k^n$
vertex-/edge-symmetric	yes/yes	yes/no unless $n=2$
connectivity	$2n$	$4n-2$
wide-diameter ($n \geq 3$)	$n\lfloor \frac{k}{2} \rfloor + 1$	$\leq max\{(n-1)k$ $-(n-2), k+7\}$
wide-diameter ($n=2$)	$2\lfloor \frac{k}{2} \rfloor + 1$	$\leq k$
diameter ($n \geq 3$)	$n\lceil \frac{k}{2} \rceil$	$\leq \frac{k}{4}(n+1)$ (k even) $\leq \frac{k}{4}(n+1) + \frac{n}{4}$ (k odd)
diameter ($n=2$)	$2\lceil \frac{k}{2} \rceil$	$\lfloor \frac{k}{3} \rfloor + \lceil \frac{k-1}{3} \rceil$
routing time	$O(nk)$	$O(nk)$

Figure 2. A comparison between Q_n^k and $AQ_{n,k}$.

In this paper, we further investigate the topological properties of augmented k-ary n-cubes; in particular, pancyclicity and panconnectivity. Path and cycle networks are fundamental in parallel computing; for not only is there a multitude of algorithms specifically designed for linear arrays of processors and cycles of processors but paths and cycles appear as data structures in many more algorithms for parallel machines whose processors are inter-connected in a variety of topologies. For example, having a collection of processors connected in a cycle means that all-to-all message passing can be undertaken by 'daisy-chaining' messages around the cycle. In Section 2, we provide the basic definitions and concepts relating to this paper. Our main results are proven in Sections 3 and 4. In particular, in Section 3 we prove that any augmented k-ary n-cube $AQ_{n,k}$ is edge-pancyclic, and in Section 4 that $AQ_{2,k}$ is panconnected. In Section 5, we present our conclusions and discuss panconnectivity in $AQ_{n,k}$ when $n \geq 3$.

II. Basic definitions

We assume throughout that arithmetic on tuple elements is modulo k, and we denote tuples of elements by bold type. Recall the definition of the *k-ary n-cube* Q_n^k: the vertex set $V(Q_n^k)$ is $\{(a_n, a_{n-1}, \ldots, a_1) : 0 \leq a_i \leq k-1\}$; and the edge set $E(Q_n^k)$ is $\{(\mathbf{u}, \mathbf{v}) : \mathbf{u} = (u_n, u_{n-1}, \ldots, u_1), \mathbf{v} = (v_n, v_{n-1}, \ldots, v_1)$, either $u_i = v_i - 1$ or $u_i = v_i + 1$, for some i, and $u_j = v_j$, for all $i \neq j\}$. We regard all graphs defined in this paper as undirected.

An augmented k-ary n-cube is defined as follows.

Definition 1: Let $n \geq 1$ and $k \geq 3$ be integers. The *augmented k-ary n-cube* $AQ_{n,k}$ has k^n vertices, each labelled by an n-bit string $(a_n, a_{n-1}, \ldots, a_1)$, with $0 \leq a_i \leq k-1$, for $1 \leq i \leq n$. There is an edge joining vertex $\mathbf{u} = (u_n, u_{n-1}, \ldots, u_1)$ to vertex $\mathbf{v} = (v_n, v_{n-1}, \ldots, v_1)$ if, and only if:

- $v_i = u_i - 1$ (resp. $v_i = u_i + 1$), for some i such that $1 \leq i \leq n$, and $v_j = u_j$, for all j such that $1 \leq j \leq n$ and $j \neq i$; or
- for some i such that $2 \leq i \leq n$, $v_i = u_i - 1, v_{i-1} = u_{i-1} - 1, \ldots, v_1 = u_1 - 1$ (resp. $v_i = u_i + 1, v_{i-1} = u_{i-1} + 1, \ldots, v_1 = u_1 + 1$), and $v_j = u_j$, for all $j > i$.

The augmented k-ary n-cube $AQ_{n,k}$ can also be recursively defined as it was in the Introduction (the proof of this fact is a simple induction) and the essential properties of $AQ_{n,k}$ have already been given in Fig. 2. Note that we can partition $AQ_{n,k}$ recursively as follows: we refer to the subgraph of $AQ_{n,k}$ induced by the vertices whose first component is i, for some fixed $i \in \{0, 1, \ldots, k-1\}$, as $AQ_{n-1,k}^i$, and this subgraph is clearly a copy of $AQ_{n-1,k}$.

A graph $G = (V, E)$ is *pancyclic* (resp. *m-pancyclic*) if it contains a cycle of every length between 3 (resp. m) and $|V|$ (inclusive). If a graph $G = (V, E)$ is such that given any edge e, it contains a cycle passing through e of every length between 3 and $|V|$ then we say that G is *edge-pancyclic*. Let $d_G(u, v)$ denote the length of a shortest path in G joining vertex u and vertex v. A graph is *panconnected* (resp. *m-panconnected*) if for every pair of distinct vertices u and v of V, there is a path of every length between $d_G(u, v)$ (resp. m) and $|V| - 1$ joining u and v.

