Traces in complex hyperbolic geometry

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A complex hyperbolic orbifold M can be written as $\mathbf{H}^2_{\mathbb{C}}/\Gamma$ where Γ is a discrete, faithful representation of $\pi_1(M)$ to $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$. The group $\mathrm{SU}(2,1)$ is a triple cover of the group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ and (taking subgroups if necessary) we view Γ as a subgroup of $\mathrm{SU}(2,1)$. Our main goal is to discuss the connection between the geometry of M and traces of Γ. We do this in two specific cases. First, we consider the case where Γ is a free group on two generators, which we view as the fundamental group of a three holed sphere. We indicate how to use this analysis to give Fenchel-Nielsen coordinates on the complex hyperbolic quasi-Fuchsian space of a surface of genus $g \geq 2$. Secondly, we consider the case where Γ is a triangle group generated by complex reflections in three complex lines. We keep in mind similar results from the more familiar setting of Fuchsian and Kleinian groups and we explain those examples from our point of view.

1. Introduction

It is well known that any Riemann surface Σ of genus $g \geq 2$ may be written as $\Sigma = \mathbf{H}^2/\Gamma$ where Γ is a Fuchsian representation ρ of $\pi_1(\Sigma)$ to $\mathrm{PSL}(2,\mathbb{R})$, the orientation preserving isometries of the hyperbolic plane $\mathbf{H}^2$. One may lift ρ to a representation of $\pi_1(\Sigma)$ to $\mathrm{SL}(2,\mathbb{R})$. There is close relationship between the geometry of the surface and the representation $\Gamma = \rho\big(\pi_1(\Sigma)\big)$. For example, the lengths of closed geodesics on Σ may be written simply in terms of the traces of elements of Γ. Thus one may find useful geometric information on the possible hyperbolic metrics on the surface by studying either the representation variety or the character variety. This idea goes back to the work of Fricke in the nineteenth century.

Consider the *representation variety* $\mathrm{Hom}\big(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{R})\big)/\mathrm{SL}(2,\mathbb{R})$ of conjugacy classes of representations of $\pi_1(\Sigma)$ to $\mathrm{SL}(2,\mathbb{R})$. (One must take care with this quotient. The quotient by conjugation, that is by the group of inner automorphisms, is not Hausdorff and so in fact one takes the "maximal Hausdorff quotient space"; see the discussion on page 97 of Goldman [7].) One component of the representation variety is Teichmüller space $\mathcal{T}(\Sigma)$, which is the space of marked hyperbolic metrics on Σ (see page 98 of [7]). Fenchel-Nielsen coordinates [6] give a global set of parameters for $\mathcal{T}(\Sigma)$. These coordinates are the hyperbolic lengths of $3g-3$ simple closed geodesics on Σ and $3g-3$ twist parameters. The length parameters may be studied directly using traces but the twist parameters do not have an obvious direct interpretation in terms of the representation variety. A natural question is whether one can find $6g-6$ lengths (or traces) to give global parameters. A theorem proved by Okumura [18] and Schmutz [29] says that in fact one needs an extra length, or trace, to give global coordinates.

Similarly, one may consider the representation variety $\mathrm{Hom}\big(\pi_1(\Sigma),\mathrm{SL}(2,\mathbb{C})\big)/\mathrm{SL}(2,\mathbb{C})$ of conjugacy classes of representations of $\pi_1(\Sigma)$ to $\mathrm{SL}(2,\mathbb{C})$. Of particular interest are the *quasi-Fuchsian* representations. These are characterised by being discrete, faithful, type-preserving and geometrically finite. Complex Fenchel-Nielsen coordinates were defined by Tan [30] and Kourouniotis [12]. The main difference is that the lengths and twists are now complex numbers. For both the lengths and twists, the real part is just the ordinary hyperbolic length and the imaginary part measures how the normal plane is rotated as we move around the geodesic. This relationship is discussed in detail by Parker and Series [25]. A second key difference is that, while complex Fenchel Nielsen coordinates distinguish non-conjugate representations, it is in general not clear what subset of $\mathbb{C}^{6g-6}$ corresponds to quasi-Fuchsian space.

A triangle group is the group generated by reflections in the sides of a spherical, Euclidean or hyperbolic triangle. This triangle, and hence the group, is completely determined up to isometry (or similarity in the Euclidean case) by the internal angles of the triangle. These angles may be found using the trace of the product of reflections in the associated sides. So once again, traces lead to geometrical information and to a set of parameters for these groups. More specifically, one may write the Gram matrix G (or cosine matrix) of a triangle. This is a symmetric matrix whose diagonal entries are all 1 and the off diagonal entries are $-\cos(\theta_i)$ where θ_i for $i=1,\,2,\,3$ are the internal angles; see Davis [3]. The Gram matrix is G positive definite, singular, of signature $(2,1)$ respectively if the sum of the internal angles is greater than, equal to or less than π respectively (see

Theorem 6.8.12 of [3]). One may then write down a representation of the triangle group into $SO(G)$, the group of unimodular, orthogonal matrices preserving the bilinear form associated to G. When G has signature $(2,1)$ then $\mathrm{SO}(G)$ is the isometry group of the hyperbolic plane $\mathbf{H}^2$. This representation of the triangle group is completely determined up to conjugacy by the three internal angles. In particular, one may write down the trace of any group element as an integer polynomial in the variables $2\cos(\theta_i)$.

The purpose of this article is to investigate how these ideas may be extended to complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$. This is a natural generalisation of the hyperbolic plane to higher *complex* dimensions. As well as many similarities, there are many fascinating differences between the generalisation of the hyperbolic plane to higher real dimensions and to higher complex dimensions. Complex hyperbolic space has variable (quarter pinched) negative curvature and this causes many things to become harder than in the constant curvature setting. However the complex structure simplifies many things. Particular instances of these similarities and differences will occur throughout these notes. We use results from real hyperbolic geometry to inspire us when formulating complex hyperbolic problems. Quite often this leads us to results in the complex world with a similar overall structure but whose details differ considerably from the analogous results from the real world.

The holomorphic isometry group of $\mathbf{H}^2_{\mathbb{C}}$ is the projective group $\mathrm{PU}(2,1)$. It is more convenient to lift to a triple cover $\mathrm{SU}(2,1)$. Hence we want to find out what types of geometrical information may be deduced from information about traces. In principle, this relationship is very similar to the connection between geometrical information about $\mathbf{H}^2$ or $\mathbf{H}^3$ and traces in $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$. In practice this relationship is more subtle. Part of the subtlety arises from the fact that we are dealing with 3×3 matrices and part because $\mathbf{H}^2_{\mathbb{C}}$ has variable negative curvature.

The background material is quite standard. Readers may find more detail in the book of Goldman [8] and the forthcoming book of Parker [21]. The discussion of traces for two generator groups follows work of Lawton [13] and Will [33], [34]. The application of this work to Fenchel-Nielsen coordinates is related to work of Parker and Platis [23]. As such it is part of a wider area of complex hyperbolic quasi-Fuchsian groups. We will not discuss this topic in detail. Instead we refer readers to the survey article [24]. Finally, our treatment of traces for triangle groups follows Sandler [28] and Pratoussevitch [27]. It has applications for the construction of lattices, see Parker and Paupert [22].

2. Background

2.1. *Hermitian and unitary matrices*

The material in this section is completely standard; see Goldman [8], Chen and Greenberg [2] or Parker [21].

Let $A = (a_{ij})$ be a $k \times l$ complex matrix. The *Hermitian transpose* of A is the $l \times k$ complex matrix $A^* = (\overline{a}_{ji})$ formed by complex conjugating each entry of A and then taking the transpose. As with ordinary transpose, the Hermitian transpose of a product is the product of the Hermitian transposes in the reverse order. That is $(AB)^* = B^*A^*$. Clearly $(A^*)^* = A$ and, since $I^* = I$, if A is invertible we also have $(A^*)^{-1} = (A^{-1})^*$.

A $k \times k$ complex matrix H is said to be *Hermitian* if it equals its own Hermitian transpose $H = H^*$. Let H be a Hermitian matrix and μ an eigenvalue of H with eigenvector $\mathbf{z} \neq \mathbf{0}$. We claim that μ is real. In order to see this, observe that

$$\mu \mathbf{z}^* \mathbf{z} = \mathbf{z}^*(\mu \mathbf{z}) = \mathbf{z}^* H \mathbf{z} = \mathbf{z}^* H^* \mathbf{z} = (H\mathbf{z})^* \mathbf{z} = (\mu \mathbf{z})^* \mathbf{z} = \overline{\mu} \mathbf{z}^* \mathbf{z}.$$

Since $\mathbf{z}^*\mathbf{z}$ is real and non-zero, we see that μ is real. Suppose that H is a non-singular Hermitian matrix (that is, all its eigenvalues are non-zero) with p positive eigenvalues and q negative ones. Then we say that H has *signature* (p, q).

To each $k \times k$ Hermitian matrix H we can associate a *Hermitian form* $\langle \cdot, \cdot \rangle : \mathbb{C}^k \times \mathbb{C}^k \longrightarrow \mathbb{C}$ given by $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z}$ (note that we change the order) where $\mathbf{w}$ and $\mathbf{z}$ are column vectors in $\mathbb{C}^k$. Hermitian forms are sesquilinear, that is they are linear in the first factor and conjugate linear in the second factor. In other words, for $\mathbf{z}$, $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{w}$ column vectors in $\mathbb{C}^k$ and λ a complex scalar, we have

$$\begin{aligned} \langle \mathbf{z}_1 + \mathbf{z}_2, \mathbf{w} \rangle &= \langle \mathbf{z}_1, \mathbf{w} \rangle + \langle \mathbf{z}_2, \mathbf{w} \rangle, \\ \langle \lambda \mathbf{z}, \mathbf{w} \rangle &= \lambda \langle \mathbf{z}, \mathbf{w} \rangle, \\ \langle \mathbf{w}, \mathbf{z} \rangle &= \overline{\langle \mathbf{z}, \mathbf{w} \rangle}. \end{aligned}$$

From these we see that

$$\begin{aligned} \langle \mathbf{z}, \mathbf{z} \rangle &\in \mathbb{R}, \\ \langle \mathbf{z}, \lambda \mathbf{w} \rangle &= \overline{\lambda} \langle \mathbf{z}, \mathbf{w} \rangle, \\ \langle \lambda \mathbf{z}, \lambda \mathbf{w} \rangle &= |\lambda|^2 \langle \mathbf{z}, \mathbf{w} \rangle. \end{aligned}$$

If H has signature (p, q) then we say that $\langle \cdot, \cdot \rangle$ also has signature (p, q). Let $\mathbb{C}^{p,q}$ be a complex vector space of (complex) dimension $p + q$ equipped with a non-degenerate Hermitian form $\langle \cdot, \cdot \rangle$ of signature (p, q). This means that $\langle \cdot, \cdot \rangle$ is given by a non-singular $(p+q) \times (p+q)$ Hermitian matrix H with p positive eigenvalues and q negative eigenvalues. Later we will see that choosing a different form of signature (p,q) leads to an isomorphic space. In what follows we will be interested in the case where $k=3$, $p=2$ and $q=1$, but we will keep a running example of the case where $k=2$ and $p=q=1$. We will see that, depending on the choice of Hermitian form, this example yields the Poincaré disc or half plane model of $\mathbf{H}^2$. For a discussion of higher dimensional (real) hyperbolic space from the Hermitian point of view, see [19].

Example 2.1: Consider

$$H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad H_0' = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.1}$$

It is clear that H_0 and H_0' are both Hermitian. Moreover, as H_0 is diagonal it is immediate that it has signature $(1,1)$. It is not hard to check that H_0' also has signature $(1,1)$.

Let H be a Hermitian form of signature (p,q) and let A be a matrix that preserves the corresponding Hermitian form. In other words, for all $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{C}^{p,q}$ we have

$$\mathbf{w}^* A^* H A \mathbf{v} = \langle A\mathbf{v},\, A\mathbf{w} \rangle = \langle \mathbf{v},\, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{v}. \tag{2.2}$$

We say that such a matrix A is *unitary* with respect to H. It is clear that the collection of all matrices that are unitary with respect to H form a group, the *unitary group of H*, denoted $\mathrm{U}(H)$. Sometimes we wish to consider unimodular unitary matrices and we denote the corresponding group by $\mathrm{SU}(H)$. By letting $\mathbf{v}$ and $\mathbf{w}$ run through a basis of of $\mathbb{C}^{p,q}$ we see that (2.2) implies $A^*HA = H$. In other words, $H^{-1}A^*HA = I$ and so $A^{-1} = H^{-1}A^*H$.

