# CM POINTS AND WEIGHT 3/2 MODULAR FORMS

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## 1. INTRODUCTION

The theta correspondence has been an important tool in the theory of automorphic forms with plentiful applications to arithmetic questions.

In this paper, we consider a specific theta lift for an isotropic quadratic space $V$ over $\mathbb{Q}$ of signature $(1,2)$. The theta kernel we employ associated to the lift has been constructed by Kudla-Millson (e.g., [29, 30]) in much greater generality for $\mathrm{O}(p,q)$ ($\mathrm{U}(p,q)$) to realize generating series of cohomological intersection numbers of certain, 'special' cycles in locally symmetric spaces of orthogonal (unitary) type as holomorphic Siegel (Hermitian) modular forms. In our case for $\mathrm{O}(1,2)$, the underlying locally symmetric space $M$ is a modular curve, and the special cycles, parametrized by positive integers $N$, are the classical CM points $Z(N)$; i.e., quadratic irrationalities of discriminant $-N$ in the upper half plane.

We survey the results of [16] and of our joint work with Bruinier [12] on using this particular theta kernel to define lifts of various kinds of functions $F$ on the underlying modular curve $M$. The theta lift is given by

$$I(\tau,F)=\int_M F(z)\theta(\tau,z), \tag{1.1}$$

where $\tau\in\mathbb{H}$, the upper half plane, $z\in M$, and $\theta(\tau,z)$ is the theta kernel in question. Then $I(\tau,F)$ is a (in general non-holomorphic) modular form of weight $3/2$ for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. One key feature of the theta kernel is its very rapid decay on $M$, which distinguishes it from other theta kernels which are usually moderately increasing. Consequently, we can lift some rather nonstandard, even exponentially increasing, functions $F$.

Note that Kudla and Millson, who focus entirely on the (co)homological aspects of their general lift, study in this situation only the lift of the constant function 1 in the compact case of a Shimura curve, when $V$ is anisotropic.

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One feature of our work is that it provides a uniform approach to several topics and (previously known) results, which so far all have been approached by (entirely) different methods. We discuss the following cases in some detail:

(i) The lift of the constant function 1. Then $I(\tau, 1)$ realizes the generating series of the (geometric) degree of the 0-cycles $Z(N)$ as the holomorphic part of a non-holomorphic modular form. As a special case, we recover Zagier's well known Eisenstein series $\mathcal{F}(\tau, s)$ of weight $3/2$ at $s = 1/2$ (in our normalization) whose Fourier coefficients of positive index are given by the Kronecker-Hurwitz class numbers $H(N)$ [40, 21].

(ii) The lift of a modular function $f$ of weight 0 on $M$. In that case, we obtain a generalization with a completely different proof of Zagier's influential result [42] on the generating series of the traces of the singular moduli, that is, the sum of values of the classical $j$-invariant over the CM points of a given discriminant. Moreover, our method provides a generalization to modular curves of arbitrary genus.

(iii) The lift of the logarithm of the Petersson metric $\log \|\Delta\|$ of the discriminant function $\Delta$. This was suggested to us by U. Kühn. In that case, the lift $I(\tau, \|\log \Delta\|)$ turns out to be the *derivative* of Zagier's Eisenstein series $\mathcal{F}'(\tau, s)$ at $s = 1/2$. Furthermore, one can interpret the Fourier coefficients as the *arithmetic* degree of the ($\mathbb{Z}$ extension of the) CM cycles. This provides a different approach for the result of (Kudla, Rapoport and) Yang [39] in this case, part of Kudla's general program on realizing generating series in arithmetic geometry as modular forms, in particular as derivatives of Eisenstein series. Their result in the modular curve case grew out of their extensive and deep work on the analogous but more involved case for Shimura curves [32, 33].

(iv) The lift of a weight 0 Maass cusp form $f$ on $M$. For this input, our lift is equivalent to a theta lift introduced by Maass [34], which was studied and applied by Duke [14] (to obtain equidistribution results for the CM points and certain geodesics in $M$) and Katok and Sarnak [22] (to obtain nonnegativity of the L-function of $f$ at the center of the criticial strip).

The paper is mostly expository; for convenience of the reader and for future use, we briefly discuss the construction of the theta kernel and also give general formulas for the Fourier coefficients.

However, we also discuss a few new aspects. Namely:

(v) For any meromorphic modular form $f$, we give an explicit formula for the positive Fourier coefficients of the lift $I(\tau, \log \|f\|)$ of the logarithm of the Petersson metric of $f$ in the case when the divisor of $f$ is *not* (necessarily) disjoint to one of the 0-cycles $Z(N)$. In particular, for the $j$-invariant, we realize the logarithm of the *norm* of the singular moduli as the Fourier coefficients of a non-holomorphic modular form of weight 3/2. Recall that the norms of the singular moduli were studied by Gross-Zagier [20].

In this context and also in view of (iii) it will be interesting to consider the lift for the logarithm of the Petersson metric of a Borcherds product [2]. We will come back to this point in the near future.

(vi) Bringmann, Ono, and Rouse [7] consider the intersection of a modular curve with a Hirzebruch-Zagier curve $T_N$ in a Hilbert modular curve. Based on our work, they realize the generating series of the traces of the singular moduli on these intersections as a weakly holomorphic modular form of weight 2. They proceed to find some beautiful formulas involving Hilbert class polynomials.

In the last section of this paper, we show how one can obtain such generating series in the context of the Kudla-Millson machinery and generalize this aspect of [7] to the intersection of a modular curve with certain special divisors inside locally symmetric spaces associated to $\mathrm{O}(n, 2)$.

Finally, one comment on the usage of this particular kernel function for the lift. The lift $I$ is designed to produce *holomorphic* generating series, while often theta series and integrals associated to indefinite quadratic forms give rise to non-holomorphic modular forms. Furthermore, the lift focuses a priori only on the positive coefficients which correspond to the CM points, while the negative coefficients (which correspond to the geodesics in $M$) often vanish. For the geodesics, in the Kudla-Millson theory [29, 30], there is another lift for signature $(2, 1)$ with weight 2 forms as input, which produces generating series of periods over the geodesics, see also [18]. This lift is closely related to Shintani's theta lift [36].

Finally note that J. Bruinier [9] wrote up a survey on some aspects of our work as well. I also thank him and U. Kühn for comments on the present paper. We also thank the Centre de Recerca Matemàtica in Bellaterra/Spain for its hospitality during fall 2005.

## 2. Basic Notions

2.1. **CM points.** Let $V$ be a rational vector space of dimension 3 with a non-degenerate symmetric bilinear form $(\ ,\ )$ of signature $(1, 2)$. We assume that $V$ is given by

$$V = \{X \in M_2(\mathbb{Q}); \operatorname{tr}(X) = 0\} \tag{2.1}$$

with $(X,Y) = \operatorname{tr}(XY)$ and associated quadratic form $q(X) = \frac{1}{2}(X,X) = \det(X)$. We let $\underline{G} = \operatorname{Spin} V \simeq \operatorname{SL}_2$, which acts on $V$ by $g.X := gXg^{-1}$. We set $G = \underline{G}(\mathbb{R})$ and let $D = G/K$ be the associated symmetric space, where $K = \operatorname{SO}(2)$ is the standard maximal compact subgroup of $G$. We have $D \simeq \mathbb{H} = \{z \in \mathbb{C};\ \Im(z) > 0\}$. Let $L \subset V(\mathbb{Q})$ be an integral lattice of full rank and let $\Gamma$ be a congruence subgroup of $G$ which takes $L$ to itself. We write $M = \Gamma\backslash D$ for the attached locally symmetric space, which is a modular curve. Throughout the paper let $p$ be a prime or $p = 1$. For simplicity, we assume that the lattice $L$ is given by

$$L = \left\{ [a,b,c] := \begin{pmatrix} b & -2c \\ 2ap & -b \end{pmatrix} :\ a,b,c \in \mathbb{Z} \right\}. \tag{2.2}$$

(For arbitrary even lattices, see [12]). Then we can take $\Gamma = \Gamma_0^*(p)$, the extension of the Hecke subgroup $\Gamma_0(p)$ by the Fricke involution $W_p$. Note that then $M$ has only one cusp.