As regards the k-ary n-cube, panconnectivity and pancyclicity issues have only recently been resolved, as we now explain. The situation for Q_n^k is confused as when k is even, Q_n^k is bipartite and consequently cannot be pancyclic nor panconnected. For bipartite graphs, the notions of bipancyclicity and bipannconnectivity are more relevant, where a bipartite graph $G = (V, E)$ is *bipancyclic* if it contains a cycle of every even length between 4 and $|V|$, and *bipannconnected* if for every distinct pair of vertices u and v of V, there is a path of every even length between $d_G(u, v)$ and $|V| - 1$ joining u and v (although the notions of bipancyclicity and bipannconnectivity are primarily designed for bipartite graphs, they are still relevant for non-bipartite graphs). In [14] it was shown that Q_n^k is bipanconnected and bipancyclic, when $k \geq 3$ and $n \geq 2$, and also that when k is odd, Q_n^k is m-panconnected, for $m = \frac{n(k-1)+2k-6}{2}$, and $(k-1)$-pancyclic (these bounds are optimal).

We shall use a specific technique whilst building our paths and cycles. Suppose that ρ is some path or cycle $v_1, v_2, \ldots, v_m$, for some m (where there is an edge (v_m, v_1) if ρ is a cycle). We say that ρ can be *progressively shortened* to a path of length m', say, if we can iteratively replace a sub-path x, y, z, say, in ρ with the edge x, z until we obtain a path or cycle of length m'.

The following lemma from [16] will prove useful.

Lemma 2: The following are automorphisms of $AQ_{n,k}$:

(a) the mapping taking the vertex $(v_n, v_{n-1}, \ldots, v_1)$ to $(v_n - a_n, v_{n-1} - a_{n-1}, \ldots, v_1 - a_1)$, where $(a_n, a_{n-1}, \ldots, a_1) \in \{0, 1, \ldots, k-1\}^n$ is fixed;
(b) the mapping taking the vertex $(v_n, v_{n-1}, \ldots, v_1)$ to $(\epsilon v_n, \epsilon v_{n-1}, \ldots, \epsilon v_1)$, where $\epsilon \in \{+1, -1\}$ is fixed.

The following are automorphisms of $AQ_{2,k}$:

(c) the mapping taking the vertex (i, j) to the vertex $(j - i, j)$, if $i \leq j$, and the vertex (i, j) to the vertex $(k - (i - j), j)$, if $i > j$;

(d) the mapping taking the vertex (i, j) to the vertex (j, i).

III. Pancyclicity of $AQ_{n,k}$

We prove in this section that when $n \geq 2$ and $k \geq 3$, $AQ_{n,k}$ is edge-pancyclic. We begin with $AQ_{2,k}$.

Proposition 3: Let $\mathbf{u} = (0, 0)$, $\mathbf{v} = (1, 1)$, and $\mathbf{w} = (0, 1)$ be vertices of $AQ_{2,k}$, where $k \geq 3$. There is a Hamiltonian cycle C in $AQ_{2,k}$ that contains the edges $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}, \mathbf{w})$ and that we can progressively shorten until we obtain the cycle $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Proof: We break our proof into two cases, depending upon the parity of k.

Case 1: k is even.

Consider the following Hamiltonian cycle C of $AQ_{2,k}$:

$$\begin{aligned}&\mathbf{u}, \mathbf{v}, (2,2), \ldots, (k-1,k-1), (k-1,0), (k-1,1),\\ &\quad \ldots, (k-1,k-2), (k-2,k-3), (k-2,k-4),\\ &\quad \ldots, (k-2,k-1), (k-3,k-2), \ldots, (2,3),\\ &\quad (1,2), (1,3), \ldots, (1,0), (0,k-1), (0,k-2),\\ &\quad \ldots, (0,2), \mathbf{w}.\end{aligned}$$

See Fig. 3(a) for a visualization of the above cycle. We can progressively shorten the cycle by first shortening the cycle as it runs through zone A and then as it runs through zone B, so that we finally obtain the cycle $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Case 2: k is odd.

Consider the following Hamiltonian cycle C of $AQ_{2,k}$:

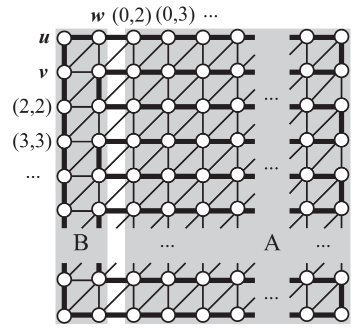
$$\begin{aligned}&\mathbf{u}, \mathbf{v}, (2,2), \ldots, (k-1,k-1), (k-2,k-1),\\ &\quad (k-1,0), (k-2,0), (k-1,1), \ldots, (k-2,k-4),\\ &\quad (k-1,k-3), (k-1,k-2), (k-2,k-3),\\ &\quad (k-3,k-4), (k-3,k-5), \ldots, (k-3,k-2),\\ &\quad \ldots, (2,3), (1,2), (1,3), \ldots, (1,0), (0,k-1),\\ &\quad (0,k-2), \ldots, (0,2), \mathbf{w}.\end{aligned}$$