Example 2.2: Consider the Hermitian forms H_0 and H_0' in (2.1). Suppose that $A \in \mathrm{SU}(H_0)$. Then

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1} = H_0^{-1} A^* H_0 = \begin{pmatrix} \overline{a} & -\overline{c} \\ -\overline{b} & \overline{d} \end{pmatrix}.$$

Therefore $b = \overline{c}$ and $d = \overline{a}$. Hence $1 = ad - bc = |a|^2 - |c|^2$. Hence

$$\mathrm{SU}(H_0) = \left\{ \begin{pmatrix} a & \overline{c} \\ c & \overline{a} \end{pmatrix} \;:\; a,\, c \in \mathbb{C},\ |a|^2 - |c|^2 = 1 \right\}.$$

Similarly, suppose $A' \in \mathrm{SU}(H_0')$. Then

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A'^{-1} = {H_0'}^{-1} A'^* H_0' = \begin{pmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{pmatrix}.$$

Therefore a, b, c, d are all real. Hence

$$\mathrm{SU}(H_0') = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ a,\, b,\, c,\, d \in \mathbb{R},\ ad - bc = 1 \right\}.$$

That is, $\mathrm{SU}(H_0') = \mathrm{SL}(2, \mathbb{R})$.

Given two Hermitian forms H and H' of the same signature we can pass between them using a *Cayley transform* C. That is, we can write

$$H' = C^* H C.$$

The existence of the Cayley transform follows from Sylvester's law of inertia. The Cayley transform C is not unique for we may precompose and postcompose by any unitary matrix preserving the relevant Hermitian form. It is clear that if A is unitary with respect to H then $A' = C^{-1}AC$ is unitary with respect to H'. In order to see this, observe that, using $(C^{-1}AC)^* = C^* A^* C^{*-1}$, we have

$$A'^* H' A' = (C^{-1}AC)^* (C^* H C)(C^{-1}AC) = C^* A^* H A C = C^* H C = H'.$$

This means that one does not need to specify the form but only the signature. Hence we can talk about $\mathbb{C}^{p,q}$, which is the vector space $\mathbb{C}^{p+q}$ together with any Hermitian form of signature (p, q). The choice of a particular Hermitian form is equivalent to a choice of basis. Also, we write $\mathrm{U}(p, q)$ and $\mathrm{SU}(p, q)$ for the unitary and special unitary groups for a form of signature (p, q) and in doing so we do not need to specify a particular form.

Example 2.3: Consider H_0 and H_0' given by (2.1) and

$$C_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

Then it is easy to check that $H_0' = C_0^* H_0 C_0$. Furthermore suppose

$$A = \begin{pmatrix} a & \overline{c} \\ c & \overline{a} \end{pmatrix} \in \mathrm{SU}(H_0).$$

Then

$$A' = C_0^{-1} A C_0 = \begin{pmatrix} \Re(a) - \Im(c) & \Im(a) + \Re(c) \\ -\Im(a) + \Re(c) & \Re(a) + \Im(c) \end{pmatrix} \in \mathrm{SU}(H_0') = \mathrm{SL}(2, \mathbb{R}).$$

2.2. *Complex hyperbolic space and its isometries*

We now define complex hyperbolic space. We first define the projective model and then go on to specialise to the unit ball model and the Siegel domain model. This is simultaneously a complex version of the projective and Klein-Beltrami models of ordinary (real) hyperbolic space and also a generalisation to higher complex dimensions of the Poincaré disc and half plane models of the hyperbolic plane. We restrict ourselves to the case of complex hyperbolic 2-space. These definitions can be generalised to higher dimensions in an obvious way.

Let H be a 3×3, non-singular Hermitian form of signature $(2,1)$. If $\mathbf{z} \in \mathbb{C}^{2,1}$ then we know that $\langle \mathbf{z}, \mathbf{z} \rangle$ is real. Thus we may define subsets V_-, V_0 and V_+ of $\mathbb{C}^{2,1}$ by

$$V_- = \left\{ \mathbf{z} \in \mathbb{C}^{2,1} | \langle \mathbf{z}, \mathbf{z} \rangle < 0 \right\}, \tag{2.3}$$

$$V_0 = \left\{ \mathbf{z} \in \mathbb{C}^{2,1} - \{\mathbf{0}\} | \langle \mathbf{z}, \mathbf{z} \rangle = 0 \right\}, \tag{2.4}$$

$$V_+ = \left\{ \mathbf{z} \in \mathbb{C}^{2,1} | \langle \mathbf{z}, \mathbf{z} \rangle > 0 \right\}. \tag{2.5}$$

We say that $\mathbf{z} \in \mathbb{C}^{2,1}$ is *negative, null* or *positive* if $\mathbf{z}$ is in V_-, V_0 or V_+ respectively. Motivated by special relativity, these are sometimes called time-like, light-like and space-like.

Example 2.4: Consider $\mathbb{C}^{1,1}$ with the Hermitian form given by H_0 in (2.1). Then

$$V_- = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \; : \; |z_1|^2 < |z_2|^2 \right\}, \quad V_0 = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; : \; |z_1|^2 = |z_2|^2 \right\}.$$

Likewise for $\mathbb{C}^{1,1}$ with the Hermitian form given by H_0',

$$V_- = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \; : \; \Im(z_1 \overline{z}_2) > 0 \right\}, \quad V_0 = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; : \; \Im(z_1 \overline{z}_2) = 0 \right\}.$$

Define an equivalence relation on $\mathbb{C}^{2,1} - \{\mathbf{0}\}$ by $\mathbf{z} \sim \mathbf{w}$ if and only if there is a non-zero complex scalar λ so that $\mathbf{w} = \lambda \mathbf{z}$. Let $\mathbb{P} : \mathbb{C}^{2,1} - \{\mathbf{0}\} \longmapsto \mathbb{CP}^2$ denote the standard projection map defined by $\mathbb{P}(\mathbf{z}) = [\mathbf{z}]$ where $[\mathbf{z}]$ is the equivalence class of $\mathbf{z}$. On the chart of $\mathbb{C}^{2,1}$ with $z_3 \neq 0$ the projection map $\mathbb{P}$ is given by

$$\mathbb{P} : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \longmapsto \begin{pmatrix} z_1/z_3 \\ z_2/z_3 \end{pmatrix} \in \mathbb{C}^2.$$

Because $\langle \lambda \mathbf{z}, \lambda \mathbf{z} \rangle = |\lambda|^2 \langle \mathbf{z}, \mathbf{z} \rangle$ we see that for any non-zero complex scalar λ the point $\lambda \mathbf{z}$ is negative, null or positive if and only if $\mathbf{z}$ is. The *projective model* of complex hyperbolic space is defined to be the collection of negative lines in $\mathbb{C}^{2,1}$ and its boundary is defined to be the collection of null lines. In other words $\mathbf{H}^2_{\mathbb{C}}$ is $\mathbb{P}V_-$ and $\partial\mathbf{H}^2_{\mathbb{C}}$ is $\mathbb{P}V_0$,

We define the other two standard models of complex hyperbolic space by considering two standard Hermitian forms on $\mathbb{C}^{2,1}$. We call these the first and second Hermitian forms. Let $\mathbf{z}$, $\mathbf{w}$ be the column vectors $(z_1,\, z_2,\, z_3)^t$ and $(w_1,\, w_2,\, w_3)^t$ respectively. The *first Hermitian form* is defined to be:

$$\langle \mathbf{z}, \mathbf{w}\rangle_1 = z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3. \tag{2.6}$$

It is given by the Hermitian matrix H_1:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.7}$$

The *second Hermitian form* is defined to be:

$$\langle \mathbf{z}, \mathbf{w}\rangle_2 = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1. \tag{2.8}$$

It is given by the Hermitian matrix H_2:

$$H_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.9}$$

Both of these forms have the property that each vector in V_- has non-zero third entry. Therefore, we can take the section defined by $z_3 = 1$. This gives a unique point on each complex line in V_-. In other words given $z = (z_1, z_2) \in \mathbb{C}^2$, we define its *standard lift* to $\mathbb{C}^{2,1}$ to be the column vector

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}$$

in $\mathbb{C}^{2,1}$. Clearly $\mathbb{P}$ sends $\mathbf{z}$ back to z. We consider what it means for $\langle \mathbf{z}, \mathbf{z}\rangle$ to be negative for the first and second Hermitian forms, respectively.

For the first Hermitian form, we see that the standard lift of $z = (z_1, z_2) \in \mathbb{C}^2$ is in V_- if and only if $|z_1|^2 + |z_2|^2 < 1$. In other words, $\mathbb{P}V_- = \mathbf{H}^2_{\mathbb{C}}$ is the *unit ball* in $\mathbb{C}^2$. Likewise, the standard lift of z is in V_0 if and only if $|z_1|^2 + |z_2|^2 = 1$ and so $\mathbb{P}V_0 = \partial\mathbf{H}^2_{\mathbb{C}}$ is the unit 3-sphere S^3 in $\mathbb{C}^2$.

For the second Hermitian form, the standard lift of z is negative if and only if

$$z_1 + |z_2|^2 + \overline{z}_2 = 2\Re(z_1) + |z_2|^2 < 0.$$

Thus $\mathbb{P}V_-$ is a paraboloid in $\mathbb{C}^2$, called the *Siegel domain*. Similarly z in $\mathbb{P}V_0 \cap \mathbb{C}^2$ satisfies $2\Re(z_1) + |z_2|^2 = 0$. However, not all points in $\mathbb{P}(V_0)$ lie in $\mathbb{C}^2 \subset \mathbb{CP}^2$. We have to add an extra point, denoted ∞, on the boundary of the Siegel domain. The standard lift of ∞ is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Of course ∞ is not the only point in $\mathbb{CP}^2 - \mathbb{C}^2$, that is it is not the only point "at infinity". In this respect it is different from the point ∞ on the boundary of the upper half plane model of the hyperbolic plane, which is the only point of $\mathbb{CP}^1$ that is not in $\mathbb{C}$.

Example 2.5: For $z \in \mathbb{C}$ the standard lift of z to $\mathbb{C}^{1,1}$ is

$$\mathbf{z} = \begin{bmatrix} z \\ 1 \end{bmatrix} \in \mathbb{C}^{1,1}.$$

If $\mathbb{C}^{1,1}$ has the Hermitian form given by H_0 from (2.1), then we see that $z \in \mathbb{P}V_-$ if and only if $|z| < 1$ and $z \in \mathbb{P}V_0$ if and only if $|z| = 1$. Thus $\mathbb{P}V_-$ is the unit disc and $\mathbb{P}V_0$ is the unit circle in $\mathbb{C}$.

Similarly for H_0', the point $z \in \mathbb{P}V_-$ if and only if $\Im(z) > 0$ and $z \in \mathbb{P}V_0 \cap \mathbb{C}$ if and only if z is real. We must add an extra point ∞ whose standard lift is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

There are other Hermitian forms which are widely used in the literature. In particular, Chen and Greenberg (page 67 of [2]) give a close relative of the second Hermitian form, namely the one given by the matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

When we consider complex hyperbolic triangle groups in Section 5 we will see that, following Mostow, it is most convenient to use the Gram matrix as the Hermitian form. This form will always be defined, but only for certain points in the parameter space will it have the correct signature.

So far we have only defined complex hyperbolic space as a set of points. In order to understand its geometry we must give it a metric. This metric is called the *Bergman metric*, given by the following element of arc length:

$$ds^2 = \frac{-4}{\langle \mathbf{z}, \mathbf{z}\rangle^2} \det \begin{pmatrix} \langle \mathbf{z}, \mathbf{z}\rangle & \langle d\mathbf{z}, \mathbf{z}\rangle \\ \langle \mathbf{z}, d\mathbf{z}\rangle & \langle d\mathbf{z}, d\mathbf{z}\rangle \end{pmatrix}. \tag{2.10}$$

The choice of the constant 4 in the above formula means that the holomorphic sectional curvature of $\mathbf{H}^2_{\mathbb{C}}$ is -1. Compare this with equation (3.3) on page 73 and Theorem 3.1.9 of Goldman [8]. The similar formula in equation (19.2) on page 135 of Mostow [16] has the constant 1 here and this leads to holomorphic sectional curvature -4.