We identify $D$ with the space of lines in $V(\mathbb{R})$ on which the form $(\ ,\ )$ is positive:

$$D \simeq \{z \subset V(\mathbb{R});\ \dim z = 1 \text{ and } (\ ,\ )|_z > 0\}. \tag{2.3}$$

We pick as base point of $D$ the line $z_0$ spanned by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. For $z \in \mathbb{H}$, we choose $g_z \in G/K$ such that $g_z i = z$; the action is the usual linear fractional transformation on $\mathbb{H}$. Then $z \longmapsto g_z z_0$ gives rise to a $G$-equivariant isomorphism $\mathbb{H} \simeq D$. The positive line associated to $z = x + iy \in \mathbb{H}$ is generated by $X(z) := g_z.\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We let $(\ ,\ )_z$ be the minimal majorant of $(\ ,\ )$ associated to $z \in D$. One easily sees $(X,X)_z = (X, X(z))^2 - (X,X)$.

The classical CM points are now given as follows. For $X = [a,b,c] \in V$ such that $q(X) = 4acp - b^2 = N > 0$, we put

$$D_X = \operatorname{span}(X) \in D. \tag{2.4}$$

It is easy to see that $D_X$ is explicitly given by the point $\frac{-b+i\sqrt{N}}{2ap}$ in the upper half plane. The stabilizer $\Gamma_X$ of $X$ in $\Gamma$ is finite. We then denote by $Z(X)$ the image of $D_X$ in $M$, counted with multiplicity $\frac{1}{|\overline{\Gamma}_X|}$. Here $\overline{\Gamma}_X$ denotes the image of $\Gamma_X$ in $\operatorname{PGL}_2(\mathbb{Z})$. Furthermore, $\Gamma$ acts on $L_N = \{X \in L;\ q(X) = N\}$ with finitely many orbits. The CM points of discriminant $-N$ are given by

$$Z(N) = \sum_{X \in \Gamma\backslash L_N} Z(X). \tag{2.5}$$

We can interpret this in terms of positive definite binary quadratic forms as well. For $N>0$ a positive integer, we let $\mathcal{Q}_{N,p}$ be the set of positive definite binary quadratic forms of the form $apX^2+bXY+cY^2$ of discriminant $-N=b^2-4acp$ with $a,b,c,\in\mathbb{Z}$. Then $\Gamma=\Gamma_0^*(p)$ acts on $\mathcal{Q}_{N,p}$ in the usual way, and the obvious map from $\mathcal{Q}_{N,p}$ to $L_N$ is $\Gamma_0^*(p)$-equivariant, and $L_N$ is in bijection with $\mathcal{Q}_{N,p}\coprod -\mathcal{Q}_{N,p}$. (The vector $X=[a,b,c]\in L_N$ with $a<0$ corresponds to a negative definite form).

For a $\Gamma$-invariant function $F$ on $D\simeq\mathbb{H}$, we define its trace by

$$\mathbf{t}_F(N)=\sum_{z\in Z(N)}F(z)=\sum_{X\in\Gamma\backslash L_N}\frac{1}{|\overline{\Gamma}_X|}F(D_X). \tag{2.6}$$

2.2. **The Theta Lift.** In [29], Kudla and Millson explicitly construct a Schwartz function $\varphi_{KM}=\varphi$ on $V(\mathbb{R})$ valued in $\Omega^{1,1}(D)$, the differential $(1,1)$-forms on $D$. It is given by

$$\varphi(X,z)=\left((X,X(z))^2-\frac{1}{2\pi}\right)e^{-\pi(X,X)_z}\,\omega, \tag{2.7}$$

where $\omega=\frac{dx\wedge dy}{y^2}=\frac{i}{2}\frac{dz\wedge d\bar{z}}{y^2}$. We have $\varphi(g.X,gz)=\varphi(X,z)$ for $g\in G$. We define

$$\varphi^0(X,z)=e^{\pi(X,X)}\varphi(X,z)=\left((X,X(z))^2-\frac{1}{2\pi}\right)e^{-2\pi R(X,z)}\,\omega, \tag{2.8}$$

with $R(X,z)=\frac{1}{2}(X,X)_z-\frac{1}{2}(X,X)$. Note $R(X,z)=0$ if and only if $z=D_X$, i.e., if $X$ lies in the line generated by $X(z)$.

For $\tau=u+iv\in\mathbb{H}$, we put $g'_\tau=\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)\begin{pmatrix}v^{1/2}&0\\0&v^{-1/2}\end{pmatrix}$, and we define

$$\varphi(X,\tau,z)=\varphi^0(\sqrt{v}X,z)e^{2\pi iq(X)\tau}. \tag{2.9}$$

Then, see [30, 16], the theta kernel

$$\theta(\tau,z):=\sum_{X\in L}\varphi(X,\tau,z) \tag{2.10}$$

defines a *non-holomorphic* modular form of weight 3/2 with values in $\Omega^{1,1}(M)$, for the congruence subgroup $\Gamma_0(4p)$. By [16, 12] we have

$$\theta(\tau,z)=O(e^{-Cy^2})\qquad\text{as}\qquad y\to\infty, \tag{2.11}$$

uniformly in $x$, for some constant $C>0$.

In this paper, we discuss for certain $\Gamma$-invariant functions $F$ with possible logarithmic singularities inside $D$, the theta integral

$$I(\tau,F):=\int_M F(z)\theta(\tau,z). \tag{2.12}$$

Note that by (2.11), $I(\tau,F)$ typically converges even for exponentially increasing $F$. It is clear that $I(\tau,F)$ defines a (in general non-holomorphic) modular form on the upper half plane of weight $3/2$. The Fourier expansion is given by

$$I(\tau,F)=\sum_{N=-\infty}^{\infty}a_N(v)q^N \tag{2.13}$$

with

$$a_N(v)=\int_M\sum_{X\in L_N}F(z)\varphi^0(\sqrt{v}X,z). \tag{2.14}$$

For the computation of the Fourier expansion of $I(\tau,f)$, Kudla's construction of a Green function $\xi^0$ associated to $\varphi^0$ is crucial, see [26]. We let

$$\xi^0(X,z)=-Ei(-2\pi R(X,z))=\int_1^{\infty}e^{-2\pi R(X,z)t}\frac{dt}{t}, \tag{2.15}$$

where $Ei(w)$ denotes the exponential integral, see [1]. For $q(X)>0$, the function $\xi^0(X,z)$ has logarithmic growth at the point $D_X$, while it is smooth on $D$ if $q(X)\leq 0$.

We let $\partial$, $\bar\partial$ and $d$ be the usual differentials on $D$ and set $d^c=\frac{1}{4\pi i}(\partial-\bar\partial)$.

**Theorem 2.1** (Kudla [26], Proposition 11.1)**.** *Let $X$ be a nonzero vector in $V$. Set $D_X=\emptyset$ if $q(X)\leq 0$. Then, outside $D_X$, we have*

$$dd^c\xi^0(X,z)=\varphi^0(X,z). \tag{2.16}$$

*In particular, $\varphi^0(X,z)$ is exact for $q(X)\leq 0$. Furthermore, if $q(X)>0$ or if $q(X)<0$ and $q(X)\notin -(\mathbb{Q}^\times)^2$ (so that $\overline{\Gamma}_X$ is infinite cyclic), we have for a smooth function $F$ on $\Gamma_X\backslash D$ that*

$$\int_{\Gamma_X\backslash D}F(z)\varphi^0(X,z)=\delta_{D_X}(F)+\int_{\Gamma_X\backslash D}(dd^cF(z))\,\xi^0(X,z). \tag{2.17}$$

*Here $\delta_{D_X}$ denotes the delta distribution concentrated at $D_X$. By Propositions 2.2 and 4.1 of* [12] *and their proofs, (2.17) does not only hold for compactly support functions $F$ on $\Gamma_X\backslash D$, but also for functions of "linear-exponential" growth on $\Gamma_X\backslash D$.*

In Proposition 4.11, we will give an extension of Theorem 2.1 to $F$ having logarithmic singularities inside $D$.