See Fig. 3(b) for a visualization of the above cycle. We can progressively shorten the cycle by shortening the cycle as it runs through zone A and then as it runs through zone B, so that we finally obtain the cycle $\mathbf{u}, \mathbf{v}, \mathbf{w}$. ■

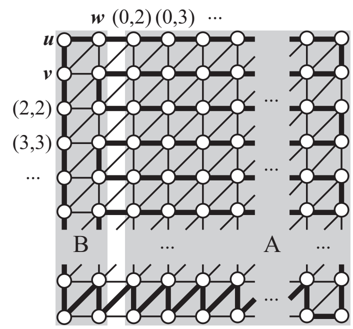
An immediate corollary of Proposition 3 is that $AQ_{2,k}$ is pancyclic. However, as $AQ_{2,k}$ is edge-symmetric [16], it is trivially also edge-pancyclic.

Corollary 4: $AQ_{2,k}$ is edge-pancyclic.

Note that we could define an 'augmented grid' by omitting the 'wrap-around' edges in $AQ_{2,k}$ (this augmented grid is actually the network visualized in Fig 3). As can be seen from the proof of Proposition 3, this augmented grid is



(a) k is even.



(b) k is odd.

Fig. 3. A Hamiltonian cycle in $AQ_{2,k}$.

pancyclic too (though not edge-pancyclic; for just consider the edge $(\mathbf{v}, \mathbf{w})$ in Fig. 3).

Now for the general case. We begin with a useful lemma.

Lemma 5: Let $n \geq 2$ and $k \geq 3$. If $(\mathbf{u}, \mathbf{v})$ is an edge of $AQ_{n,k}$ then there are paths of lengths 2 and 3 joining $\mathbf{u}$ and $\mathbf{v}$.

Proof: By Lemma 2, we may assume w.l.o.g. that $\mathbf{u} = (0, 0, \ldots, 0)$.

Case (*a*): $\mathbf{v} = (0, \ldots, 0, 1, 0, \ldots, 0)$, where all components are 0 except for the ith, which is 1, and $i \neq 1$.

Define $\mathbf{w} = (0, \ldots, 0, 1, 1, \ldots, 1)$; that is, the jth component is 0, if $j > i$, with all other components 1. Define $\mathbf{x} = (0, \ldots, 0, 0, 1, \ldots, 1)$; that is, the jth component is 0, if $j \geq i$, with all other components 1. Define $\mathbf{y} = \mathbf{w}$.

Case (*b*): $\mathbf{v} = (0, \ldots, 0, 1)$, where all components are 0 except for the first, which is 1.

Define $\mathbf{w} = (0, \ldots, 0, 1, 1)$; that is, all components are 0 except for the first two which are 1. Define $\mathbf{x} = (0, \ldots, 0, 1, 0)$; that is, the second component is 1, with all other components 0. Define $\mathbf{y} = \mathbf{w}$.

Case (*c*): $\mathbf{v} = (0, \ldots, 0, 1, 1, \ldots, 1)$, where $i > 1$ and where the jth component is 1 (resp. 0) if, and only if, $i \geq j$ (resp. $i < j$).

Define $\mathbf{w} = (0, \ldots, 0, 0, 1, \ldots, 1)$; that is, $\mathbf{w}$ is identical to $\mathbf{v}$ in every component except that the ith component of $\mathbf{w}$ is 0. Define $\mathbf{x} = (0, \ldots, 0, 1)$; that is, the first component is 1, with all other components 0. Define $\mathbf{y} = (0, \ldots, 0, 1, 1, \ldots, 2)$; that is, the jth component is 1, if $1 < j \leq i$, with all other components 0 except for the first component which is 2.

In each case, $\mathbf{u}, \mathbf{w}, \mathbf{v}$ is a path in $AQ_{n,k}$, as is $\mathbf{u}, \mathbf{x}, \mathbf{y}, \mathbf{v}$. The result follows as by Lemma 2, all other cases (for $\mathbf{v}$) are isomorphic to one of the above cases. ■

We now consider pancyclicity in $AQ_{n,k}$.

Theorem 6: Let $\mathbf{u} = (0, 0, \ldots, 0)$ be a vertex of $AQ_{n,k}$, where $n \geq 2$ and $k \geq 3$. Let $\mathbf{v}$ be any neighbour of $\mathbf{u}$. There exists a neighbour $\mathbf{w}$ of $\mathbf{u}$, different from $\mathbf{v}$, such that for every m such that $5 \leq m \leq k^n$, there is a cycle of length m in $AQ_{n,k}$ containing the edge $(\mathbf{u}, \mathbf{v})$ as well as the edge $(\mathbf{u}, \mathbf{w})$.

Proof: Let $\mathbf{v} = (v_n, v_{n-1}, \ldots, v_1)$. We will prove the theorem by induction on n. The base case, when $n = 2$, is given by Proposition 3 and Lemma 2. Fix $n > 2$ and consider $AQ_{n,k}$, where $k \geq 3$. Partition $AQ_{n,k}$ into $AQ^0_{n-1,k}, AQ^1_{n-1,k}, \ldots, AQ^{k-1}_{n-1,k}$ by fixing the first component of every vertex of $AQ^i_{n-1,k}$ at i; for ease of notation, denote each $AQ^i_{n-1,k}$ by AQ^i.