Alternatively, the Bergman metric is given by the distance function $\rho(\cdot,\cdot)$ defined by the formula

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w}\rangle \langle \mathbf{w}, \mathbf{z}\rangle}{\langle \mathbf{z}, \mathbf{z}\rangle \langle \mathbf{w}, \mathbf{w}\rangle}.$$

See equation (3.4) on page 77 of Goldman [8]. The different normalisation given by Mostow means that $\rho/2$ is replaced with ρ in formula (19.4') on page 136 of [16].

For the ball model and Siegel domain model one can find the distance between points z and w by plugging their standard lifts $\mathbf{z}$ and $\mathbf{w}$ into the above formula. However, as may easily be seen, this formula is independent of which lifts $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{C}^{2,1}$ of z and w we choose. Thus the distance function is well defined in these two models.

Example 2.6: Consider $z \in \mathbb{C}$ with $|z| < 1$. We have seen that for the Hermitian form H_0 this point is in $z \in \mathbb{P}V_-$. Moreover, the standard lift of z and its derivative are

$$\mathbf{z} = \begin{bmatrix} z \\ 1 \end{bmatrix}, \quad d\mathbf{z} = \begin{bmatrix} dz \\ 0 \end{bmatrix}.$$

Plugging this vector and the Hermitian form given by H_0 into (2.10) gives

$$ds^2 = \frac{-4}{\big(|z|^2-1\big)^2} \det \begin{pmatrix} |z|^2 - 1 & \overline{z}\, dz \\ z\, d\overline{z} & dz\, d\overline{z} \end{pmatrix} = \frac{4|dz|^2}{\big(1-|z|^2\big)^2}.$$

This is just the Poincaré metric on the unit disc.

Similarly consider $z \in \mathbb{C}$ with $\Im(z) > 0$. This is in $\mathbb{P}V_-$ for the Hermitian form H_0'. Plugging its standard lift into (2.10) gives

$$ds^2 = \frac{-4}{\big(-2\Im(z)\big)^2} \det \begin{pmatrix} -2\Im(z) & i dz \\ -i d\overline{z} & 0 \end{pmatrix} = \frac{|dz|^2}{\big(\Im(z)\big)^2}.$$

This is the Poincaré metric on the upper half plane.

Note that in both these examples we have the constant 4 and constant curvature -1

Unitary matrices in $\mathrm{U}(2,1)$ act on $\mathbb{C}^{2,1}$ preserving V_+, V_0 and V_-. They also preserve the Bergman metric since it is given solely in terms of the Hermitian form. Therefore unitary matrices act as isometries on complex hyperbolic space. Let us see this action explicitly. Let $z = (z_1, z_2)$ be a point in $\mathbb{C}^2$ and let $\mathbf{z}$ be its standard lift to $\mathbb{C}^{2,1}$. Then $A \in \mathrm{U}(2,1)$ acts as follows:

$$A(z) = \mathbb{P}(A\mathbf{z}).$$

In other words, if

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$$

then

$$A(z_1, z_2) = \left(\frac{az_1 + bz_2 + c}{gz_1 + hz_2 + j}, \frac{dz_1 + ez_2 + f}{gz_1 + hz_2 + j} \right).$$

This is just a linear fractional transformation in two variables.

Example 2.7: Consider $A \in SU(H_0)$:

$$A = \begin{pmatrix} a & \overline{c} \\ c & \overline{a} \end{pmatrix}.$$

Arguing as above, we see that A acts on the unit disc s the Möbius transformation in $\mathrm{PSU}(H_0)$

$$A(z) = \mathbb{P} \begin{pmatrix} a & \overline{c} \\ c & \overline{a} \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{az + \overline{c}}{cz + \overline{a}}.$$

Similarly for H_0', the matrix $A \in \mathrm{SU}(H_0') = \mathrm{SL}(2, \mathbb{R})$ acts on the upper half plane as a Möbius transformation in $\mathrm{PSL}(2, \mathbb{R})$.

Any matrix in $\mathrm{U}(2,1)$ which is a (non-zero) complex scalar multiple of the identity maps each line through the origin in $\mathbb{C}^{2,1}$ to itself and so acts trivially on complex hyperbolic space. Since this matrix is unitary with respect to $\langle \cdot, \cdot \rangle$ then the scalar must have unit norm. Because of this, we define the projective unitary group $\mathrm{PU}(2,1) = \mathrm{U}(2,1)/\mathrm{U}(1)$ where $\mathrm{U}(1)$ is canonically identified with $\{e^{i\theta} I | 0 \leq \theta < 2\pi\}$, where I is the identity matrix in $\mathrm{U}(2,1)$. Sometimes it will be useful to consider $\mathrm{SU}(2,1)$, the group of matrices with determinant 1 which are unitary with respect to $\langle\cdot,\cdot\rangle$. The group $\mathrm{SU}(2,1)$ is a 3-fold covering of $\mathrm{PU}(2,1)$:

$$\mathrm{PU}(2,1) = \mathrm{SU}(2,1)/\{I,\, \omega I,\, \omega^2 I\}$$

where $\omega = (-1+i\sqrt{3})/2$ is a cube root of unity. This is completely analogous to the fact that $\mathrm{SL}(2,\mathbb{C})$ is a double cover of $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{I,-I\}$. Cube roots of unity are used because $\mathrm{SU}(2,1)$ consists of 3×3 matrices.

3. The geometry of isometries

3.1. *Introduction*

It is well known that the dynamical behaviour of hyperbolic isometries in $\mathrm{PSL}(2,\mathbb{C})$ may be classifies as elliptic, parabolic or loxodromic (hyperbolic) and the trace of the corresponding matrix in $\mathrm{SL}(2,\mathbb{C})$ distinguishes between these classes. Moreover, for non-parabolic isometries, the geometry of the action in terms of rotation angle or hyperbolic translation length may be read off directly from the trace. Likewise, following Goldman [8], one may use the trace of an element of $\mathrm{SU}(2,1)$ to decide whether the corresponding complex hyperbolic isometry in $\mathrm{PU}(2,1)$ is elliptic, parabolic or loxodromic. Furthermore, one may deduce information about the geometry of the action of the isometry from this trace. For loxodromic isometries this is straightforward, but for elliptic and parabolic isometries there is some subtlety involved, which we will explain.

3.2. *Classification of elements of* $\mathbf{SU(2,1)}$ *by their trace*

In this section we will classify elements of $\mathrm{SU}(2,1)$ by their trace. Elements of $\mathrm{SU}(2,1)$ are holomorphic complex hyperbolic isometries and the familiar trichotomy from real hyperbolic geometry applies in this setting too; see Chen and Greenberg [2]. A holomorphic complex hyperbolic isometry A is said to be:

(i) *loxodromic* if it fixes exactly two points of $\partial\mathbf{H}^2_{\mathbb{C}}$;
(ii) *parabolic* if it fixes exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$;
(iii) *elliptic* if it fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$.

Following Goldman, [8], we now show how $A \in \mathrm{SU}(2,1)$ may be classified as loxodromic, parabolic or elliptic using $\mathrm{tr}(A)$. First observe that, since $A^{-1} = H^{-1}A^*H$, we must have

$$\mathrm{tr}(A^{-1}) = \mathrm{tr}(A^*) = \overline{\mathrm{tr}(A)}.$$

Hence, if we denote the trace of A by $\mathrm{tr}(A)=\tau$, then $\mathrm{tr}(A^{-1})=\overline{\tau}$. Putting this in the expression for the characteristic polynomial of $A\in\mathrm{SU}(2,1)$ from Lemma 4.5 below gives:

$$\mathrm{ch}_A(x)=x^3-\mathrm{tr}(A)x^2+\mathrm{tr}(A^{-1})x-1=x^3-\tau x^2+\overline{\tau}x-1.$$

We want to find out when $A\in\mathrm{SU}(2,1)$ has repeated eigenvalues. In other words, we want to find conditions on τ for which $\mathrm{ch}_A(x)=0$ has repeated solutions. This is true if and only if $\mathrm{ch}_A(x)$ and its derivative $\mathrm{ch}'_A(x)$ have a common root. Clearly

$$\mathrm{ch}'_A(x)=3x^2-2\tau x+\overline{\tau}.$$

According to Lemma 3.3 on page 53 of Kirwan [10], two polynomials have a common root if and only if their resultant vanishes.

The *resultant* $R\big(p(x),q(x)\big)$ of two polynomials $p(x)$ and $q(x)$ of degree m and n, respectively, is the determinant of the $(m+n)\times(m+n)$ matrix defined as follows. Write the coefficients of $p(x)$ in the first row followed by $n-1$ zeros. In the next row the coefficients are displaced one place to the right, with one zero to the left and $n-2$ to the right. Continue in this fashion until the nth row is $n-1$ zeros followed by the coefficients of $p(x)$. For the last m rows we do the same thing with $p(x)$ and $q(x)$ interchanged. For details, see Definition 3.2 on page 52 of [10].

Since $\mathrm{ch}_A(x)$ and $\mathrm{ch}'_A(x)$ have degrees 3 and 2 respectively, the resultant is a 5×5 determinant. Applying the above recipe, we see that

$$R\big(\mathrm{ch}_A(x),\mathrm{ch}'_A(x)\big)=\det\begin{pmatrix}1 & -\tau & \overline{\tau} & -1 & 0\\ 0 & 1 & -\tau & \overline{\tau} & -1\\ 3 & -2\tau & \overline{\tau} & 0 & 0\\ 0 & 3 & -2\tau & \overline{\tau} & 0\\ 0 & 0 & 3 & -2\tau & \overline{\tau}\end{pmatrix}.$$

Evaluating this determinant, we see that

$$R\big(\mathrm{ch}_A(x),\mathrm{ch}'_A(x)\big)=-|\tau|^4+8\Re(\tau^3)-18|\tau|^2+27.$$

This leads to the following theorem of Goldman, Theorem 6.2.4 of [8]:

Theorem 3.1: *Let $f(\tau)=|\tau|^4-8\Re(\tau^3)+18|\tau|^2-27$. Let $A\in\mathrm{SU}(2,1)$ then:*

(i) A has an eigenvalue λ with $|\lambda|\neq1$ if and only if $f\big(\mathrm{tr}(A)\big)>0$,
(ii) A has a repeated eigenvalue if and only if $f\big(\mathrm{tr}(A)\big)=0$,
(iii) A has distinct eigenvalues of unit modulus if and only if $f\big(\mathrm{tr}(A)\big)<0$.

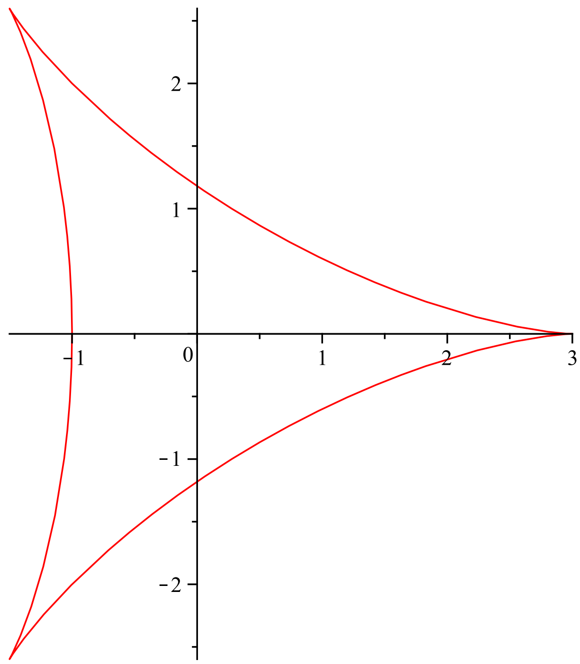

Fig. 1. *The deltoid given by $f(\tau) = 0$. The region with $f(\tau) < 0$ is inside and that with $f(\tau) > 0$ is outside.*

The curve $f(\tau) = 0$ is a classical curve called a *deltoid*, see Chapter 8 of Lockwood [14] or page 26 of Kirwan [10] where it is written in terms of $x = \Re(\tau)$ and $y = \Im(\tau)$. The points outside correspond to case (i) in the theorem. This may be seen by considering A with eigenvalues r, r^{-1} and 1 where $r > 1$, which implies $\mathrm{tr}(A)$ lies in the interval $(3, \infty)$, and considering A with eigenvalues $e^{i\theta}$, $e^{-i\theta}$ and 1 whose trace lies in $(-1, 3)$. The rest follows by continuity.