By the usual unfolding argument, see [12], section 4, we have

**Lemma 2.2.** *Let $N>0$ or $N<0$ such that $N\notin -(\mathbb{Q}^\times)^2$. Then*

$$a_N(v)=\sum_{X\in\Gamma\backslash L_N}\int_{\Gamma_X\backslash D}F(z)\varphi^0(\sqrt{v}X,z).$$

*If $F$ is smooth on $X$, then by Theorem 2.17 we obtain*

$$a_N(v) = \mathbf{t}_F(N) + \sum_{X\in\Gamma\backslash L_N} \frac{1}{|\overline{\Gamma}_X|} \int_D (dd^c F(z)) \cdot \xi^0(\sqrt{v}X, z), \qquad (N > 0)$$

$$a_N(v) = \sum_{X\in\Gamma\backslash L_N} \int_{\Gamma_X\backslash D} (dd^c F(z)) \cdot \xi^0(\sqrt{v}X, z) \qquad (N < 0, \quad N \notin -(\mathbb{Q}^\times)^2)$$

For $N = -m^2$, unfolding is (typically) not valid, since in that case $\overline{\Gamma}_X$ is trivial. In the proof of Theorem 7.8 in [12] we outline

**Lemma 2.3.** *Let $N = -m^2$. Then*

$$\begin{aligned} a_N(v) = \sum_{X\in\Gamma\backslash L_N} &\frac{1}{2\pi i}\int_M d\left(F(z)\sum_{\gamma\in\Gamma}\partial\xi^0(\sqrt{v}X,\gamma z)\right) \\ &+ \frac{1}{2\pi i}\int_M d\left(\bar\partial F(z)\sum_{\gamma\in\Gamma}\xi^0(\sqrt{v}X,\gamma z)\right) \\ &- \frac{1}{2\pi i}\int_M (\partial\bar\partial F(z))\sum_{\gamma\in\Gamma}\xi^0(\sqrt{v}X,\gamma z). \end{aligned}$$

Note that with our choice of the particular lattice $L$ in (2.2), we actually have $\#\Gamma\backslash L_{-m^2} = m$, and as representatives we can take $\{\left(\begin{smallmatrix} m & 2k \\ & -m\end{smallmatrix}\right); k = 0, \ldots, m-1\}$.

Finally, we have

$$a_0(v) = \int_M F(z) \sum_{X\in L_0} \varphi^0(\sqrt{v}X, z). \tag{2.18}$$

We split this integral into two pieces $a_0'$ for $X = 0$ and $a''(v) = a_0(v) - a_0'$ for $X \neq 0$. However, unless $F$ is at most mildly increasing, the two individual integrals will not converge and have to be regularized in a certain manner following [2, 12]. For $a_0''(v)$, we have only one $\Gamma$-equivalence class of isotropic lines in $L$, since $\Gamma$ has only one cusp. We denote by $\ell_0 = \mathbb{Q}X_0$ the isotropic line spanned by the primitive vector in $L$, $X_0 = \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0\end{smallmatrix}\right)$. Note that the pointwise stabilizer of $\ell_0$ is $\Gamma_\infty$, the usual parabolic subgroup of $\Gamma$. We obtain

**Lemma 2.4.**

$$a_0' = -\frac{1}{2\pi}\int_M^{reg} F(z)\omega, \tag{2.19}$$

$$a_0''(v) = \frac{1}{2\pi i}\int_M^{reg} d\left(F(z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sideset{}{'}\sum_{n=-\infty}^{\infty}\partial\xi^0(\sqrt{v}nX_0,\gamma z)\right) + \frac{1}{2\pi i}\int_M^{reg} d\left(\bar\partial F(z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sideset{}{'}\sum_{n=-\infty}^{\infty}\xi^0(\sqrt{v}nX_0,\gamma z)\right) - \frac{1}{2\pi i}\int_M^{reg}(\partial\bar\partial F(z))\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sideset{}{'}\sum_{n=-\infty}^{\infty}\xi^0(\sqrt{v}nX_0,\gamma z). \tag{2.20}$$

*Here* $\sum'$ *indicates that the sum only extends over* $n \neq 0$.

## 3. The lift of modular functions

**3.1. The lift of the constant function.** The modular trace of the constant function $F = 1$ is already very interesting. In that case, the modular trace of index $N$ is the (geometric) degree of the 0-cycle $Z(N)$:

$$\mathbf{t}_1(N) = \deg Z(N) = \sum_{X\in\Gamma\backslash L_N}\frac{1}{|\overline{\Gamma}_X|}. \tag{3.1}$$

For $p = 1$, this is twice the famous *Kronecker-Hurwitz* class number $H(N)$ of positive definite binary integral (not necessarily primitive) quadratic forms of discriminant $-N$. From that perspective, we can consider $\deg Z(N)$ for a general lattice $L$ as a generalized class number. On the other hand, $\deg Z(N)$ is essentially the number of length $N$ vectors in the lattice $L$ modulo $\Gamma$. So we can think about $\deg Z(N)$ also as the direct analogue of the classical representation numbers by quadratic forms in the positive definite case.

**Theorem 3.1** ([16]). *Recall that we write* $\tau = u + iv \in \mathbb{H}$. *Then*

$$I(\tau,1) = \mathrm{vol}(X) + \sum_{N=1}^{\infty}\deg Z(N)q^N + \frac{1}{8\pi\sqrt{v}}\sum_{n=-\infty}^{\infty}\beta(4\pi v n^2)q^{-n^2}.$$

*Here* $\mathrm{vol}(X) = -\frac{1}{2\pi}\int_X \omega \in \mathbb{Q}$ *is the (normalized) volume of the modular curve* $M$. *Furthermore,* $\beta(s) = \int_1^\infty e^{-st}t^{-3/2}dt$.

In particular, for $p = 1$, we recover Zagier's well known Eisenstein series $\mathcal{F}(\tau)$ of weight 3/2, see [40, 21]. Namely, we have

**Theorem 3.2.** *Let* $p = 1$, *so that* $\deg Z(N) = 2H(N)$. *Then*

$$\frac{1}{2}I(\tau,1) = \mathcal{F}(\tau) = -\frac{1}{12} + \sum_{N=1}^{\infty}H(N)q^N + \frac{1}{16\pi\sqrt{v}}\sum_{n=-\infty}^{\infty}\beta(4\pi n^2 v)q^{-n^2}$$

**Remark 3.3.** We can view Theorem 3.1 on one hand as the generalization of Zagier's Eisenstein series. On the other hand, we can consider Theorem 3.2 as a special case of the Siegel-Weil formula, realizing the theta integral as an Eisenstein series. Note however that here Theorem 3.2 arises by explicit computation and comparison of the Fourier expansions on both sides. For a more intrinsic proof, see Section 3.3 below.

**Remark 3.4.** Lemma 2.2 immediately takes care of a large class of coefficients. However, the calculation of the Fourier coefficients of index $-m^2$ is quite delicate and represents the main technical difficulty for Theorem 3.1, since the usual unfolding argument is not allowed. We have two ways of computing the integral. In [16], we employ a method somewhat similar to Zagier's method in [41], namely we appropriately regularize the integral in order to unfold. In [12], we use Lemma 2.3, i.e., explicitly the fact that for negative index, the Schwartz function $\varphi_{KM}(x)$ (with $(x,x) < 0$) is *exact* and apply Stokes' Theorem.

**Remark 3.5.** In joint work with O. Imamoglu [17], we are currently considering the analogue of the present situation to general hyperbolic space $(1,q)$. We study a similar theta integral for constant and other input. In particular, we realize the generating series of certain 0-cycles inside hyperbolic manifolds as Eisenstein series of weight $(q+1)/2$.