Case (*a*): $v_n = 0$; thus, $\mathbf{v}' = (v_{n-1}, v_{n-2}, \ldots, v_1)$ is a neighbour of $\mathbf{u}' = (0, 0, \ldots, 0)$ in $AQ_{n-1,k}$.

We assume, as our induction hypothesis, that there is a neighbouring vertex $\mathbf{w}' = (w_{n-1}, w_{n-2}, \ldots, w_1)$ of $\mathbf{u}'$ in $AQ_{n-1,k}$, different from $\mathbf{v}'$, such that for every m for which $5 \leq m \leq k^{n-1}$, there is a cycle C_m of length m in $AQ_{n-1,k}$ containing both of the edges $(\mathbf{u}', \mathbf{v}')$ and $(\mathbf{u}', \mathbf{w}')$. For ease of notation, define the following vertices in $AQ_{n,k}$, for each $i \in \{0, 1, \ldots, k-1\}$: $\mathbf{u}^i = (i, 0, 0, \ldots, 0)$; $\mathbf{v}^i = (i, v_{n-1}, v_{n-2}, \ldots, v_1)$; and $\mathbf{w}^i = (i, w_{n-1}, w_{n-2}, \ldots, w_1)$ (in particular, $\mathbf{u} = \mathbf{u}^0$ and $\mathbf{v} = \mathbf{v}^0$).

For each $i \in \{0, 1, \ldots, k-1\}$ and for each m for which $5 \leq m \leq k^{n-1}$, denote by C^i_m the natural embedding of the cycle C_m in AQ^i (and so, in particular, C^i_m contains the edges $(\mathbf{u}^i, \mathbf{v}^i)$ and $(\mathbf{u}^i, \mathbf{w}^i)$). Fix $j \in \{1, 2, \ldots, k-1\}$. For each $0 \leq i \leq j$, choose the cycle $C^i_{m_i}$ in AQ^i, where $5 \leq m_i \leq k^{n-1}$. Join these cycles together as follows:

- remove the edge $(\mathbf{u}^0, \mathbf{w}^0)$ from $C^0_{m_0}$;
- remove the edges $(\mathbf{u}^i, \mathbf{w}^i)$ and $(\mathbf{u}^i, \mathbf{v}^i)$ from C^i_m, for $1 \leq i \leq j-1$;
- if j is even then remove the edge $(\mathbf{u}^j, \mathbf{v}^j)$ from $C^j_{m_j}$, and if j is odd then remove the edge $(\mathbf{u}^j, \mathbf{w}^j)$ from $C^j_{m_j}$;
- if $0 \leq i \leq j-1$ and i is even then include the edges $(\mathbf{u}^i, \mathbf{u}^{i+1})$ and $(\mathbf{w}^i, \mathbf{w}^{i+1})$;
- if $0 \leq i \leq j-1$ and i is odd then include the edges $(\mathbf{u}^i, \mathbf{u}^{i+1})$ and $(\mathbf{v}^i, \mathbf{v}^{i+1})$.

The resulting cycle has length $m_0 + m_1 + \ldots + m_j$ and contains the edges $(\mathbf{u}^0, \mathbf{v}^0)$ and $(\mathbf{u}^0, \mathbf{u}^1)$. By choosing j and the m_i's appropriately, for every m such that $7 \leq m \leq k^n$, we can obtain a cycle of length m in $AQ_{n,k}$ containing both of the edges $(\mathbf{u}^0, \mathbf{v}^0)$ and $(\mathbf{u}^0, \mathbf{u}^1)$; that is, the edges $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}, \mathbf{u}^1)$. A typical cycle can be visualized as in Fig. 4 (where we have assumed that j is odd).

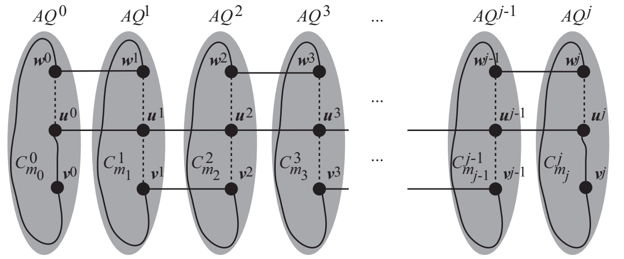


Fig. 4. A typical cycle in $AQ_{n,k}$.

All that remains to do is to find cycles of lengths 5 and 6 containing the edges $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}, \mathbf{u}^1)$. There is a cycle $\mathbf{u}, \mathbf{v}, \mathbf{v}^1, \mathbf{u}^1$ of length 4, and Lemma 5 yields cycles of lengths 5 and 6 containing $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}, \mathbf{u}^1)$ (we simply replace the edge $(\mathbf{u}^1, \mathbf{v}^1)$ with paths of lengths 2 and 3 in AQ^1).

Case (*b*): $v_n = 1$ and $\mathbf{v}' = (v_{n-1}, v_{n-2}, \ldots, v_1) = (1, 1, \ldots, 1)$.