It is easy to see that part (ii) of Theorem 3.1 follows from the construction of $f(\tau)$. We now look at the other cases separately.

Lemma 3.2: *Let $A \in \mathrm{SU}(2,1)$ and let λ be an eigenvalue of A. Then $\overline{\lambda}^{-1}$ is an eigenvalue of A.*

Proof: We know that A preserves the Hermitian form defined by H. Hence, $A^*HA = H$ and so $A = H^{-1}(A^*)^{-1}H$. Thus A has the same set of eigenvalues as $(A^*)^{-1}$ (they are conjugate). Since the characteristic polynomial of A^* is the complex conjugate of the characteristic polynomial of A, we see that if λ is an eigenvalue of A then $\overline{\lambda}$ is an eigenvalue of A^*. Therefore $\overline{\lambda}^{-1}$ is an eigenvalue of $(A^*)^{-1}$ and hence of A. □

Corollary 3.3: *Suppose $A \in \mathrm{SU}(2,1)$ has an eigenvalue λ with $|\lambda| \neq 1$. Then $\overline{\lambda}^{-1}$ is a distinct eigenvalue and the third eigenvalue is $\overline{\lambda}\lambda^{-1}$ of absolute value 1. Moreover, A is loxodromic.*

Proof: Using Lemma 3.2 we see $\overline{\lambda}^{-1}$ is an eigenvalue. Since $|\lambda| \neq 1$ it is not equal to λ. As the product of the eigenvalues is 1 we obtain the third eigenvalue.

If $\mathbf{v} \neq \mathbf{0}$ is an eigenvector corresponding to λ then

$$\langle \mathbf{v}, \mathbf{v} \rangle = \langle A\mathbf{v}, A\mathbf{v} \rangle = \langle \lambda\mathbf{v}, \lambda\mathbf{v} \rangle = |\lambda|^2 \langle \mathbf{v}, \mathbf{v} \rangle.$$

As $|\lambda| \neq 1$ we see that $\mathbf{v} \in V_0$ and $\mathbb{P}\mathbf{v}$ is a fixed point of A on $\partial\mathbf{H}^2_{\mathbb{C}}$. Similarly, a non-zero $\overline{\lambda}^{-1}$-eigenvector corresponds to a second fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$. Finally, a non-zero $\overline{\lambda}\lambda^{-1}$-eigenvector lies in V_+ and is a normal vector for the complex line through the two fixed points on $\partial\mathbf{H}^2_{\mathbb{C}}$. Hence A has precisely two fixed points on $\partial\mathbf{H}^2_{\mathbb{C}}$ and is loxodromic. □

Lemma 3.4: *Suppose that $A \in \mathrm{SU}(2,1)$ has an eigenvalue λ with $|\lambda| \neq 1$. Then $f\big(\mathrm{tr}(A)\big) > 0$.*

Proof: Suppose that $re^{i\theta}$ is an eigenvalue of A where r is positive and $r \neq 1$. Then by Corollary 3.3 we see that the other eigenvalues are $(\overline{re^{i\theta}})^{-1} = r^{-1}e^{i\theta}$ and $e^{-2i\theta}$. Therefore $\tau = re^{i\theta} + r^{-1}e^{i\theta} + e^{-2i\theta}$. Hence

$$\begin{aligned} |\tau|^2 &= (r + r^{-1})^2 + 2(r + r^{-1})\cos(3\theta) + 1, \\ \Re(\tau^3) &= (r + r^{-1})^3\cos(3\theta) + 3(r + r^{-1})^2 + 3(r + r^{-1})\cos(3\theta) + \cos(6\theta). \end{aligned}$$

From this it is easy to see that

$$f\big(re^{i\theta} + r^{-1}e^{i\theta} + e^{-2i\theta}\big) = (r - r^{-1})^2\big(r + r^{-1} - 2\cos(3\theta)\big)^2 > 0. \qquad \square$$

Lemma 3.5: *Suppose that $A \in \mathrm{SU}(2,1)$ has three distinct eigenvalues, all of unit modulus. Then $f\big(\mathrm{tr}(A)\big) < 0$.*

Proof: We write the eigenvalues as $e^{i\theta}$, $e^{i\phi}$, $e^{i\psi}$ where θ, ϕ and ψ are distinct and $e^{i\theta+i\phi+i\psi}=1$. Then $\tau=e^{i\theta}+e^{i\phi}+e^{i\psi}$ and

$$\begin{aligned}|\tau|^2 &= 3+2\cos(\theta-\phi)+2\cos(\phi-\psi)+2\cos(\psi-\theta),\\ \Re(\tau^3) &= \cos(3\theta)+\cos(3\phi)+\cos(3\psi)\\ &\qquad +6\cos(\theta-\phi)+6\cos(\phi-\psi)+6\cos(\psi-\theta)+6.\end{aligned}$$

But,

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta-\phi-\psi)\\ &= \cos(\theta-\phi)\cos(\phi-\psi)+\sin(\theta-\phi)\sin(\phi-\psi).\end{aligned}$$

Hence

$$\begin{aligned}\Re(\tau^3) &= \cos(\theta-\phi)\cos(\phi-\psi)+\sin(\theta-\phi)\sin(\phi-\psi)\\ &\qquad +\cos(\phi-\psi)\cos(\psi-\theta)+\sin(\phi-\psi)\sin(\psi-\theta)\\ &\qquad +\cos(\psi-\theta)\cos(\theta-\phi)+\sin(\psi-\theta)\sin(\theta-\phi)\\ &\qquad +6\cos(\theta-\phi)+6\cos(\phi-\psi)+6\cos(\psi-\theta)+6.\end{aligned}$$

Using this we calculate

$$f\big(e^{i\theta}+e^{i\phi}+e^{i\psi}\big)=-4\Big(\sin(\theta-\phi)+\sin(\phi-\psi)+\sin(\phi-\psi)\Big)^2<0.\quad\square$$

As these two lemmas exhaust all the possibilities when A has distinct eigenvalues, we have proved Theorem 3.1.

We will discuss loxodromic maps in the next two sections. We now briefly discuss elliptic maps. Suppose first that A has three distinct eigenvalues of unit modulus.

Proposition 3.6: *Suppose that $A\in\mathrm{SU}(2,1)$ has distinct eigenvalues $e^{i\theta}$, $e^{i\phi}$ and $e^{i\psi}$. Then A has a unique fixed point in $\mathbf{H}^2_{\mathbb{C}}$ corresponding to one of the eigenspaces. There are then three distinct conjugacy classes of elliptic maps with this trace.*

If the fixed point corresponds to the $e^{i\theta}$ eigenspace then A acts on the tangent space at this point by a unitary matrix with eigenvalues $e^{i\phi-i\theta}$ and $e^{i\psi-i\theta}$.

Proof: Since A is diagonalisable, we may consider its action on a basis of eigenvectors. The first part follows directly.

For the second part, we consider the action of $e^{-i\theta}A$ on $\mathbb{C}^{2,1}$ and restrict this to the tangent space. The result follows. $\square$

We now consider what happens if A has a repeated eigenvalue and so $\mathrm{tr}(A)$ lies on the deltoid. When all three eigenvalues are the same they must be a cube root of unity. These traces are the three vertices of the deltoid. Such maps are either parabolic or act as the identity on $\mathbf{H}^2_{\mathbb{C}}$. We will not discuss these maps here. We now consider the case where A has exactly two distinct eigenvalues.

Proposition 3.7: *Suppose that $A \in \mathrm{SU}(2,1)$ has two distinct eigenvalues, one of them repeated. Then the eigenvalues of A are $e^{i\psi}$, $e^{i\psi}$, $e^{-2i\psi}$ for some ψ with $3\psi \not\equiv 0$ mod 2π. Moreover, one of the following three possibilities arises:*

(i) A fixes a complex line L in $\mathbf{H}^2_{\mathbb{C}}$ and rotates a normal vector to L by -3ψ;
(ii) A fixes a point in $\mathbf{H}^2_{\mathbb{C}}$ and acts as $e^{3i\psi}I$ on the tangent space at this point;
(iii) A fixes a point on $\partial\mathbf{H}^2_{\mathbb{C}}$ and there is a complex line L with this point on its boundary so that A acts as a parabolic map on L and rotates a normal vector to L by -3ψ.

Proof: Suppose that A has a repeated eigenvalue λ. Since $\det(A) = 1$, it is clear that the third eigenvalue is λ^{-2}. Using Lemma 3.2 we see that $\{\lambda, \lambda, \lambda^{-2}\} = \{\overline{\lambda}^{-1}, \overline{\lambda}^{-1}, \overline{\lambda}^{2}\}$. This is a contradiction if $|\lambda| \neq 1$. Thus $|\lambda| = 1$ and the eigenvalues are $e^{i\psi}$, $e^{i\psi}$, $e^{-2i\psi}$ as claimed.

Now we discuss the possible conjugacy classes of A. Note that the $e^{-2i\psi}$-eigenspace of A is Hermitian orthogonal to the $e^{i\psi}$-eigenspace and so cannot be contained in V_0 (or else H would be singular). Suppose first that A is diagonalisable. First suppose that the $e^{-2i\psi}$-eigenspace is in V_+. Then the $e^{i\psi}$-eigenspace is indefinite, in that it contains vectors in V_-, V_0 and V_+. Its image under $\mathbb{P}$ is a complex line in $\mathbf{H}^2_{\mathbb{C}}$ fixed by A. Secondly, suppose that the $e^{-2i\psi}$-eigenspace is in V_- and so corresponds to an isolated fixed point of A in $\mathbf{H}^2_{\mathbb{C}}$. Then the $e^{i\psi}$-eigenspace is in V_+ and corresponds to the tangent space to $\mathbf{H}^2_{\mathbb{C}}$ at the fixed point. Thus A acts on this tangent space as $e^{3i\psi}I$.

Now suppose that A is not diagonalisable. Putting A into Jordan normal form, we see that there are non-zero vectors $\mathbf{v}$ and $\mathbf{u}$ so that $A\mathbf{v} = e^{i\psi}\mathbf{v}$ and $A\mathbf{u} = e^{i\psi}\mathbf{u} + \mathbf{v}$. Thus

$$\langle \mathbf{u}, \mathbf{v}\rangle = \langle A\mathbf{u}, A\mathbf{v}\rangle = \langle e^{i\psi}\mathbf{u} + \mathbf{v}, e^{i\psi}\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + e^{-i\psi}\langle \mathbf{v}, \mathbf{v}\rangle.$$

Hence $\mathbf{v} \in V_0$ and corresponds to a fixed point of A on $\partial\mathbf{H}^2_{\mathbb{C}}$. The hyperplane spanned by $\mathbf{u}$ and $\mathbf{v}$ corresponds to a complex line L in $\mathbf{H}^2_{\mathbb{C}}$ and A acts on this line as a parabolic map with fixed point $\mathbb{P}\mathbf{v}$. The $e^{-2i\psi}$ eigenspace of A is spanned by a polar vector of A and so A acts on a normal vector to L as multiplication by $e^{-3i\psi}$. □

The maps described in Proposition 3.6 are called *regular elliptic*. Elliptic maps of the type given in Proposition 3.7 (i) are called *complex reflections in a line* and will be discussed in Section 5.2. Elliptic maps of the type given in Proposition 3.7 (ii) are called *complex reflections in a point*. The maps described in Proposition 3.7 (iii) are called *screw parabolic* or *ellipto-parabolic*.