3.2. **The lift of modular functions and weak Maas forms.** In [11], we introduced the space of weak Maass forms. For weight 0, it consists of those $\Gamma$-invariant and harmonic functions $f$ on $D \simeq \mathbb{H}$ which satisfy $f(z) = O(e^{Cy})$ as $z \to \infty$ for some constant $C$. We denote this space by $H_0(\Gamma)$. A form $f \in H_0(\Gamma)$ can be written as $f = f^+ + f^-$, where the Fourier expansions of $f^+$ and $f^-$ are of the form

(3.2)
$$f^+(z) = \sum_{n\in\mathbb{Z}} b^+(n)e(nz) \qquad \text{and} \qquad f^-(z) = b^-(0)v + \sum_{n\in\mathbb{Z}-\{0\}} b^-(n)e(n\bar{z}),$$

where $b^+(n) = 0$ for $n \ll 0$ and $b^-(n) = 0$ for $n \gg 0$. We let $H_0^+(\Gamma)$ be the subspace of those $f$ that satisfy $b^-(n) = 0$ for $n \geq 0$. It consists for those $f \in H_0(\Gamma)$ such that $f^-$ is exponentially decreasing at the cusps. We define a $\mathbb{C}$-antilinear map by $(\xi_0 f)(z) = y^{-2}\overline{L_0 f(z)} = R_0\overline{f(z)}$. Here $L_0$ and $R_0$ are the weight 0 Maass lowering and raising operators. Then the significance of $H_0^+(\Gamma)$ lies in the fact, see [11], Section 3, that $\xi_0$ maps $H_0^+(\Gamma)$ onto $S_2(\Gamma)$, the space of weight 2 cusp forms for $\Gamma$. Furthermore, we let $M_0^!(\Gamma)$ be the space of modular functions for $\Gamma$ (or weakly holomorphic modular forms for $\Gamma$ of weight 0). Note that $\ker\xi = M_0^!(\Gamma)$. We therefore have a short exact sequence

$$0 \longrightarrow M_0^!(\Gamma) \longrightarrow H_0^+(\Gamma) \xrightarrow{\xi_0} S_2(\Gamma) \longrightarrow 0\,. \tag{3.3}$$

**Theorem 3.6** ([12], Theorem 1.1)**.** *For $f \in H_0^+(\Gamma)$, assume that the constant coefficient $b^+(0)$ vanishes. Then*

$$I(\tau, f) = \sum_{N>0} \mathbf{t}_f(N) q^N + \sum_{n\geq 0} \big(\sigma_1(n) + p\sigma_1(n/p)\big) b^+(-n) - \sum_{m>0}\sum_{n>0} m b^+(-mn) q^{-m^2}$$

*is a weakly holomorphic modular form (i.e., meromorphic with the poles concentrated inside the cusps) of weight $3/2$ for the group $\Gamma_0(4p)$. If $a(0)$ does not vanish, then* <u>*in addition*</u> *non-holomorphic terms as in Theorem 3.1 occur, namely*

$$\frac{1}{8\pi\sqrt{v}} b^+(0) \sum_{n=-\infty}^{\infty} \beta(4\pi v n^2) q^{-n^2}.$$

For $p = 1$, we let $J(z) := j(z) - 744$ be the normalized Hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$. Here $j(z)$ is the famous $j$-invariant. The values of $j$ at the CM points are of classical interest and are known as *singular moduli.* For example, they are algebraic integers. In fact, the values at the CM points of discriminant $D$ generate the Hilbert class field of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$. Hence its modular trace (which can be also be considered as a suitable Galois trace) is of particular interest. Zagier [42] realized the generating series of the traces of the singular moduli as a weakly holomorpic modular form of weight $3/2$. For $p = 1$, Theorem 3.6 recovers this influential result of Zagier [42].

**Theorem 3.7** (Zagier [42])**.** *We have that*

$$-q^{-1} + 2 + \sum_{N=1}^{\infty} \mathbf{t}_J(N) q^N$$

*is a weakly holomorphic modular form of weight $3/2$ for $\Gamma_0(4)$.*

**Remark 3.8.** The proof of Theorem 3.6 follows Lemmas 2.2, 2.3, and 2.4. The formulas given there simplify greatly since the input $f$ is harmonic (or even holomorphic) and $\overline{\partial} f$ is rapidly decreasing (or even vanishes). Again, the coefficients of index $-m^2$ are quite delicate. Furthermore, $a_0''(v)$ vanishes unless $b_0^+$ is nonzero, while we use a method of Borcherds [2] to explicitly compute the average value $a_0'$ of $f$. (Actually, for $a_0'$, Remark 4.9 in [12] only covers the holomorphic case, but the same argument as in the proof of Theorem 7.8 in [12] shows that the calculation is also valid for $H_0^+$).

**Remark 3.9.** Note that Zagier's approach to the above result is quite different. To obtain Theorem 3.7, he explicitly constructs a weakly holomorphic modular form of weight 3/2, which turns to be the generating series of the traces of the singular moduli. His proof heavily depends on the fact that the Riemann surface in question, $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, has genus 0. In fact, Zagier's proof extends to other genus 0 Riemann surfaces, see [23, 24].

Our approach addresses several questions and issues which arise from Zagier's work:

- We show that the condition 'genus 0' is irrelevant in this context; the result holds for (suitable) modular curves of any genus.
- A geometric interpretation of the constant coefficient is given as the regularized average value of $f$ over $M$, see Lemma 2.4. It can be explicitly computed, see Remark 3.8 above.
- A geometric interpretation of the coefficient(s) of negative index is given in terms of the behavior of $f$ at the cusp, see Definition 4.4 and Theorem 4.5 in [12].
- We settle the question when the generating series of modular traces for a weakly holomorphic form $f \in M_0^!(\Gamma)$ is part of a weakly holomorphic form of weight 3/2 (as it is the case for $J(z)$) or when it is part of a *non*holomorphic form (as it is the case for the constant function $1 \in M_0^!(\Gamma)$). This behavior is governed by the (non)vanishing of the constant coefficient of $f$.

**Remark 3.10.** Theorem 3.6 has inspired several papers of K. Ono and his collaborators, see [4, 5, 7]. In Section 5, we generalize some aspects of [7].

**Remark 3.11.** As this point we are not aware of any particular application of the above formula in the case when $f$ is a weak Maass form and not weakly holomorphic. However, it is important to see that the result does not (directly) depend on the underlying complex structure of $D$. This suggests possible generalizations to locally symmetric spaces for other orthogonal groups when they might or might not be an underlying complex structure, most notably for hyperbolic space associated to signature $(1, q)$, see [17]. The issue is to find appropriate analogues of the space of weak Maass forms in these situations.

In any case, the space of weak Maass forms has already displayed its significance, for example in the work of Bruinier [8], Bruinier-Funke [11], and Bringmann-Ono [6].

3.3. **The lift of the weight 0 Eisenstein Series.** For $z \in \mathbb{H}$ and $s \in \mathbb{C}$, we let

$$\mathcal{E}_0(z,s) = \frac{1}{2}\zeta^*(2s+1) \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (\Im(\gamma z))^{s+\frac{1}{2}}$$

be the Eisenstein series of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$. Here $\Gamma_\infty$ is the standard stabilizer of the cusp $i\infty$ and $\zeta^*(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ is the completed Riemann Zeta function. Recall that with the above normalization, $\mathcal{E}_0(z,s)$ converges for $\Re(s) > 1/2$ and has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1/2$ with residue $1/2$.

**Theorem 3.12** ([12], Theorem 7.1)**.** *Let $p = 1$. Then*

$$I(\tau, \mathcal{E}_0(z,s)) = \zeta^*(s + \frac{1}{2})\mathcal{F}(\tau, s).$$

*Here we use the normalization of Zagier's Eisenstein series as given in* [39]*, in particular $\mathcal{F}(\tau) = \mathcal{F}(\tau, \frac{1}{2})$.*

We proof this result by switching to a mixed model of the Weil representation and using not more than the definition of the two Eisenstein series involved. In particular, we do not have to compute the Fourier expansion of the Eisenstein series. One can also consider Theorem 3.12 and its proof as a special case of the extension of the Siegel Weil formula by Kudla and Rallis [31] to the divergent range. Note however, that our case is actually not covered in [31], since for simplicity they only consider the integral weight case to avoid dealing with metaplectic coverings.