We assume, as our induction hypothesis, that there is a neighbouring vertex $\mathbf{w}' = (w_{n-1}, w_{n-2}, \ldots, w_1)$ of $\mathbf{u} = (0, 0, \ldots, 0)$ in $AQ_{n-1,k}$, different from $\mathbf{v}'$, such that for every m for which $5 \leq m \leq k^{n-1}$, there is a cycle C_m of length m in $AQ_{n-1,k}$ containing both edges $(\mathbf{u}', \mathbf{v}')$ and $(\mathbf{u}', \mathbf{w}')$. By applying a suitable automorphism to each AQ^i (via Lemma 2), we may assume that for each m such that $5 \leq m \leq k^{n-1}$, we can find a cycle C^i_m of length m in AQ^i that contains the edges $((i, i, i, \ldots, i), (i, i+1, i+1, \ldots, i+1))$ and $((i, i, i, \ldots, i), (i, w_{n-1}+i, w_{n-2}+i, \ldots, w_1+i))$ (with arithmetic modulo k). For ease of notation, define the following vertices of $AQ_{n,k}$, for each $i \in \{0, 1, \ldots, k-1\}$: $\mathbf{u}^i = (i, i, i, \ldots, i)$; $\mathbf{v}^i = (i, i+1, i+1, \ldots, i+1)$; and $\mathbf{w}^i = (i, w_{n-1}+i, w_{n-2}+i, \ldots, w_1+i)$ (in particular, $\mathbf{u} = \mathbf{u}^0$ and $\mathbf{v} = \mathbf{u}^1$).

We now proceed exactly as we did in Case (*a*) and obtain that for every m such that $5 \leq m \leq k^n$, there is a cycle of length m in $AQ_{n,k}$ containing both of the edges $(\mathbf{u}^0, \mathbf{u}^1)$ and $(\mathbf{u}^0, \mathbf{v}^0)$; that is, the edges $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}, \mathbf{v}^0)$.

Case (*c*): $v_n = 1$ and $\mathbf{v}' = (v_{n-1}, v_{n-2}, \ldots, v_1) = (0, 0, \ldots, 0)$.

Define the vertex $\mathbf{x} = (0, 0, \ldots, 0, 1)$ in $AQ_{n,k}$; that is, the first component of $\mathbf{x}$ is 1, with all other components 0. Also, define the vertex $\mathbf{x}' = (0, 0, \ldots, 0, 1)$ of $AQ_{n-1,k}$ similarly. We assume, as our induction hypothesis, that there is a neighbour $\mathbf{w}' = (w_{n-1}, w_{n-2}, \ldots, w_1)$ of $\mathbf{u}' = (0, 0, \ldots, 0)$ in $AQ_{n-1,k}$, different from $\mathbf{x}'$, such that for every m for which $5 \leq m \leq k^{n-1}$, there is a cycle C_m of length m in $AQ_{n-1,k}$ containing both of the edges $(\mathbf{u}', \mathbf{x}')$ and $(\mathbf{u}', \mathbf{w}')$. For ease of notation, define the following vertices of $AQ_{n,k}$, for each $i \in \{0, 1, \ldots, k-1\}$: $\mathbf{u}^i = (i, 0, 0, \ldots, 0)$; $\mathbf{v}^i = (i, 0, 0, \ldots, 0, 1)$; and $\mathbf{w}^i = (i, w_{n-1}, w_{n-2}, \ldots, w_2, w_1)$ (in particular, $\mathbf{u} = \mathbf{u}^0$, $\mathbf{v} = \mathbf{u}^1$, and $\mathbf{x} = \mathbf{v}^0$).

We now proceed exactly as we did in Case (*a*) and obtain that for every m such that $5 \leq m \leq k^n$, there is a cycle of length m in $AQ_{n,k}$ containing both of the edges $(\mathbf{u}^0, \mathbf{u}^1)$ and $(\mathbf{u}^0, \mathbf{v}^0)$; that is, the edges $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}, \mathbf{x})$.

The result follows by induction as every other case for the neighbour $\mathbf{v}$ of $\mathbf{u}$ is isomorphic to one of the cases considered above. ■

Lemmas 2 and 5 and Theorem 6 yield the main result of this section.

Corollary 7: For $n \geq 2$ and $k \geq 3$, $AQ_{n,k}$ is edge-pancyclic.

IV. Panconnectivity in $AQ_{2,k}$

In this section, we show that when $k \geq 3$, $AQ_{2,k}$ is panconnected. Just as we did before, partition $AQ_{2,k}$ into $AQ^0, AQ^1, \ldots, AQ^{k-1}$ by fixing the first component of every vertex of AQ^i at i (so, each AQ^i is a cycle of length k). To prove that $AQ_{2,k}$ is panconnected, we work in vertex-induced subgraphs of $AQ_{2,k}$, namely the graphs B_i induced by the vertices of $AQ^0, AQ^1, \ldots, AQ^i$, where $3 \leq i \leq k-1$.