3.3. *Traces and eigenvalues for loxodromic maps*

A loxodromic matrix A in $\mathrm{SL}(2,\mathbb{C})$ has eigenvalues λ and λ^{-1} where $|\lambda| > 1$. Furthermore writing $\mathrm{tr}(A) = \tau$ then $\tau = \lambda + \lambda^{-1}$. The map sending λ to τ is a conformal map from the exterior of the unit disc to the complex plane slit along the real axis from -2 to 2 (inclusive).

In this section we want to generalise this result to loxodromic maps in $\mathrm{SU}(2,1)$. The map from eigenvalues to traces is no longer holomorphic but, but we are able to show that it is a diffeomorphism from the exterior of the unit disc onto the set of points in $\mathbb{C}$ with $f(\tau) > 0$, that is the exterior of the deltoid, compare Theorem 3.1. If $\tau = \mathrm{tr}(A)$ then recall the discriminant function $f(\tau)$ of Theorem 3.1

$$f(\tau) = |\tau|^4 - 8\Re(\tau^3) + 18|\tau|^2 - 27. \tag{3.1}$$

Our main result is:

Proposition 3.8: *Let A be a loxodromic map in $\mathrm{SU}(2,1)$ with eigenvalue λ with $|\lambda| > 1$. Then the function Φ that gives the trace in terms of the eigenvalue*

$$\Phi : \left\{\lambda \in \mathbb{C} \ : \ |\lambda| > 1\right\} \longmapsto \left\{\tau \in \mathbb{C} \ : \ f(\tau) > 0\right\}$$

given by

$$\Phi(\lambda) = \tau = \lambda + \overline{\lambda}\lambda^{-1} + \overline{\lambda}^{-1}$$

is a diffeomorphism. Moreover, $\Phi(\omega\lambda) = \omega\Phi(\lambda)$, where ω is a cube root of unity, and so this diffeomorphism is well defined for elements of $\mathrm{PSU}(2,1)$.

Before we prove this result, we need to show that the map Φ is a surjection.

Lemma 3.9: *Suppose that* $\tau \in \mathbb{C}$ *satisfies* $f(\tau) > 0$ *then there exists* $\lambda \in \mathbb{C}$ *with* $|\lambda| > 1$ *so that* $\tau = \lambda + \overline{\lambda}\lambda^{-1} + \overline{\lambda}^{-1}$.

Proof: If we can find such a $\lambda = re^{i\theta}$ then, as in Lemma 3.4, we must find r and θ solving

$$\begin{aligned} |\tau|^2 &= (r + r^{-1})^2 + 2(r + r^{-1})\cos(3\theta) + 1, \\ \Re(\tau^3) &= (r + r^{-1})^3 \cos(3\theta) + 3(r + r^{-1})^2 + 3(r + r^{-1})\cos(3\theta) + \cos(6\theta). \end{aligned}$$

Eliminating $\cos(3\theta)$ from these equations, we must find $x = (r + r^{-1})^2 > 4$ solving $g(x) = 0$ where

$$g(x) = x^3 - \bigl(3 + |\tau|^2\bigr)x^2 + \bigl(3 + 2\Re(\tau^3) - |\tau|^2\bigr)x - \bigl(|\tau|^2 - 1\bigr)^2.$$

Moreover, since

$$\bigl((r + r^{-1}) - 1\bigr)^2 \leq |\tau|^2 \leq \bigl((r + r^{-1}) + 1\bigr)^2$$

such a solution must satisfy

$$\bigl(|\tau| - 1\bigr)^2 \leq x = (r + r^{-1})^2 \leq \bigl(|\tau| + 1\bigr)^2.$$

Note that since $f(\tau) > 0$ we must have $|\tau| > 1$ and so $\bigl(|\tau| + 1\bigr)^2 > 4$.

We now evaluate $g(x)$ at $x = 4$, $x = \bigl(|\tau| - 1\bigr)^2$ and $x = \bigl(|\tau| + 1\bigr)^2$:

$$\begin{aligned} g(4) &= 27 - 18|\tau|^2 + 8\Re(\tau^3) - |\tau|^4 \ = \ -f(\tau) \ < \ 0, \\ g\bigl((|\tau| - 1)^2\bigr) &= -2\bigl(|\tau| - 1\bigr)^2\bigl(|\tau^3| - \Re(\tau^3)\bigr) \ \leq \ 0, \\ g\bigl((|\tau| + 1)^2\bigr) &= 2\bigl(|\tau| + 1\bigr)^2\bigl(|\tau^3| + \Re(\tau^3)\bigr) \ \geq \ 0. \end{aligned}$$

Therefore by the intermediate value theorem, there exists $x = x_0$ with $g(x_0) = 0$ so that

$$x_0 > 4, \quad x_0 \geq \bigl(|\tau| - 1\bigr)^2, \quad x_0 \leq \bigl(|\tau| + 1\bigr)^2.$$

From this we can solve $(r + r^{-1})^2 = x_0$ to obtain

$$r = \frac{\sqrt{x_0} + \sqrt{x_0 - 4}}{2} > 1.$$

Substituting into the first of our equations, we obtain

$$\cos(3\theta) = \frac{|\tau|^2 - (r + r^{-1})^2 - 1}{2(r + r^{-1})} = \frac{|\tau|^2 - x_0 - 1}{2\sqrt{x_0}}.$$

The right hand side lies in $[-1,1]$ by construction. So we can solve to find 3θ. Finally, by considering $\arg(\tau)$ we can solve for θ.

Writing $\lambda = re^{i\theta}$ gives the result. □

Proof: (Proposition 3.8.) Write

$$X = \{\lambda \in \mathbb{C} \ : \ |\lambda| > 1\}, \qquad Y = \{\tau \in \mathbb{C} \ : \ f(\tau) > 0\}.$$

From Lemma 3.4 we see that the image of X under Φ is contained in Y and by Lemma 3.9 we see that Φ maps X onto Y.

We calculate the Jacobian of $\tau(\lambda)$:

$$\begin{aligned} |J_\tau(\lambda)| &= \left|\frac{\partial\tau}{\partial\lambda}\right|^2 - \left|\frac{\partial\tau}{\partial\overline{\lambda}}\right|^2 \\ &= |1-\overline{\lambda}\lambda^{-2}|^2 - |\lambda^{-1} - \overline{\lambda}^{-2}|^2 \\ &= \bigl(1-|\lambda|^{-2}\bigr)\Bigl(1-2|\lambda|^{-1}\cos\bigl(3\arg(\lambda)\bigr) + |\lambda|^{-2}\Bigr). \end{aligned}$$

This is clearly different from 0 whenever $|\lambda| > 1$.

Therefore Φ is a local diffeomorphism from X onto Y. It is clear that, when $\lambda \in X$ then λ tends to infinity if and only if τ tends to infinity. Likewise, from the proof of Lemma 3.4, it is clear that Φ extends continuously to a map from the unit circle $\{\lambda \in \mathbb{C} \ : \ |\lambda| = 1\}$ to the set $\{\tau \in \mathbb{C} \ : \ f(\tau) = 0\}$. Hence Φ extends continuously to a map from $\overline{X}$ to $\overline{Y}$ and is therefore proper. Thus, by Ehresmann's fibration theorem we see that Φ is a locally trivial fibration (that is, when thought of as a map from an annulus to itself, it is a covering map). Because Φ is a bounded distance from the identity for large values of $|\lambda|$ we see that it has winding number 1 and so Φ is a global diffeomorphism. □

3.4. *Eigenvalues and complex displacement for loxodromic maps*

A loxodromic element A of $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SU}(1,1)$ with eigenvalues λ and λ^{-1} where $|\lambda| > 1$ corresponds to a hyperbolic isometry, which we also denote by A, in $\mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{PU}(1,1)$, respectively. Since A is loxodromic, it has two fixed points on the boundary of the hyperbolic plane and these are the projections of the eigenspaces. The geodesic joining these two fixed points is called the *axis* of A, and is denoted $\tilde{\alpha}$. Then $\mathbf{H}^1_{\mathbb{C}}/\langle A\rangle$ is a hyperbolic cylinder (geometrically a catenoid) and $\alpha = \tilde{\alpha}/\langle A \rangle$ is the hyperbolic geodesic around its waist with hyperbolic length ℓ where

$$|\lambda| = e^{\ell/2}, \quad \big|\mathrm{tr}(A)\big| = 2\cosh\big(\ell/2\big).$$

In other words A translates along its axis by a hyperbolic translation length of ℓ. The ambiguity in the sign of $\mathrm{tr}(A)$ exactly corresponds to the choice of lift from $\mathrm{PSL}(2,\mathbb{R})$ to $\mathrm{SL}(2,\mathbb{R})$ or from $\mathrm{PSU}(1,1)$ to $\mathrm{SU}(1,1)$ respectively.

Similarly, when A is in $\mathrm{SL}(2,\mathbb{C})$ its trace corresponds to a complex length. More precisely, suppose $\mathrm{tr}(A) = \lambda + \lambda^{-1}$ where $|\lambda| > 1$. Then once again $|\lambda| = e^{\ell/2}$. To find the argument of λ, for any $z \in \tilde{\alpha}$, consider a tangent vector ξ in $T_z(\mathbf{H}^3)$ orthogonal to $\tilde{\alpha}$, the axis of A. Then A sends ξ in $T_z(\mathbf{H}^3)$ to a tangent vector $\xi e^{i\phi}$ in $T_{A(z)}(\mathbf{H}^3)$. In other words, A translates along $\tilde{\alpha}$ by a hyperbolic distance ℓ and rotates the tangent space by an angle ϕ. Then

$$\lambda = e^{\ell/2 + i\phi/2}, \quad \mathrm{tr}(A) = 2\cosh\big(\ell/2 + i\phi/2\big).$$

Since ϕ is defined mod 2π we see that the imaginary part of $\ell/2 + i\phi/2$ is defined mod π. This introduces an ambiguity of ± 1 in the trace and this corresponds exactly to the ambiguity introduced when lifting A from $\mathrm{PSL}(2,\mathbb{C})$ to $\mathrm{SL}(2,\mathbb{C})$; see Parker and Series [25].

In this section we illustrate how the geometric action of $A \in \mathrm{SU}(2,1)$ is recorded by $\mathrm{tr}(A)$. In principle the relationship is very similar to the case of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ but the functions involved are more complicated. The main result of this section is:

Proposition 3.10: *Let $A \in \mathrm{SU}(2,1)$ be a loxodromic map with axis $\tilde{\alpha}$. Let $\lambda \in \mathbb{C}$ be the eigenvalue of A with $|\lambda| > 1$. Suppose that A has a Bergman translation length ℓ along $\tilde{\alpha}$ and rotates complex lines normal to $\tilde{\alpha}$ by an angle ϕ. Then*

$$\lambda = e^{\ell/2 - i\phi/3} \tag{3.2}$$

and

$$\mathrm{tr}(A) = 2\cosh\big(\ell/2\big)e^{-i\phi/3} + e^{2i\phi/3}. \tag{3.3}$$

Furthermore, since ϕ is defined mod 2π, the arguments of λ and τ are only given mod $2\pi/3$ and so these formulae are only well defined on $\mathrm{PU}(2,1)$.

Proof: It will be convenient to use the Hermitian form H_2 and to conjugate within $\mathrm{SU}(H_2)$ so that A is diagonal:

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \overline{\lambda}\lambda^{-1} & 0 \\ 0 & 0 & \overline{\lambda}^{-1} \end{bmatrix}.$$

The action of A on $\mathbf{H}^2_{\mathbb{C}}$ is given by

$$A(z_1, z_2) = \mathbb{P}\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \overline{\lambda}\lambda^{-1} & 0 \\ 0 & 0 & \overline{\lambda}^{-1} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} = \mathbb{P}\begin{bmatrix} \lambda z_1 \\ \overline{\lambda}\lambda^{-1} z_2 \\ \overline{\lambda}^{-1} \end{bmatrix} = \left(|\lambda|^2 z_1, \frac{\overline{\lambda}^3}{|\lambda|^2}\, z_2 \right).$$

The axis $\tilde{\alpha}$ of A is given by

$$\tilde{\alpha} = \big\{ (-x, 0) \in \mathbb{C}^2 \ : \ x > 0 \big\}.$$

Let $\mathbf{x}$ be the standard lift of $(-x, 0)$ in $\tilde{\alpha}$. Let ℓ be the Bergman translation length of A along its axis then

$$\begin{aligned} \cosh\big(\ell/2\big) &= \cosh\Big(\rho\big(A(-x,0), (-x,0)\big)/2\Big) \\ &= \left| \frac{\langle A\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} \right| \\ &= \left| \frac{-\lambda x - \overline{\lambda}^{-1} x}{-2x} \right| \\ &= \big(|\lambda| + |\lambda|^{-1}\big)/2. \end{aligned}$$

Therefore, once again we have $|\lambda| = e^{\ell/2}$.