Taking residues at $s = 1/2$ on both sides of Theorem 3.12 one obtains again

**Theorem 3.13.**

$$I(\tau, 1) = \tfrac{1}{2}\mathcal{F}(\tau, \tfrac{1}{2}),$$

*as asserted by the Siegel-Weil formula.*

From our point of view, one can consider Theorem 3.2/3.13 as some kind of *geometric* Siegel-Weil formula (Kudla): The geometric degrees of the 0-cycles $Z(N)$ in (regular) (co)homology form the Fourier coefficients of the special value of an Eisenstein series. For the analogous (compact) case of a Shimura curve, see [32].

### 3.4. **Other Inputs.**

3.4.1. *Maass cusp forms.* We can also consider $I(\tau, f)$ for $f \in L^2_{cusp}(\Gamma\backslash D)$, the space of cuspidal square integrable functions on $\Gamma\backslash D = M$. In that case, the lift is closely related to another theta lift $I_M$ first introduced by Maass [34] and later reconsidered by Duke [14] and Katok and Sarnak [22]. The Maass lift uses a similar theta kernel associated to a quadratic space of signature $(2,1)$ and maps rapidly decreasing functions on $M$ to forms of weight $1/2$. In fact, in [34, 22] only Maass forms are considered, that is, eigenfunctions of the hyperbolic Laplacian $\Delta$.

To describe the relationship between $I$ and $I_M$, we need the operator $\xi_k$ which maps forms of weight $k$ to forms of "dual" weight $2-k$. It is given by

$$\xi_k(f)(\tau) = v^{k-2}\overline{L_k f(\tau)} = R_{-k} v^k \overline{f(\tau)}, \tag{3.4}$$

where $L_k$ and $R_{-k}$ are the usual Maass lowering and raising operators. In [12], we establish an explicit relationship between the two kernel functions and obtain

**Theorem 3.14** ([12])**.** *For $f \in L^2_{cusp}(\Gamma\backslash D)$, we have*

$$\xi_{1/2} I_M(\tau, f) = -\pi I(\tau, f).$$

*If $f$ is an eigenfunction of $\Delta$ with eigenvalue $\lambda$, then we also have*

$$\xi_{3/2} I(\tau, f) = -\frac{\lambda}{4\pi} I_M(\tau, f).$$

**Remark 3.15.** The theorem shows that the two lifts are essentially equivalent on Maass forms. However, the theta kernel for $I_M$ is moderately increasing. Hence one cannot define the Maass lift on $H_0^+$, at least not without regularization. On the other hand, since $I(\tau, f)$ is holomorphic for $f \in H_0^+$, we have $\xi_{3/2} I(\tau, f) = 0$ (which would be the case $\lambda = 0$).

**Remark 3.16.** Duke [14] uses the Maass lift to establish an equidistribution result for the CM points and also certain geodesics in $M$ (which in our context correspond to the negative coefficients). Katok and Sarnak [22] use the fact that the periods over these geodesics correspond to the values of $L$-functions at the center of the critical strip to extend the nonnegativity of those values to Maass Hecke eigenforms. It seems that for these applications one could have also used our lift $I$.

3.4.2. *Petersson metric of (weakly) holomorphic modular forms.* Similarly, one could study the lift for the Petersson metric of a (weakly) holomorphic modular form $f$ of weight $k$ for $\Gamma$. For such an $f$, we define its Petersson metric by $\|f(z)\| = |f(z)y^{k/2}|$. Then by Lemma 2.2 the holomorphic part of the positive Fourier coefficients of $I(\tau, \|f\|)$ is given by the $\mathbf{t}_{\|f\|}(N)$. It would be very interestig to find an application for this modular trace.

It should be also interesting to consider the lift of the Petersson metric for a *meromorphic* modular form $f$ or in weight 0, of a meromorphic modular function itself. Of course, in these cases, the integral is typcially divergent and needs to be normalized. To find an appropriate normalization would be interesting in its own right.

3.4.3. *Other Weights.* Zagier [42] also discusses a few special cases of traces for a (weakly holomorphic) modular form $f$ of negative weight $-2k$ (for small $k$) by considering the modular trace of $R_{-2} \circ R_{-4} \circ \cdots \circ R_{-2k} f$, where $R_\ell$ denotes the raising operator for weight $\ell$. For $k$ even, Zagier obtains a correspondence in which forms of weight $-2k$ correspond to forms of positive weight $3/2 + k$. Zagier's student Fricke [15] following our work [12] introduces theta kernels similar to ours to realize Zagier's correspondence via theta liftings. It would be interesting to see whether his approach can be understood in terms of the extension of the Kudla-Millson theory to cycles with coefficients by Funke and Millson [19]. For $k$ odd, Zagier's correspondence takes a different form, namely forms of weight $-2k$ correspond to forms of negative weight $1/2 - k$. For this correspondence, one needs to use a different approach, constructing other theta kernels.

## 4. The lift of $\log \|f\|$

In this section, we study the lift for the logarithm of the Petersson metric of a meromorphic modular form $f$ of weight $k$ for $\Gamma$. We normalize the Petersson metric such that it is given by

$$\|f(z)\| = e^{-kC/2} |f(z)(4\pi y)^{k/2}|,$$

with $C = \frac{1}{2}(\gamma + \log 4\pi)$. Here $\gamma$ is Euler's constant.

The motivation to consider such input comes from the fact that the positive Fourier coefficients of the lift will involve the trace $\mathbf{t}_{\log\|f\|}(N)$. It is well known that such a trace plays a significant role in arithmetic geometry as we will also see below.

### 4.1. The lift of $\log \|\Delta\|$.

We first consider the discriminant function

$$\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right)^{24}.$$

Via the Kronecker limit formula

$$-\frac{1}{12} \log |\Delta(z) y^6| = \lim_{s \to \frac{1}{2}} \left(\mathcal{E}_0(z,s) - \zeta^*(2s-1)\right) \tag{4.1}$$

we can use Theorem 3.12 to compute the lift $I(\tau, \|\Delta\|)$. Namely, we take the constant term of the Laurent expansion at $s = 1/2$ on both sides of Theorem 3.12 and obtain

**Theorem 4.1.** *We have*

$$-\frac{1}{12} I\left(\tau, \log \|\Delta(z)\|\right) = \mathcal{F}'(\tau, \frac{1}{2}).$$

On the other hand, we can give an interpretation in arithmetic geometry in the context of the program of Kudla, Rapoport and Yang, see e.g. [33]. We give a very brief sketch. For more details, see [39, 32, 12]. We let $\mathcal{M}$ be the Deligne-Rapoport compactification of the moduli stack over $\mathbb{Z}$ of elliptic curves, so $\mathcal{M}(\mathbb{C})$ is the orbifold $SL_2(\mathbb{Z})\backslash\mathbb{H}\cup\infty$. We let $\widehat{CH}^1_{\mathbb{R}}(\mathcal{M})$ be the extended arithmetic Chow group of $\mathcal{M}$ with real coefficients and let $\langle\, ,\, \rangle$ be the extended Gillet-Soulé intersection pairing, see [37, 3, 13, 25]. The normalized metrized Hodge bundle $\widehat{\omega}$ on $\mathcal{M}$ defines an element

$$\widehat{c}_1(\widehat{\omega}) = \tfrac{1}{12}(\infty, -\log\|\Delta(z)\|^2) \in \widehat{CH}^1_{\mathbb{R}}(\mathcal{M}). \tag{4.2}$$

For $N \in \mathbb{Z}$ and $v > 0$, Kudla, Rapoport and Yang construct elements $\widehat{\mathcal{Z}}(N,v) = (\mathcal{Z}(N), \Xi(N,v)) \in \widehat{CH}^1_{\mathbb{R}}(\mathcal{M})$. Here for $N > 0$ the complex points of $\mathcal{Z}(N)$ are the CM points $Z(N)$ and $\xi(N,v) = \sum_{X\in L_N} \xi^0(\sqrt{v}X)$ is a Green's function for $Z(N)$. In [12] we indicate

**Theorem 4.2** ([12])**.**

$$-\frac{1}{12} I\left(\tau, \log\left(\|\Delta(z)\|\right)\right) = 4\sum_{N\in\mathbb{Z}} \langle \widehat{\mathcal{Z}}(N,v), \widehat{\omega}\rangle q^N.$$

We therefore recover

**Theorem 4.3** ((Kudla-Rapoport-)Yang [39])**.** *For the generating series of the arithmetic degrees $\langle\widehat{\mathcal{Z}}(N,v), \widehat{\omega}\rangle$, we have*

$$\sum_{N\in\mathbb{Z}} \langle \widehat{\mathcal{Z}}(N,v), \widehat{\omega}\rangle q^N = \frac{1}{4}\mathcal{F}'(\tau, \frac{1}{2}).$$

**Remark 4.4.** One can view our treatment of the above result as some kind of arithmetic Siegel-Weil formula in the given situation, realizing the "arithmetic theta series" (Kudla) of the arithmetic degrees of the cycles $\mathcal{Z}(N)$ on the left hand side of Theorem 4.3 as an honest theta integral (and as the derivative of an Eisenstein series).