Throughout, $\mathbf{u}$ and $\mathbf{v}$ are two arbitrary, distinct vertices of $AQ_{2,k}$. Let d be the length of a shortest path in $AQ_{2,k}$ joining $\mathbf{u}$ and $\mathbf{v}$. We will show that there are paths of all lengths between d and $k^2 - 1$ joining $\mathbf{u}$ and $\mathbf{v}$ in $AQ_{2,k}$. By Lemma 2, w.l.o.g. we may suppose that $\mathbf{u} = (0, 0)$ and that $\mathbf{v} = (i, j)$, where $i \leq j$. Let us begin by supposing that $\mathbf{v} \in B_3$. We shall describe a Hamiltonian path from $\mathbf{u}$ to $\mathbf{v}$ in B_3. There are various cases to consider, depending upon the vertex $\mathbf{v}$.

<u>Case 1</u>: $i = 0$.

A Hamiltonian path from $\mathbf{u}$ to $\mathbf{v}$ in B_3 is depicted in Fig. 5(a).

<u>Case 2</u>: $i = 1$.

A Hamiltonian path from $\mathbf{u}$ to $\mathbf{v}$ in B_3 when $\mathbf{v} \neq (1, 1)$ is depicted in Fig. 5(b), and one from $\mathbf{u}$ to $(1, 1)$ in Fig. 5(c).

<u>Case 3</u>: $i = 2$.

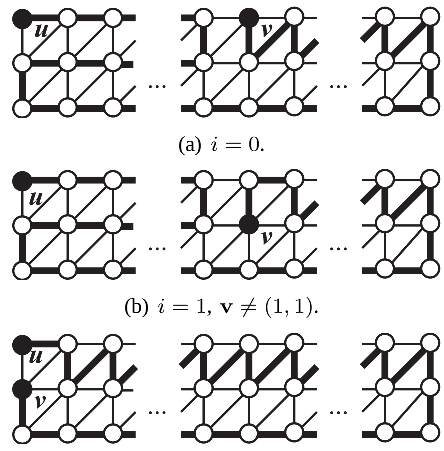
A Hamiltonian path from $\mathbf{u}$ to $\mathbf{v}$ in B_3 when $\mathbf{v} \neq (2, 2)$ is depicted in Fig. 6(a), and one from $\mathbf{u}$ to $(2, 2)$ in Fig. 6(b).

It is trivial to verify that all of the paths constructed above can be progressively shortened until they have length j (remember, $i \leq j$).

Let us now suppose that $\mathbf{v} \in B_4$. We shall describe a Hamiltonian path from $\mathbf{u}$ to $\mathbf{v}$ in B_4. There are various cases to consider, depending upon the vertex $\mathbf{v}$.

<u>Case 1</u>: $i = 0$.

A Hamiltonian path from $\mathbf{u}$ to $\mathbf{v}$ in B_4 is depicted in Fig. 7(a).

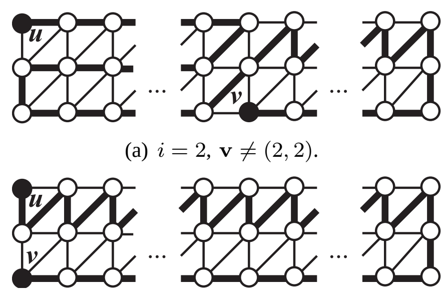


(a) $i = 0$.

(b) $i = 1$, $\mathbf{v} \neq (1, 1)$.

(c) $i = 1$, $\mathbf{v} = (1, 1)$.

Fig. 5. Hamiltonian paths in B_3 when $i = 0, 1$.



(a) $i = 2$, $\mathbf{v} \neq (2, 2)$.

(b) $i = 2$, $\mathbf{v} = (2, 2)$.

Fig. 6. Hamiltonian paths in B_3 when $i = 2$.

<u>Case 2</u>: $i = 1$.

A Hamiltonian path from $\mathbf{u}$ to $\mathbf{v}$ in B_4 is depicted in Fig. 7(b).

<u>Case 3</u>: $i = 2$.

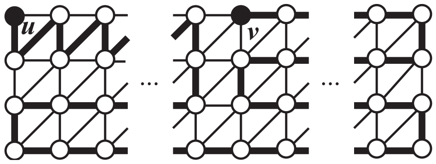
A Hamiltonian path from $\mathbf{u}$ to $\mathbf{v}$ in B_4 is depicted in Fig. 7(c).

<u>Case 4</u>: $i = 3$.

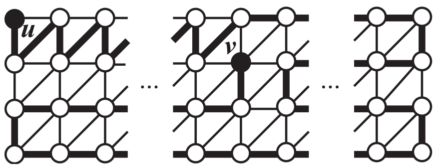
A Hamiltonian path from $\mathbf{u}$ to $\mathbf{v}$ in B_4 when $\mathbf{v} \neq (4, 4)$ is depicted in Fig. 8(a), and one from $\mathbf{u}$ to $(4, 4)$ in Fig. 8(b).

Again, it is trivial to verify that all of the paths constructed above can be progressively shortened until they have length j.