We now consider the argument of λ. The axis $\tilde{\alpha}$ is contained in a unique complex line, the *complex axis* $\tilde{\alpha}_{\mathbb{C}}$. With our normalisation,

$$\tilde{\alpha}_{\mathbb{C}} = \big\{ (z, 0) \in \mathbb{C}^2 \ : \ \Re(z) < 0 \big\}.$$

For any point $(-x, 0) \in \tilde{\alpha}$ let ξ be a tangent vector in $T_{(-x,0)}(\mathbf{H}^2_{\mathbb{C}})$ orthogonal to $\tilde{\alpha}_{\mathbb{C}}$. Since

$$A(z_1, z_2) = \big(|\lambda|^2 z_1, \overline{\lambda}^2 \lambda^{-1} z_2\big),$$

we see that $\xi \in T_{(-x,0)}(\mathbf{H}^2_{\mathbb{C}})$ is sent to $\xi e^{i\phi}$ then

$$\phi = \arg\big(\overline{\lambda}^2 \lambda^{-1}\big) = -3\arg(\lambda).$$

Hence $\arg(\lambda) = -\phi/3$. Thus we obtain (3.2).

Finally, since $\mathrm{tr}(A) = \lambda + \overline{\lambda}\lambda^{-1} + \overline{\lambda}^{-1}$, we obtain (3.3). □

Corollary 3.11: *Let A be as in Proposition 3.10. The function* (3.3) *relating ℓ and ϕ to* $\mathrm{tr}(A)$ *is a local diffeomorphism.*

Proof: Since it is clear that $\lambda = e^{\ell/2 - i\phi/3}$ is a local diffeomorphism the result follows by composing this map with the function relating λ and $\mathrm{tr}(A)$, and then using Proposition 3.8.

In fact it is jut as simple to calculate the Jacobian directly. Using (3.3), the real and imaginary parts of $\mathrm{tr}(A)$ are:

$$\begin{aligned}\mathrm{Re}\big(\mathrm{tr}(A)\big) &= 2\cosh(\ell/2)\cos(\phi/3) + \cos(2\phi/3),\\ \mathrm{Im}\big(\mathrm{tr}(A)\big) &= -2\cosh(\ell/2)\sin(\phi/3) + \sin(2\phi/3).\end{aligned}$$

Therefore:

$$\begin{aligned}\big|J_\tau(\ell,\phi)\big| &= \det\begin{pmatrix} \sinh(\frac{\ell}{2})\cos(\frac{\phi}{3}) & -\frac{2}{3}\cosh(\frac{\ell}{2})\sin(\frac{\phi}{3}) - \frac{2}{3}\sin(\frac{2\phi}{3}) \\ -\sinh(\frac{\ell}{2})\sin(\frac{\phi}{3}) & -\frac{2}{3}\cosh(\frac{\ell}{2})\cos(\frac{\phi}{3}) + \frac{2}{3}\cos(\frac{2\phi}{3}) \end{pmatrix}\\ &= -(2/3)\sinh(\ell/2)\big(\cosh(\ell/2) - \cos(\phi)\big).\end{aligned}$$

This is clearly non-zero when $\ell > 0$. □

4. Two generator groups and Fenchel-Nielsen coordinates

4.1. *Introduction*

There is a long tradition of studying subgroups of $\mathrm{SL}(2,\mathbb{C})$ by relating the traces of group elements to their geometry. This goes back to work of Vogt and Fricke who showed that a non-elementary two generator subgroup of $\mathrm{SL}(2,\mathbb{C})$ is determined up to conjugation by the traces of the generators and their product. For a precise statement of this result see Theorem A of Goldman [9]. One goal of this section is to extend this result to two generator subgroups of $\mathrm{SU}(2,1)$ and our treatment follows work of Lawton [13] and Will [33], [34]. The method we use begins by discussing trace relations in $\mathrm{M}(3,\mathbb{C})$, then specialising to $\mathrm{SL}(3,\mathbb{C})$ before finally giving our results for $\mathrm{SU}(2,1)$.

We are also interested in the geometry of two generator subgroups of $\mathrm{SU}(2,1)$. In this section we concentrate on the case where the generators and their product are all loxodromic. The fundamental group of a three-holed sphere is a free group on two generators. The generators and their product correspond to the three boundary components. Since we require that these three elements are loxodromic, we can use the results of section 3.4 to give geometric information about the corresponding three-holed sphere. As an application, we discuss how to generalise Fenchel-Nielsen coordinates to complex hyperbolic representations of surface groups. This follows work of Parker and Platis [23].

4.2. *Trace identities in* $\mathbf{M}(3,\mathbb{C})$

In this section we derive some trace identities for 3×3 matrices. This follows Lawton [13] and Will [33], [34]. The first lemma follows by writing $\mathrm{tr}(A)$, $\mathrm{tr}(A^2)$ and $\mathrm{tr}(A^3)$ as homogeneous polynomials in the eigenvalues of A and then solving for the coefficients of the characteristic polynomial.

Lemma 4.1: *Let* $A \in \mathrm{M}(3,\mathbb{C})$*. Then the characteristic polynomial* $\mathrm{ch}_A(x)$ *of* A *is*

$$x^3 - \mathrm{tr}(A)x^2 + \frac{\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)}{2}\,x - \frac{\mathrm{tr}(A)^3 - 3\mathrm{tr}(A)\mathrm{tr}(A^2) + 2\mathrm{tr}(A^3)}{6}.$$

For any $A \in \mathrm{M}(3,\mathbb{C})$ define $\mathrm{ch}(A)$ to be the following matrix (here I is the 3×3 identity matrix):

$$\begin{aligned}\mathrm{ch}(A) = A^3 - \mathrm{tr}(A)A^2 &+ \tfrac{1}{2}\big(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)\big)\,A \\ &-\tfrac{1}{6}\big(\mathrm{tr}(A)^3 - 3\mathrm{tr}(A)\mathrm{tr}(A^2) + 2\mathrm{tr}(A^3)\big)\,I.\end{aligned} \tag{4.1}$$

Then the Cayley-Hamilton theorem states that $\mathrm{ch}(A) = O$, the 3×3 zero matrix. We use a process known as trilinearisation on this identity to obtain the following:

Proposition 4.2: *Let* A, B, $C \in \mathrm{M}(3,\mathbb{C})$*. Then*

$$\begin{aligned}O = ABC &+ ACB + BAC + BCA + CAB + CBA \\ &-\mathrm{tr}(A)\big(BC+CB\big) - \mathrm{tr}(B)\big(AC+CA\big) - \mathrm{tr}(C)\big(AB+BA\big) \\ &+\big(\mathrm{tr}(B)\mathrm{tr}(C) - \mathrm{tr}(BC)\big)A + \big(\mathrm{tr}(A)\mathrm{tr}(C) - \mathrm{tr}(AC)\big)B \\ &+\big(\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB)\big)C \\ &-\big(\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(C) + \mathrm{tr}(ABC) + \mathrm{tr}(CBA)\big)I \\ &+\big(\mathrm{tr}(A)\mathrm{tr}(BC) + \mathrm{tr}(B)\mathrm{tr}(AC) + \mathrm{tr}(C)\mathrm{tr}(AB)\big)I.\end{aligned}$$

Proof: Using the Cayley-Hamilton theorem, as indicated above, for any A, B, $C \in \mathrm{M}(3,\mathbb{C})$ we have

$$\begin{aligned}O = \mathrm{ch}(A+B+C) &- \mathrm{ch}(A+B) - \mathrm{ch}(B+C) - \mathrm{ch}(A+C) \\ &+\mathrm{ch}(A) + \mathrm{ch}(B) + \mathrm{ch}(C).\end{aligned}$$

To obtain the result, we expand this expression and simplify, using $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. □

Corollary 4.3: *For any A, $B \in \mathrm{M}(3,\mathbb{C})$ we have:*

$$
\begin{aligned}
O = {} & ABA^{-1} + B + A^{-1}BA \\
& -\mathrm{tr}(A)\big(BA^{-1} + A^{-1}B\big) - \mathrm{tr}(A^{-1})\big(AB + BA\big) + \mathrm{tr}(A)\mathrm{tr}(A^{-1})B \\
& +\big(\mathrm{tr}(B)\mathrm{tr}(A^{-1}) - \mathrm{tr}(BA^{-1})\big)A + \big(\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB)\big)A^{-1} \\
& -\big(\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1}) + \mathrm{tr}(B) - \mathrm{tr}(A)\mathrm{tr}(BA^{-1}) - \mathrm{tr}(A^{-1})\mathrm{tr}(AB)\big)I;
\end{aligned}
$$

$$
\begin{aligned}
O = {} & ABA + A^2B + BA^2 - \mathrm{tr}(A)\big(AB + BA\big) - \frac{1}{2}\mathrm{tr}(B)A^2 \\
& +\big(\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB)\big)A + \frac{1}{2}\big(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)\big)B \\
& -\frac{1}{2}\big(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)\big)\mathrm{tr}(B)I + \big(\mathrm{tr}(A)\mathrm{tr}(AB) - \mathrm{tr}(A^2B)\big)I.
\end{aligned}
$$

Proof: For the first identity put $C = A^{-1}$ into the expression from Proposition 4.2 and use $\mathrm{tr}(I) = 3$. For the second put $C = A$ into Proposition 4.2 and divide by 2. □

Corollary 4.4: *For any A, $B \in \mathrm{M}(3,\mathbb{C})$ we have*

$$
\begin{aligned}
& \mathrm{tr}[A,B] + \mathrm{tr}[A^{-1},B] \\
& = \mathrm{tr}(A)\mathrm{tr}(A^{-1}) + \mathrm{tr}(B)\mathrm{tr}(B^{-1}) + \mathrm{tr}(A)\mathrm{tr}(A^{-1})\mathrm{tr}(B)\mathrm{tr}(B^{-1}) - 3 \\
& \quad +\mathrm{tr}(AB)\mathrm{tr}(A^{-1}B^{-1}) - \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1}B^{-1}) \\
& \quad -\mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B)\mathrm{tr}(AB^{-1}) \\
& \quad -\mathrm{tr}(A^{-1})\mathrm{tr}(B)\mathrm{tr}(AB^{-1}) - \mathrm{tr}(A)\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1}B).
\end{aligned}
$$

Proof: Multiplying the first expression from Corollary 4.3 on the right by B^{-1} gives

$$
\begin{aligned}
O = {} & ABA^{-1}B^{-1} + I + A^{-1}BAB^{-1} \\
& -\mathrm{tr}(A)\big(BA^{-1}B^{-1} + A^{-1}\big) - \mathrm{tr}(A^{-1})\big(A + BAB^{-1}\big) \\
& +\mathrm{tr}(A)\mathrm{tr}(A^{-1})I + \big(\mathrm{tr}(B)\mathrm{tr}(A^{-1}) - \mathrm{tr}(BA^{-1})\big)AB^{-1} \\
& +\big(\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB)\big)A^{-1}B^{-1} \\
& -\big(\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1}) + \mathrm{tr}(B)\big)B^{-1} \\
& +\big(\big(\mathrm{tr}(A)\mathrm{tr}(BA^{-1})\mathrm{tr}(A^{-1})\mathrm{tr}(AB)\big)B^{-1}.
\end{aligned}
$$

Taking traces gives the result. □

4.3. *Trace identities in* $\mathbf{SL(3,\mathbb{C})}$

When A is in $\mathrm{SL}(3,\mathbb{C})$ its characteristic polynomial may be written in a form that is somewhat simpler the form given in Lemma 4.1.