Our proof is different than the one given in [39]. We use two different ways of 'interpreting' the theta lift, the Kronecker limit formula, and unwind the basic definitions and formulas of the Gillet-Soulé intersection pairing. The proof given in [39] is based on the explicit computation of both sides, which is not needed with our method. The approach and techniques in [39] are the same as the ones Kudla, Rapoport, and Yang [32] employ in the analogous situation for 0-cycles in Shimura curves. In that case again, the generating series of the arithmetic degrees of the analogous cycles is the *derivative* of a certain Eisenstein series.

It needs to be stressed that the present case is considerably easier than the Shimura curve case. For example, in our situation the finite primes play no role, since the CM points do not intersect the cusp over $\mathbb{Z}$. Moreover, our approach is not applicable in the Shimura curve case, since there are no Eisenstein series (and no Kronecker limit formula). See also Remark 4.10 below.

Finally note that by Lemma 2.2 we see that the main (holomorphic) part of the positive Fourier coefficients of the lift is given by $\mathbf{t}_{\log \|\Delta(z)y^6\|}(N)$, which is equal to the Faltings height of the cycle $\mathcal{Z}(N)$. For details, we refer again the reader to [39].

4.2. **The lift for general $f$.** In this section, we consider $I(\tau, \log\|f\|)$ for a general meromorphic modular form $f$. Note that while $\log\|f\|$ is of course integrable, we cannot evaluate $\log\|f\|$ at the divisor of $f$. So if the divisor of $f$ is not disjoint to (one) of the 0-cycles $Z(N)$, we need to expect complications when computing the Fourier expansion of $I(\tau, \log\|f\|)$.

We let $t$ be the order of $f$ at the point $D_X = z_0$, i.e., $t$ is the smallest integer such that

$$\lim_{z\to z_0} (z-z_0)^{-t} f(z) =: f^{(t)}(z_0) \notin \{0,\infty\}.$$

Note that the value $f^{(t)}(z_0)$ does depend on $z_0$ itself and not just on the $\Gamma$-equivalence class of $z_0$. If $f$ has order $t$ at $z_0$ we put

$$||f^{(t)}(z_0)|| = e^{-C(t+k/2)}|f^{(t)}(z_0)(4\pi y_0)^{t+k/2}|$$

**Lemma 4.5.** *The value $||f^{(t)}(z_0)||$ depends only on the $\Gamma$-equivalence class of $z_0$, i.e.,*

$$||f^{(t)}(\gamma z_0)|| = ||f^{(t)}(z_0)||$$

*for $\gamma\in\Gamma$.*

*Proof.* It's enough to do the case $t\geq 0$. For $t<0$, consider $1/f$. We successively apply the raising operator $R_\ell = 2i\frac{\partial}{\partial\tau} + \ell y^{-1}$ to $f$ and obtain

$$\left(-\frac{1}{2}i\right)^t R_{k+t-2}\circ\cdots\circ R_k f(z) = f^{(t)}(z) + \text{lower derivatives of } f. \tag{4.3}$$

But $|R_{k+t-2}\cdots R_k e^{-C(t+k/2)} f(z)(4\pi)y^{t+k/2}|$ has weight 0 and its value at $z_0$ is equal to $||f^{(t)}(z_0)||$ since the lower derivatives of $f$ vanish at $z_0$. □

**Theorem 4.6.** *Let $f$ be a meromorphic modular form of weight $k$. Then for $N>0$, the $N$-th Fourier coefficient of $I(\tau,\log\|f\|)$ is given by*

$$a_N(v) = \sum_{z\in Z(N)} \frac{1}{|\bar\Gamma_z|}\left(\log||f^{(ord(f,z))}(z)|| - \frac{ord(f,z)}{2}\log((4\pi)^2 Nv) + \frac{k}{16\pi i} J(4\pi N v)\right),$$

*where*

$$J(t) = \int_0^\infty e^{-tw}[(w+1)^{\frac12}-1]w^{-1}dw.$$

We give the proof of Theorem 4.6 in the next section.

**Remark 4.7.** We will leave the computation of the other Fourier coefficients for another time. Note however, that the coefficient for $N < 0$ such that $N \notin -(\mathbb{Q})^2$ can be found in [32], section 12.

**Remark 4.8.** The constant coefficient $a_0'$ of the lift is given by

$$\int_M^{reg} \log ||f(z)|| \frac{dx\, dy}{y^2}, \tag{4.4}$$

see Lemma 2.4. An explicit formula can be obtained by means of Rohrlich's modular Jensen's formula [35], which holds for $f$ holomorphic on $D$ and not vanishing at the cusp. For an extension of this formula in the context of arithmetic intersection numbers, see e.g. Kühn [25]. See also Remark 4.10 below.

**Example 4.9.** In the case of the classical $j$-invariant the modular trace of the logarithm of the $j$-invariant is the logarithm of the *norm* of the singular moduli, i.e.,

$$\mathbf{t}_{\log|j|}(N) = \log|\prod_{z \in Z(N)} j(z)|. \tag{4.5}$$

Recall that the norms of the singular moduli were studied by Gross-Zagier [20]. On the other hand, we have $j(\rho) = 0$ for $\rho = \frac{1+i\sqrt{3}}{2}$ and $\frac{1}{3}\rho \in Z(3N^2)$. Hence for these indices the trace is not defined. Note that the third derivative $j'''(\rho)$ is the first non-vanishing derivative of $j$ at $\rho$. Thus

(4.6)

$$I(\tau, \log|j(z)|) = \sum_{D>0} \mathbf{t}'_{\log|j|}(D) q^D + \sum_{N=1}^{\infty} \left( \log \|j^{(3)}(\rho)\| - \frac{1}{2}\log(48\pi^2 N^2 v) \right) q^{3N^2} + \ldots.$$

Here $\mathbf{t}'_{\log|j|}(D)$ denotes the usual trace for $D \neq 3N^2$, while for $D = 3N^2$ one excludes the term corresponding to $\rho$.

Finally note that Gross-Zagier [20] in their analytic approach to the singular moduli (sections 5-7) make also essential use of the *derivative* of an Eisenstein series (of weight 1 for the Hilbert modular group).

**Remark 4.10.** It is a very interesting problem to consider the special case when $f$ is a Borcherds product. That is, when

$$\log||f(z)|| = \Phi(z, g), \tag{4.7}$$

where $\Phi(z, g)$ is a theta lift of a (weakly) holomorphic modular form of weight 1/2 via a certain regularized theta integral, see [2, 8]. The calculation of the constant coefficient $a_0'$ of the lift $I(\tau, \Phi(z, g))$ boils down (for general signature $(n, 2)$) to work of Kudla [27] and Bruinier and Kühn [10] on integrals of Borcherds forms. (The present case of a modular curve is excluded to avoid some technical difficulties). Roughly speaking, one obtains a linear combination of Fourier coefficients of the *derivative* of a certain Eisenstein series.

From that perspective, it is reasonable to expect that for the Petersson metric of Borcherds products, the full lift $I(\tau, \Phi(z,g))$ will involve the derivative of certain Eisenstein series, in particular in view of Kudla's approach in [27] via the Siegel-Weil formula. Note that the discriminant function $\Delta$ can be realized as a Borcherds product. Therefore, one can reasonably expect a new proof for Theorem 4.1. Furthermore, this method a priori is also available for the Shimura curve case (as opposed to the Kronecker limit formula), and one can hope to have a new approach to some aspects (say, at least for the Archimedean prime) of the work of Kudla, Rapoport, and Yang [32, 33] on arithmetic generating series in the Shimura curve case.