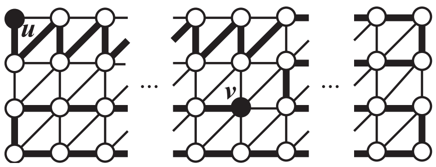
We now extend the constructions above inductively. Suppose that for some $3 \leq r \leq k-3$, given any vertex $\mathbf{v} = (i, j)$ in B_r (different from $(0, 0)$ and with $i \leq j$), we can find a Hamiltonian path ρ in B_r joining $(0, 0)$ and (i, j) that can be progressively shortened until the path has length j; moreover, we assume that there is at least one edge of ρ lying in AQ^r (this is certainly true for all paths constructed above in B_3 and B_4). Now let $\mathbf{u} = (0, 0)$ and $\mathbf{v} = (i, j)$, where $\mathbf{v}$ lies in B_{r+2} and $i \leq j$.



(a) $i = 0$.

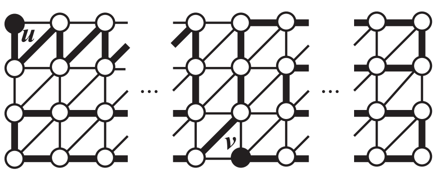


(b) $i = 1$.

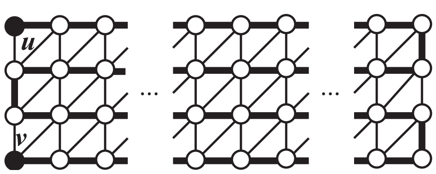


(c) $i = 2$.

Fig. 7. Hamiltonian paths in B_4 when $i = 0, 1, 2$.



(a) $i = 3$, $\mathbf{v} \neq (4, 4)$.



(b) $i = 3$, $\mathbf{v} = (4, 4)$.

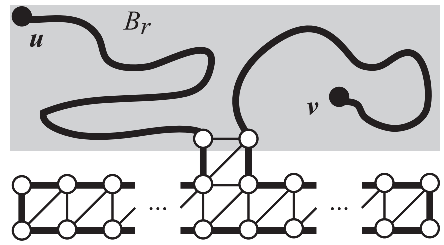
Fig. 8. Hamiltonian paths in B_4 when $i = 3$.

Case 1: (i, j) lies in B_r.

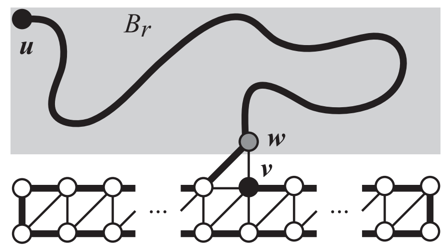
By the induction hypothesis, there is a Hamiltonian path ρ in B_r joining $(0, 0)$ and (i, j) that can be progressively shortened until the path has length j. Take any edge of the form $((r, a), (r, a+1))$ (with addition modulo k) that lies on the path ρ (such an edge exists by assumption). Extend the path ρ as in Fig. 9(a). The new Hamiltonian path in B_{r+2} can clearly be progressively shortened to obtain a path of length j.

Case 2: (i, j) lies in AQ_{r+1} (so, $i = r + 1$).

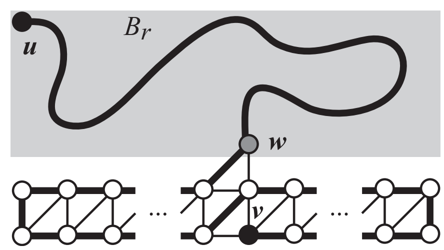
Let $\mathbf{w} = (i - 1, j - 1)$; so, $i - 1 \leq j - 1$. By the induction hypothesis, there is a Hamiltonian path ρ in B_r from $\mathbf{u}$ to $\mathbf{w}$ that can be progressively shortened until we obtain a path of length $j - 1$. The path ρ can be extended as in



(a) A simple extension.



(b) Extending to AQ^{r+1}.



(c) Extending to AQ^{r+2}.

Fig. 9. Extending Hamiltonian paths.

Fig. 9(b) (note that if $i = j = r + 1$ then we use the edge $((r, r), (r + 1, r))$). The new Hamiltonian path in B_{r+2} can clearly be progressively shortened until we obtain a path of length j.

Case 3: (i, j) lies in AQ_{r+2} (so, $i = r + 2$).

Let $\mathbf{w} = (r, j - 2)$; so, $i - 2 \leq j - 2$. By the induction hypothesis, then is a Hamiltonian path ρ in B_r from $\mathbf{u}$ to $\mathbf{w}$ that can be progressively shortened until we obtain a path of length $j - 2$. The path ρ can be extended as in Fig. 9(c). The new Hamiltonian path in B_{r+2} can clearly be progressively shortened until we obtain a path of length j.

Thus, by induction, we have the following result.

Theorem 8: We can find a Hamiltonian path ρ in $AQ_{2,k}$ joining the vertex $(0, 0)$ to any different vertex (i, j), where $i \leq j$, that can be progressively shortened until the path ρ has length j.