Lemma 4.5: *Let $A \in \mathrm{SL}(3,\mathbb{C})$. The characteristic polynomial of A is*

$$\mathrm{ch}_A(x) = x^3 - \mathrm{tr}(A)x^2 + \mathrm{tr}(A^{-1}) - 1.$$

Proof: Let λ_1, λ_2, λ_3 be the eigenvalues of A. Then $\lambda_1\lambda_2\lambda_3 = \det(A) = 1$. This gives the constant term in $\mathrm{ch}_A(x)$. Observe that λ_1^{-1}, λ_2^{-1}, λ_3^{-1} are the eigenvalues of A^{-1}. Thus, using both of these facts, we see that the linear term in $\mathrm{ch}_A(x)$ is

$$\lambda_2\lambda_3 + \lambda_1\lambda_3 + \lambda_1\lambda_2 = \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} = \mathrm{tr}(A^{-1}). \qquad \square$$

Again, using the Cayley-Hamilton theorem, we see that for $A \in \mathrm{SL}(3,\mathbb{R})$ we have

$$O = A^3 - \mathrm{tr}(A)A^2 + \mathrm{tr}(A^{-1})A - I. \tag{4.2}$$

Lemma 4.6: *Let $A \in \mathrm{SU}(2,1)$. Then*

(i) $\mathrm{tr}(A^2) = \big(\mathrm{tr}(A)\big)^2 - 2\mathrm{tr}(A^{-1})$;
(ii) $\mathrm{tr}(A^3) = \big(\mathrm{tr}(A)\big)^3 - 3\mathrm{tr}(A)\mathrm{tr}(A^{-1}) + 3$.

Proof: Multiplying equation (4.2) by A^{-1} gives:

$$A^2 = \mathrm{tr}(A)A - \mathrm{tr}(A^{-1})I + A^{-1}.$$

Taking traces we see that

$$\mathrm{tr}(A^2) = \mathrm{tr}(A)\mathrm{tr}(A) - 3\mathrm{tr}(A^{-1}) + \mathrm{tr}(A^{-1}) = \big(\mathrm{tr}(A)\big)^2 - 2\mathrm{tr}(A^{-1}),$$

which gives part (i).

Taking traces in equation (4.2) and then substituting for $\mathrm{tr}(A^2)$ gives

$$\begin{aligned}\mathrm{tr}(A^3) &= \mathrm{tr}(A)\mathrm{tr}(A^2) - \mathrm{tr}(A^{-1})\mathrm{tr}(A) + 3\\ &= \mathrm{tr}(A)\Big(\big(\mathrm{tr}(A)\big)^2 - 2\mathrm{tr}(A^{-1})\Big) - \mathrm{tr}(A^{-1})\mathrm{tr}(A) + 3\\ &= \big(\mathrm{tr}(A)\big)^3 - 3\mathrm{tr}(A)\mathrm{tr}(A^{-1}) + 3,\end{aligned}$$

which gives part (ii). $\square$

Proposition 4.7: *Let A, $B \in \mathrm{SL}(3,\mathbb{C})$. Then $\mathrm{tr}[A,B]\mathrm{tr}[B,A]$ may be expressed as a polynomial function of the traces of A, B, AB, $A^{-1}B$ and their inverses.*

The precise polynomial is given by Lawton in equation (18) of [13] or by Will on page 58 of [33]. In the case of $A \in \mathrm{SU}(2,1)$ we give a version of this polynomial in Proposition 4.10 below.

Sketch proof: This follows Will [33]; Lawton [13] has a different proof.

Write $A = MN$ and $B = NM$ in the expression in Corollary 4.4. This gives

$$\begin{aligned}
&\mathrm{tr}[MN, NM] + \mathrm{tr}[NM, MN] \\
&= 2\mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1}) + \mathrm{tr}(MN)^2\mathrm{tr}(M^{-1}N^{-1})^2 - 3 \\
&\quad +\mathrm{tr}(M^2N^2)\mathrm{tr}(M^{-2}N^{-2}) - \mathrm{tr}(MN)^2\mathrm{tr}(M^{-2}N^{-2}) \\
&\quad -\mathrm{tr}(N^{-1}M^{-1})^2\mathrm{tr}(M^2N^2) + \mathrm{tr}[M,N]\mathrm{tr}[N,M] \\
&\quad -\mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\big(\mathrm{tr}[M,N] + \mathrm{tr}[M^{-1},N]\big).
\end{aligned}$$

Using Corollary 4.4 we can express $\mathrm{tr}[M,N] + \mathrm{tr}[M^{-1},N]$ in terms of the traces of M, N, MN, $M^{-1}N$ and their inverses.

If M and N are in $\mathrm{SL}(3,\mathbb{C})$ we can use their characteristic polynomials to write

$$M^2 = \mathrm{tr}(M)M - \mathrm{tr}(M^{-1})I + M^{-1}, \quad N^2 = \mathrm{tr}(N)N - \mathrm{tr}(N^{-1})I + N^{-1}.$$

Hence

$$M^2N^2 = \big(\mathrm{tr}(M)M - \mathrm{tr}(M^{-1})I + M^{-1}\big)\big(\mathrm{tr}(N)N - \mathrm{tr}(N^{-1})I + N^{-1}\big).$$

From this we can express the trace of M^2N^2 in terms of the traces of M, N, MN, MN^{-1} and their inverses. Likewise we can express the traces of M^2N^{-2}, $M^{-2}N^2$ and $M^{-2}N^{-2}$ in terms of these traces.

Thus it suffices to express the trace of $[MN, NM]$ and $[NM, MN]$ in terms of these other traces. To do so, first write

$$[MN, NM] = MN^2MN^{-1}M^{-2}N^{-1}$$

and substitute for N^2 and M^{-2} as above. Then use Corollary 4.3 to substitute for words such as MNM, MNM^{-1}. At each stage, we write the words in terms of shorter words in the group. Eventually we obtain the desired expression. □

4.4. *Trace parameters for two generator groups of* $\mathbf{SU(2,1)}$

Let Y be a three holed sphere (also known as a "pair of pants"). Let $\pi_1 = \pi_1(Y)$ be the fundamental group of Y. If $[\alpha]$, $[\beta]$ and $[\gamma]$ are the homotopy classes in π_1 representing the boundary curves then $\alpha\beta\gamma = Id$. In fact, π_1 is a free group generated by any two of $[\alpha]$, $[\beta]$, $[\gamma]$.

It is well known that for $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SU}(1,1)$ (the holomorphic, hyperbolic isometry groups of the upper plane and Poincaré disc respectively) then the group generated by A and B is completely determined up to conjugacy by $\mathrm{tr}(A)$, $\mathrm{tr}(B)$ and $\mathrm{tr}(AB)$. Geometrically, under mild hypotheses, $\langle A, B\rangle$ corresponds to a representation ρ_0 of π_1 which gives Y a hyperbolic metric. The mild hypotheses are that $\langle A, B\rangle$ should be discrete, faithful (so free), totally loxodromic and that the axes of A, B and AB should bound a common region in the hyperbolic plane. We may choose geodesic representatives for the three boundary homotopy classes, and we denote these geodesics by α, β and γ respectively. Then by construction we have $\rho(\alpha) = A$, $\rho(\beta) = B$ and $\rho(\gamma) = C = B^{-1}A^{-1}$ with lengths given by

$$\begin{aligned}\big|\mathrm{tr}(A)\big| &= 2\cosh\big(\ell(\alpha)/2\big),\\ \big|\mathrm{tr}(B)\big| &= 2\cosh\big(\ell(\beta)/2\big),\\ \big|\mathrm{tr}(C)\big| &= 2\cosh\big(\ell(\gamma)/2\big).\end{aligned}$$

In fact, our mild hypotheses about the axes of A, B and C imply that

$$\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(C) < 0$$

and so we may choose a lift from $\mathrm{PSL}(2,\mathbb{R})$ to $\mathrm{SL}(2,\mathbb{R})$ where all three traces are negative. Conversely, given $\ell(\alpha)$, $\ell(\beta)$, $\ell(\gamma)$ in $\mathbb{R}_+$ we can construct a hyperbolic metric on Y whose boundary geodesics have these lengths. This in turn gives rise to a group $\langle A, B\rangle$ satisfying $\big|\mathrm{tr}(A)\big| = 2\cosh\big(\ell(\alpha)/2\big)$ et cetera.

Similarly, if $\langle A, B\rangle$ is a discrete, free, geometrically finite and totally loxodromic subgroup of $\mathrm{SL}(2,\mathbb{C})$ then we have a similar picture, but the lengths of the boundary curves are now complex, as discussed in the introduction to Section 3.4. The main difference here is that, not all triples of complex lengths give rise to a discrete, free, totally loxodromic, geometrically finite group.

We now want to play a similar game using complex hyperbolic representations of $\pi_1(Y)$. Again the representations we will be interested in will be discrete, free, totally loxodromic and geometrically finite; for a discussion of the latter see Bowditch [1]. We will also add the hypothesis that $\langle A, B\rangle$ is Zariski dense. A subgroup of $\mathrm{PSU}(2,1)$ is Zariski dense if and only if its action on $\mathbb{CP}^2$ does not have a global fixed point (see Remark 10 of Will [34]). Equivalently, it does not fix a point on $\overline{\mathbf{H}}^2_{\mathbb{C}}$ or preserve a complex line in $\mathbf{H}^2_{\mathbb{C}}$. Consider $\rho : \pi_1(Y) \longrightarrow \mathrm{SU}(2,1)$. Then ρ is irreducible if and only if its image is Zariski dense.

The main question is what are the data we need to completely determine $\langle A, B\rangle$ up to conjugation. Our first observation is that $\mathrm{SU}(2,1)$ has complex dimension four and so we do not expect to be able to determine $\langle A, B\rangle$ using only three complex numbers. The following theorem is a special case of a result of Wen [32], see also Lemma 5 of Lawton [13] or Corollary 3.10 of Will [33], Proposition 9 of [34]:

Theorem 4.8: *Suppose that* A, $B \in \mathrm{SU}(2,1)$ *and that* $\langle A, B\rangle$ *is Zariski dense. Then* $\langle A, B\rangle$ *is determined up to conjugation within* $\mathrm{SU}(2,1)$ *by*

$$\mathrm{tr}(A), \quad \mathrm{tr}(B), \quad \mathrm{tr}(AB), \quad \mathrm{tr}(A^{-1}B), \quad \mathrm{tr}[A,B].$$

We remark that Wen's theorem refers to A and B in $\mathrm{SL}(3,\mathbb{C})$ and also requires the traces of A^{-1}, B^{-1}, $A^{-1}B^{-1}$ and AB^{-1}. Also, this theorem has a similar flavour to the the theorem of Okumura [18] and Schmutz [29]. Namely, one would expect to only need to use four traces to describe $\langle A, B\rangle$. In fact ones needs an extra one, $\mathrm{tr}[A,B]$, and this satisfies relations with the other traces.

In what follows we want to consider A, B, $C \in \mathrm{SU}(2,1)$ with $ABC = I$. It is clear that $\mathrm{tr}(AB) = \mathrm{tr}(C^{-1}) = \overline{\mathrm{tr}(C)}$. We want to express the other parameters in a way that is symmetrical with respect to cyclic permutations of A, B and C. First we consider the trace of $A^{-1}B$.

Lemma 4.9: *Let* A, B, C *be elements of* $\mathrm{SU}(2,1)$ *so that* $ABC = I$. *Then*

$$\begin{aligned}\mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B) &= \mathrm{tr}(B^{-1}C) - \mathrm{tr}(B^{-1})\mathrm{tr}(C)\\ &= \mathrm{tr}(C^{-1}A) - \mathrm{tr}(C^{-1})\mathrm{tr}(A).\end{aligned}$$

Proof: We have already seen that

$$A^3 - \mathrm{tr}(A)A^2 + \mathrm{tr}(A^{-1})A - I = O.$$

Multiplying on the right by $A^{-1}B$ gives

$$A^2B - \mathrm{tr}(A)AB = A^{-1}B - \mathrm{tr}(A^{-1})B.$$

Taking traces and using $AB = C^{-1}$ gives

$$\mathrm{tr}(C^{-1}A) - \mathrm{tr}(C^{-1})\mathrm{tr}(A) = \mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B).$$

This shows equality between the first and third expressions. Cyclically permuting A, B and C gives the second as well. □

Therefore by using $\mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B)$ instead of $\mathrm{tr}(A^{-1}B)$ we can give symmetric parameters. Furthermore, trivially we have

$$\mathrm{tr}[A,B] = \mathrm{tr}[B,C] = \mathrm{tr}[C,A] = \overline{\mathrm{tr}[B,A]} = \overline{\mathrm{tr}[C,B]} = \overline{\mathrm{tr}[A,C]}.$$

As we have seen above in Corollary 4.4 and Proposition 4.7, the real part and absolute value of $\mathrm{tr}[A,B]$ are determined by the other parameters. We now make this explicit.