We will come back to these issues in the near future.

4.3. **Proof of Theorem 4.6.** For the proof of the theorem, we will show how Theorem 2.1 extends to functions which have a logarithmic singularity at the CM point $D_X$. This will then give the formula for the positive coefficients.

**Proposition 4.11.** *Let $q(X) = N > 0$ and let $f$ be a meromorphic modular form of weight $k$ with order $t$ at $D_X = z_0$. Then*

$$\int_D \log ||f(z)|| \cdot \varphi^0(X,z) = ||f^{(t)}(z_0)|| - \frac{t}{2}\log((4\pi)^2 N) + \int_D dd^c \log||f(z)|| \cdot \xi^0(X,z)$$
$$= ||f^{(t)}(z_0)|| - \frac{t}{2}\log((4\pi)^2 N) + \frac{k}{16\pi i}\int_D \xi^0(X,z)\frac{dxdy}{y^2}.$$

Note that by [32], section 12 we have

$$\int_D \xi^0(X,z)\frac{dxdy}{y^2} = J(4\pi N).$$

*Proof of Proposition 4.11.* The proof consists of a careful analysis and extension of the proof of Theorem 2.1 given in [26]. We will need

**Lemma 4.12.** *Let*

$$\tilde{\xi}^0(X,z) = \xi^0(X,z) + \log|z - z_0|^2.$$

*Then $\tilde{\xi}^0(X,z)$ extends to a smooth function on $D$ and*

$$\tilde{\xi}^0(X,z_0) = -\gamma - \log(4\pi N/y_0^2).$$

*In particular, writing $z - z_0 = re^{i\theta}$, we have*

$$\frac{\partial}{\partial r}\tilde{\xi}^0(X,z) = O(1)$$

*in a neighborhood of* $z_0$.

*Proof of Lemma 4.12.* This is basically Lemma 11.2 in [26]. We have

$$R(X,z) = 2N\left[\frac{r^2}{2y_0(y_0+r\cos\theta)}\right]\left[\frac{r^2}{2y_0(y_0+r\cos\theta)}+2\right]. \tag{4.8}$$

Since

$$Ei(z) = \gamma + \log(-z) + \int_0^z \frac{e^t-1}{t}dt,$$

we have

$$\begin{aligned}\tilde{\xi}^0(X,z) = -\gamma - \log\left(\left[\frac{4\pi N}{2y_0(y_0+r\cos\theta)}\right]\left[\frac{r^2}{2y_0(y_0+r\cos\theta)}+2\right]\right)\\ -\int_0^{-2\pi R(X,z)}\frac{e^t-1}{t}dt.\end{aligned} \tag{4.9}$$

The claims follow. □

For the proof of the proposition, we first note that (2.17) in Theorem 2.1 still holds for $F = \log\|f\|$ when the divisor of $f$ is disjoint to $D_X$. We now consider $\int_D dd^c \log||f(z)|| \cdot \xi^0(X,z)$. Since $\log||(z-z_0)^{-t}f(z)||$ is smooth at $z = z_0$, we see

$$\begin{aligned}\int_D dd^c \log||f(z)|| \cdot \xi^0(X,z) &= \int_D dd^c \log||(z-z_0)^{-t}f(z)|| \cdot \xi^0(X,z)\\ &= -\log||f^{(t)}(z_0)|| - tC + t\log(4\pi y_0)\\ &\quad + \int_D \log||(z-z_0)^{-t}f(z)|| \cdot \varphi^0(X,z).\end{aligned} \tag{4.10}$$

So for the proposition it suffices to proof

$$\int_D \log|z-z_0|^{-t} \cdot \varphi^0(X,z) = \frac{t}{2}\left(\gamma + \log(4\pi N/y_0^2)\right). \tag{4.11}$$

For this, we let $U_\varepsilon$ be an $\varepsilon$-neighborhood of $z_0$. We see

$$\begin{aligned}\int_{D-U_\varepsilon} dd^c \log|z-z_0|^t \cdot \xi^0(X,z) &= \int_{D-U_\varepsilon} \log|z-z_0|^t \cdot dd^c\xi^0(X,z)\\ &\quad + \int_{\partial\{D-U_\varepsilon\}} \left(\xi^0 d^c \log|z-z_0|^t - \log|z-z_0|^t d^c\xi^0\right).\end{aligned} \tag{4.12}$$

Of course $dd^c \log|z-z_0|^t = 0$ (outside $z_0$), so the integral on the left hand side vanishes. For the first term on the right hand side, we note $dd^c\xi^0 = \varphi^0$,

and using the rapid decay of $\xi^0(X)$, we obtain
(4.13)
$$\int_D \log|z-z_0|^t \varphi^0(X,z) = \lim_{\varepsilon\to 0}\int_{\partial U_\varepsilon}\left(\xi^0 d^c\log|z-z_0|^t - \log|z-z_0|^t d^c\xi^0\right).$$
For the right hand side of 4.13, we write $z-z_0 = re^{i\theta}$. Using $d^c = \frac{r}{4\pi}\frac{\partial}{\partial r}d\theta - \frac{1}{4\pi r}\frac{\partial}{\partial\theta}dr$, we see $d^c\log|z-z_0| = \frac{1}{4\pi}d\theta$. Via Lemma 4.12, we now obtain
$$\begin{aligned}
&\int_{\partial U_\varepsilon}\left(\xi^0 d^c\log|z-z_0|^t - \log|z-z_0|^t d^c\xi^0\right)\\
&= \int_0^{2\pi}\left[(-\log\varepsilon^2+\tilde{\xi}^0)\frac{t}{4\pi}d\theta - t\log\varepsilon(-\frac{1}{2\pi}d\theta + O(\varepsilon)d\theta)\right]\\
&= \int_0^{2\pi}\left[\tilde{\xi}^0\frac{t}{4\pi}d\theta - t\log\varepsilon O(\varepsilon)d\theta\right]\\
&\to \frac{t}{2}\tilde{\xi}^0(X,z_0) = -\frac{t}{2}\left(\gamma+\log(4N\pi/y_0^2)\right) \qquad \text{as } \varepsilon\to 0.
\end{aligned}$$
The proposition follows. □

## 5. Higher dimensional Analogues

We change the setting from the previous sections and let $V$ now be a rational quadratic space of signature $(n,2)$. We let $G = \mathrm{SO}_0(V(\mathbb{R}))$ be the connected component of the identity of $\mathrm{O}(V(\mathbb{R}))$. We let $D$ be the associated symmetric space, which we realize as the space of negative two planes in $V(\mathbb{R})$:
$$D = \{z \subset V(\mathbb{R});\ \dim z = 2 \,\text{and}\, (\ ,\ )|_z < 0\}. \tag{5.1}$$
We let $L$ be an even lattice in $V$ and $\Gamma$ a congruence subgroup inside $G$ stabilizing $L$. We assume for simplicity that $\Gamma$ is neat and that $\Gamma$ acts on the discriminant group $L^{\#}/L$ trivially. We set $M = \Gamma\backslash D$. It is well known that $D$ has a complex structure and $M$ is a (in general) quasi-projective variety.