Thus, in order to prove that $AQ_{2,k}$ is panconnected, all we have to do is to show that there are paths joining $\mathbf{u} = (0, 0)$ and $\mathbf{v} = (i, j)$ (where $i \leq j$) of all lengths ranging from the length of a shortest path joining $\mathbf{u}$ and $\mathbf{v}$ up to $j - 1$. With regard to candidate paths for shortest paths joining $\mathbf{u}$ and $\mathbf{v}$, the situation can be visualized in Fig. 10. An immediate observation is that if there is a shortest path from $\mathbf{u}$ to $\mathbf{v}$ that leaves the grey area (consisting of the subgraph induced by the vertices of $\{(x,y) : 0 \leq x, y \leq k-1, x \leq y\}$) then there is an analogous shortest path that does not leave the grey area. This observation can be easily verified by examining the different configurations of i and j with a view to finding a shortest path joining $\mathbf{u}$ and $\mathbf{v}$. Depending upon the relative values of i and j (with respect to each other and with respect to k and 0 also), a shortest path from $\mathbf{u}$ to $\mathbf{v}$ will be constructed in one of the following three ways:

- the first component will be increased from 0 to j and the second component will be increased from 0 to i;
- the first component will be decreased from 0 to j and the second component will be increased from 0 to i;
- the first component will be decreased from 0 to j and the second component will be decreased from 0 to i.

Note that because $i \leq j$, a shortest path from $\mathbf{u}$ to $\mathbf{v}$ need not be constructed by increasing the first component from 0 to j and decreasing the second component from 0 to i; for we can obtain a shortest path (of the same length) by decreasing the first component from 0 to j and decreasing the second component from 0 to i via a path $(0,0),(k-1,k-1),\ldots,(j,j),(j-1,j),\ldots,(i,j)$ which resides wholly within the grey area. The criteria in each of the three constructions above can be met by shortest paths residing wholly within the grey area and so we may confine ourselves to shortest paths residing wholly within the grey area (see Fig. 10 for some illustrative paths).

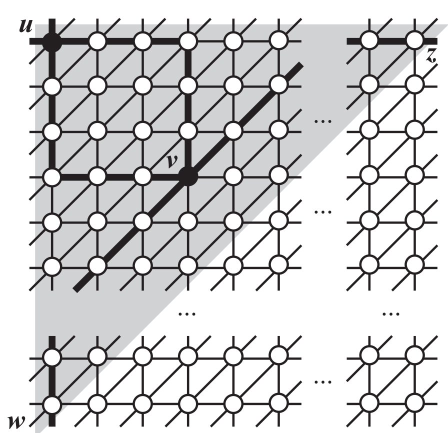


Fig. 10. Possible shortest paths joining $\mathbf{u}$ and $\mathbf{v}$.

Any such shortest path must pass through exactly one of $\mathbf{w} = (k-1,k-1)$ and $\mathbf{z} = (0,k-1)$ or must pass through neither of $\mathbf{w}$ and $\mathbf{z}$.

- If a shortest path does not pass through $\mathbf{w}$ or $\mathbf{z}$ then it has length j, and we are done (as we have already found paths joining $\mathbf{u}$ and $\mathbf{v}$ of all lengths between j and k^2-1, inclusive).
- If a shortest path passes through $\mathbf{z}$ then it has length $k-j+i$, and any such path can clearly be progressively lengthened until it has length j (if $k-j+i < j$).
- If a shortest path passes through $\mathbf{w}$ then it has length $k-i$, and any such path can clearly be progressively lengthened until it has length j (if $k-i < j$).

Thus, irrespective of the length d of a shortest path from $\mathbf{u}$ to $\mathbf{v}$, there exists a path from $\mathbf{u}$ to $\mathbf{v}$ of all lengths between d and k^2-1 (inclusive).

Consequently, we have the following result.

Theorem 9: $AQ_{2,k}$ is panconnected.

V. Conclusion

In this paper, we have shown that $AQ_{n,k}$ is edge-pancyclic, when $n \geq 2$ and $k \geq 3$, and that $AQ_{2,k}$ is panconnected. With regard to pancyclicity, the situation for $AQ_{n,k}$ is improved in comparison to the k-ary n-cube Q_n^k. With regard to panconnectivity, the situation for $AQ_{2,k}$ is improved in comparison to the k-ary 2-cube Q_2^k. Of course, the obvious question remaining to be asked is whether $AQ_{n,k}$ is panconnected when $n \geq 3$. This question is made more complicated due to the lack of a complete picture as regards the diameter of $AQ_{n,k}$ when $n \geq 3$, and more generally as regards the length of a shortest path joining two arbitrary vertices of $AQ_{n,k}$. As can be seen from [16], at present only an upper bound is known as regards the diameter of $AQ_{n,k}$ and (as remarked in [16]) deriving the diameter exactly appears to be combinatorially quite difficult. However, in preliminary investigations on this question we have tentatively obtained m-panconnectivity results for $AQ_{n,k}$ where m is $n\lceil \frac{k}{2} \rceil$ (that is, the diameter of Q_n^k). We shall continue this investigation in future.

Another topic for further research is the tolerance of $AQ_{n,k}$ to faults; for example, is it the case that $AQ_{n,k}$ remains pancyclic when a limited number of vertices or edges are removed from the network (mimicking fault processors or links in an inter-connection network based upon $AQ_{n,k}$)? We observe that $AQ_{2,k}$ can tolerate at least one faulty vertex and remain pancyclic (simply remove the vertex $\mathbf{u}$ in Fig. 3).

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