A vector $x \in V$ such that $(x,x) > 0$ defines a divisor $D_x$ by
$$D_x = \{z \in D; z \perp x\}. \tag{5.2}$$
The stabilizer $\Gamma_x$ acts on $D_x$, and we define the special divisor $Z(x) = \Gamma_x\backslash D_x \hookrightarrow M$. For $N \in \mathbb{Z}$, we set $L_N = \{x \in L;\ q(x) := \frac{1}{2}(x,x) = N\}$ and for $N > 0$, we define the composite cycle $Z(N)$ by
$$Z(N) = \sum_{x\in\Gamma\backslash L_N} Z(x). \tag{5.3}$$
For $n = 1$, these are the CM points inside a modular (or Shimura) curve discussed before, while for $n = 2$, these are (for $\mathbb{Q}$-rank 1) the famous

Hirzebruch-Zagier divisors inside a Hilbert modular surface, see [21] or [38]. On the other hand, we let $U \subset V$ be a rational positive definite subspace of dimension $n-1$. We then set

$$D_U = \{z \in D; z \perp U\}. \tag{5.4}$$

This is an embedded upper half plane $\mathbb{H}$ inside $D$. We let $\Gamma_U$ be the stabilizer of $U$ inside $\Gamma$ and set $Z(U) = \Gamma_U \backslash D_U$ which defines a modular or Shimura curve. We denote by $\iota_U$ the embedding of $Z(U)$ into $M$ (which we frequently omit). Therefore

$$D_U \cap D_x = \begin{cases} D_{U,x} & \text{if } x \notin U \\ D_U & \text{if } x \in U. \end{cases} \tag{5.5}$$

Here $D_{U,x}$ is the point (negative two plane in $V(\mathbb{R})$) in $D$, which is orthogonal to both $U$ and $x$. We denote its image in $M$ by $Z(U,x)$. Consequently $Z(U)$ and $Z(x)$ intersect transversally in $Z(U,x)$ if and only if $\gamma x \notin U$ for all $\gamma \in \Gamma$ while $Z(U) = Z(x)$ if and only if $\gamma x \in U$ for one $\gamma \in \Gamma$. This defines a (set theoretic) intersection

$$(Z(U) \cap Z(N))_M \tag{5.6}$$

in (the interior of) $M$ consisting of 0- and 1-dimensional components. For $n = 2$, the Hilbert modular surface case, this follows Hirzebruch and Zagier ([21]). For $f$ a function on the curve $Z(U)$, we let $(Z(U) \cap Z(N))_M [f]$ be the evaluation of $f$ on $(Z(U) \cap Z(N))_M$. Here on the 1-dimensional components we mean by this the (regularized) average value of $f$ over the curve, see (2.19). Now write

$$L = \sum_{i=1}^{r} (L_U + \lambda_i) \ \perp \ (L_{U^\perp} + \mu_i) \tag{5.7}$$

with $\lambda_i \in L_U^\#$ and $\mu_i \in L_{U^\perp}^\#$ such that $\lambda_1 = \mu_1 = 0$.

**Lemma 5.1.** *Let $r(N_1, L_U + \lambda_i) = \#\{x \in L_U + \lambda_i : q(x) = N_1\}$ be the representation number of the positive definite (coset of the) lattice $L_U$, and let $Z(N_2, L_{U^\perp} + \mu_i) = \sum_{\substack{x \in \Gamma_U \backslash (L_{U^\perp} + \mu_i) \\ q(x) = N_2}} Z(x)$ be the CM cycle inside the curve $Z(U)$. Let $f$ a function on the curve $Z(U)$. Then*

$$\begin{aligned} (Z(U) \cap Z(N))_M [f] = &\sum_{\substack{N_1 \geq 0, N_2 > 0 \\ N_1 + N_2 = N}} \sum_{i=1}^{r} r(N_1, L_U + \lambda_i) Z(N_2, L_{U^\perp} + \mu_i)[f] \\ &- \frac{1}{2\pi} r(N, L_U) \int_{Z(U)}^{reg} f(z) \omega. \end{aligned}$$

*Proof.* A vector $x \in L_U \cap L_N$ gives rise to a 1-dimensional intersection, and conversely a 1-dimensional intersection arises from a vector $x \in L_N$ which can be taken after translating by a suitable $\gamma \in \Gamma$ to be in $L_U$. Thus the 1-dimensional component is equal to $Z(U)$ occurring with multiplicity $r(N, L_U)$. Note that only the component $\lambda_1 = \mu_1 = 0$ occurs. This gives the second term. For the 0-dimensional components, we first take an $x = x_1 + x_2 \in L_N$ with $x_1 \in L_U + \lambda_i$ and $x_2 \in L_{U^\perp} + \mu_i$ such that $q(x_1) = N_1$ and $q(x_2) = N_2$. This gives rise to the transversal intersection point $Z(U, x)$ if $x$ cannot be $\Gamma$-translated into $U$. Note that this point lies in the CM cycle $Z(N_2, L_{U^\perp} + \mu_i)$ inside $Z(U)$. In fact, in this way, we see by changing $x_2$ by $y_2 \in L_{U^\perp} + \mu_i$ of the same length $N_2$ that the whole cycle $Z(N_2, L_{U^\perp} + \mu_i)$ lies in the transversal part of $Z(U) \cap Z(N)$. (Here we need that $\Gamma_U$ acts trivially on the cosets). Moreover, its multiplicity is the representation number $r(N_1, L_U + \lambda_i)$. (Here we need that $\Gamma_U$ acts trivially on $U$ since $\Gamma$ is neat). This gives the first term. $\square$

We now let $\varphi_V \in [\mathcal{S}(V(\mathbb{R})) \otimes \Omega^{1,1}(D)]^G$ be the Kudla-Millson Schwartz form for $V$. Then the associated theta function $\theta(\tau, \varphi_V)$ for the lattice $L$ is a modular form of weight $(n+2)/2$ with values in the differential forms of Hodge type $(1,1)$ of $M$. Moreover, for $N > 0$, the $N$-th Fourier coefficient is a Poincaré dual form for the special divisor $Z(N)$. It is therefore natural to consider the integral

$$I_V(\tau, Z(U), f) := \int_{Z(U)} f(z)\theta_V(\tau, z, L) \tag{5.8}$$

and to expect that this involves the evaluation of $f$ at $(Z(N) \cap Z(U))_M$. (Note however that the intersection of the two relative cycles $Z(U)$ and $Z(N)$ is not cohomological).

**Proposition 5.2.** *We have*

$$I_V(\tau, Z(U), f) = \sum_{i=1}^{r} \vartheta(\tau, L_U + \lambda_i) I_{U^\perp}(\tau, L_{U^\perp} + \mu_i, f).$$

*Here* $\vartheta(\tau, L_U + \lambda_i) = \sum_{x \in L_U + \lambda_i} e^{2\pi i q(x)\tau}$ *is the standard theta function of the positive definite lattice* $L_U$*, and* $I_{U^\perp}(\tau, L_{U^\perp} + \mu_i, f)$ *is the lift of* $f$ *considered in the main body of the paper for the space* $U^\perp$ *of signature* $(1,2)$ *(and the coset* $\mu_i$ *of the lattice* $L_{U^\perp}$*).*

*Proof.* Under the pullback $i_U^* : \Omega^{1,1}(D) \longrightarrow \Omega^{1,1}(D_U)$, we have, see [29], $i_U^* \varphi_V = \varphi_U^+ \otimes \varphi_{U^\perp}$, where $\varphi_U^+$ is the usual (positive definite) Gaussian on

$U$. Then

(5.9)

$$\theta_{\varphi_V}(\tau, z, L) = \sum_{x \in L} \varphi_V(X, \tau, z) = \sum_{i=1}^{r} \sum_{x \in L_U + \lambda_i} \varphi_U^+(x, \tau) \sum_{y \in L_{U^\perp} + \mu_i} \varphi_{U^\perp}(y, \tau, z),$$

which implies the assertion. □

Making the Fourier expansion $I_V(\tau, Z(U), f)$ explicit, and using Lemma 5.1 and Theorem 3.6 (in its form for cosets of a general lattice, [12]), we obtain

**Theorem 5.3.** *Let $f \in M_0^!(Z(U))$ be a modular function on $Z(U)$ such that the constant Fourier coefficient of $f$ at all the cusps of $Z(U)$ vanishes. Then $\theta_{\varphi_V}(\tau, L)$ is a weakly holomorphic modular form of weight $(n+2)/2$ whose Fourier expansion involves the generating series*

$$\sum_{N>0} \left( (Z(U) \cap Z(N))_M [f] \right) q^N$$

*of the evaluation of $f$ along $(Z(U) \cap Z(N))_M$.*

**Remark 5.4.** This generalizes a result of Bringmann, Ono, and Rouse (Theorem 1.1 of [7]), where they consider some special cases of Theorem 5.3 for $n = 2$ in the case of Hilbert modular surfaces, where the cycles $Z(N)$ and $Z(U)$ are the famous Hirzebruch-Zagier curves [21]. Note that [7] uses our Theorem 3.6 as a starting point.

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