Proposition 4.10: *Let A, B, $C \in \mathrm{SU}(2,1)$ with $ABC = I$. Let*

$$a = \mathrm{tr}(A), \quad b = \mathrm{tr}(B), \quad c = \mathrm{tr}(C), \quad d = \mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B).$$

Then

$$2\Re\big(\mathrm{tr}[A,B]\big) = |a|^2 + |b|^2 + |c|^2 + |d|^2 - abc - \overline{a}\overline{b}\overline{c} - 3$$

and

$$\begin{aligned}
&\big|\mathrm{tr}[A,B]\big|^2 \\
&= |a|^2|b|^2|c|^2 + a^2b^2\overline{c} + \overline{a}^2\overline{b}^2c + a^2\overline{b}c^2 + \overline{a}^2b\overline{c}^2 + \overline{a}b^2c^2 + a\overline{b}^2\overline{c}^2 \\
&\qquad +|a|^2|b|^2 + |b|^2|c|^2 + |a|^2|c|^2 \\
&\qquad -2ab\overline{c}^2 - 2\overline{a}\overline{b}c^2 - 2a\overline{b}^2c - 2\overline{a}b^2\overline{c} - 2\overline{a}^2bc - 2a^2\overline{b}\overline{c} \\
&\qquad +a^3 + \overline{a}^3 + b^3 + \overline{b}^3 + c^3 + \overline{c}^3 + 3abc + 3\overline{a}\overline{b}\overline{c} - 6|a|^2 - 6|b|^2 - 6|c|^2 \\
&\qquad +d\big(|a|^2b\overline{c} + a\overline{b}|c|^2 + \overline{a}|b|^2c + ab^2 + \overline{a}^2\overline{b} + a^2c + \overline{a}\,\overline{c}^2 + bc^2 + \overline{b}^2\overline{c}\big) \\
&\qquad +\overline{d}\big(|a|^2\overline{b}c + \overline{a}b|c|^2 + a|b|^2\overline{c} + \overline{a}\overline{b}^2 + a^2b + \overline{a}^2\overline{c} + ac^2 + \overline{b}\overline{c}^2 + b^2c\big) \\
&\qquad +(d^2 - 3\overline{d})\big(\overline{a}b + \overline{b}c + a\overline{c}\big) + (\overline{d}^2 - 3d)\big(a\overline{b} + b\overline{c} + \overline{a}c\big) \\
&\qquad +|d|^2\big(|a|^2 + |b|^2 + |c|^2 - 6\big) + d^3 + \overline{d}^3 + 9.
\end{aligned}$$

Proof: Using $\mathrm{tr}(A^{-1}) = \overline{\mathrm{tr}(A)} = \overline{a}$ et cetera and also $\mathrm{tr}(A^{-1}B) = d + \overline{a}b$ in the expression of Corollary 4.4 gives:

$$\begin{aligned}
2\Re\big(\mathrm{tr}[A,B]\big) &= |a|^2|b|^2 + |a|^2 + |b|^2 + |c|^2 + |d + \overline{a}b|^2 \\
&\qquad -\overline{a}b(\overline{d} + a\overline{b}) - a\overline{b}(d + \overline{a}b) - abc - \overline{a}\overline{b}\overline{c} - 3 \\
&= |a|^2 + |b|^2 + |c|^2 + |d|^2 - abc - \overline{a}\overline{b}\overline{c} - 3.
\end{aligned}$$

Similarly, using the expression for $\big|\mathrm{tr}[A,B]\big|^2$ given by Lawton in equation (18) of [13] or by Will on page 58 of [33] (see Proposition 4.7) gives the second expression. □

Putting this together gives:

Proposition 4.11: *Let A, B, C be elements of* $\mathrm{SU}(2,1)$ *with* $ABC = I$. *Then, if* $\langle A, B, C\rangle$ *is Zariski dense, it is determined up to conjugacy by*

$$\mathrm{tr}(A), \quad \mathrm{tr}(B), \quad \mathrm{tr}(C), \quad \mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B), \quad \mathrm{tr}[A,B].$$

Also, the last two of these expressions remain unchanged under cyclic permutations of A, B and C.

Moreover, the group is determined by $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathrm{tr}(C)$ *and* $\mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B)$ *together with the sign of the imaginary part of* $\mathrm{tr}[A,B]$.

Example 4.12: We now give an example that shows the traces of A, B, AB and $A^{-1}B$ do not determine the imaginary part of $\mathrm{tr}[A,B]$.

For $\theta \in (-\pi/2, \pi/2)$ let $Q(\theta) \in \mathrm{SU}(2,1)$ be the matrix

$$Q(\theta) = \frac{-e^{-i\theta/6}}{2\cos(\theta/2)} \begin{bmatrix} 1 & \sqrt{2\cos(\theta)} & -e^{i\theta} \\ \sqrt{2\cos(\theta)} & e^{-i\theta} - 1 & \sqrt{2\cos(\theta)} \\ -e^{i\theta} & \sqrt{2\cos(\theta)} & 1 \end{bmatrix}.$$

Note that $Q(\theta)^{-1} = Q(-\theta)$. For $r > 1$ and $s > 1$, define A, $B_\theta \in \mathrm{SU}(2,1)$ by

$$A = \begin{bmatrix} re^{i\phi} & 0 & 0 \\ 0 & e^{-2i\phi} & 0 \\ 0 & 0 & r^{-1}e^{i\phi} \end{bmatrix}, \quad B_\theta = Q(\theta) \begin{bmatrix} se^{i\psi} & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & s^{-1}e^{i\psi} \end{bmatrix} Q(-\theta).$$

Then we have

$$\begin{aligned} \mathrm{tr}(A) &= (r + r^{-1})e^{i\phi} + e^{-2i\phi}, \\ \mathrm{tr}(B_\theta) &= (s + s^{-1})e^{i\psi} + e^{-2i\psi}, \\ \mathrm{tr}(AB_\theta) &= \frac{1}{2 + 2\cos(\theta)}\Big((r + r^{-1})e^{i\phi} + 2\cos(\theta)e^{-2i\phi}\Big) \\ &\quad \cdot\Big((s + s^{-1})e^{i\psi} + 2\cos(\theta)e^{-2i\psi}\Big) \\ &\quad - \frac{1}{2 + 2\cos(\theta)}e^{-2i\phi-2i\psi}\Big(2\cos(\theta) + 2\cos(2\theta)\Big), \\ \mathrm{tr}(A^{-1}B_\theta) &= \frac{1}{2 + 2\cos(\theta)}\Big((r + r^{-1})e^{-i\phi} + 2\cos(\theta)e^{2i\phi}\Big) \\ &\quad \cdot\Big((s + s^{-1})e^{i\psi} + 2\cos(\theta)e^{-2i\psi}\Big) \\ &\quad - \frac{1}{2 + 2\cos(\theta)}e^{2i\phi-2i\psi}\Big(2\cos(\theta) + 2\cos(2\theta)\Big), \end{aligned}$$

$$
\begin{aligned}
&\mathrm{tr}[A, B_\theta] \\
&= 3 - \frac{1}{(2+2\cos(\theta))^2}(r-r^{-1})^2(s-s^{-1})^2 \\
&\quad - \frac{8\cos(\theta)}{(2+2\cos(\theta))^2}\Big(2-(r+r^{-1})\cos(3\phi)\Big)\Big(2-(s+s^{-1})\cos(3\psi)\Big) \\
&\quad + \frac{2\cos(\theta)}{(2+2\cos(\theta))^2}\Big(e^{i\theta}(rs^{-1}+r^{-1}s)+e^{-i\theta}(rs+r^{-1}s^{-1})\Big) \\
&\qquad \cdot\Big(r+r^{-1}-2\cos(3\phi)\Big)\Big(s+s^{-1}-2\cos(3\psi)\Big).
\end{aligned}
$$

Then it is easy to see that

$$
\mathrm{tr}(B_\theta) = \mathrm{tr}(B_{-\theta}), \quad \mathrm{tr}(AB_\theta) = \mathrm{tr}(AB_{-\theta}), \quad \mathrm{tr}(A^{-1}B_\theta) = \mathrm{tr}(A^{-1}B_{-\theta})
$$

but $\mathrm{tr}[A, B_\theta] \neq \mathrm{tr}[A, B_{-\theta}]$.

4.5. *Cross-ratios*

In [23] Parker and Platis used cross-ratios to describe the complex hyperbolic three holed spheres. In this section we will briefly outline their construction and relate them to the trace coordinates of the previous section.

Choose a signature-$(2,1)$ Hermitian form on $\mathbb{C}^{2,1}$. Let V_- and V_0 be the associated subspaces given by (2.3) and (2.4). Then $\mathbb{P}V_- = \mathbf{H}^2_{\mathbb{C}}$ and $\mathbb{P}V_0 = \partial\mathbf{H}^2_{\mathbb{C}}$. Let z_1, z_2, z_3, z_4 be four distinct points on $\partial\mathbf{H}^2_{\mathbb{C}}$ and choose lifts of them $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$, $\mathbf{z}_4$ in V_0. Following Korányi and Reimann [11], we define the *complex cross-ratio* of these four points to be

$$
\mathbb{X} = [z_1, z_2, z_3, z_4] = \frac{\langle \mathbf{z}_3, \mathbf{z}_1\rangle\langle \mathbf{z}_4, \mathbf{z}_2\rangle}{\langle \mathbf{z}_4, \mathbf{z}_1\rangle\langle \mathbf{z}_3, \mathbf{z}_2\rangle}.
$$

Since the z_i are distinct we see that $\mathbb{X}$ is finite and non-zero. We note that $\mathbb{X}$ is independent of the choice of the lifts $\mathbf{z}_j$ and is invariant under $\mathrm{SU}(2,1)$.

By choosing different orderings of our four points we may define other cross-ratios. There are some symmetries associated to certain permutations, see Property 5 on page 225 of [8]. After taking these into account, there are only three cross-ratios that remain. Given distinct points $z_1, \ldots, z_4 \in \partial\mathbf{H}^2_{\mathbb{C}}$, we define

$$
\mathbb{X}_1 = [z_1, z_2, z_3, z_4], \quad \mathbb{X}_2 = [z_1, z_3, z_2, z_4], \quad \mathbb{X}_3 = [z_2, z_3, z_1, z_4]. \tag{4.3}
$$

Then the three complex numbers $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ satisfy the following identities (see Proposition 5.2 of [23])

$$
|\mathbb{X}_2| = |\mathbb{X}_1|\,|\mathbb{X}_3|, \tag{4.4}
$$

$$
2|\mathbb{X}_1|^2\Re(\mathbb{X}_3) = |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 + 1 - 2\Re(\mathbb{X}_1 + \mathbb{X}_2). \tag{4.5}
$$

Note that the norm and real part of $\mathbb{X}_3$ are determined by $\mathbb{X}_1$ and $\mathbb{X}_2$, but that the sign of $\Im(\mathbb{X}_3)$ is not determined. Falbel has given a different parametrisation of the space of cross-ratios Proposition 2.3 of [5].

Suppose we are given loxodromic maps A and B in $\mathrm{SU}(2,1)$ with distinct fixed points. Let the attracting fixed points of A and B be a_A, a_B and the repelling fixed points r_A, r_B respectively. Suppose that these fixed points correspond to attractive eigenvectors $\mathbf{a}_A$, $\mathbf{a}_B$ and repulsive eigenvectors $\mathbf{r}_A$, $\mathbf{r}_B$ respectively. Following (4.3), we define the first, second and third cross-ratios of the loxodromic maps A and B to be

$$\mathbb{X}_1(A,B) = [a_B, a_A, r_A, r_B] = \frac{\langle \mathbf{r}_A, \mathbf{a}_B\rangle\langle \mathbf{r}_B, \mathbf{a}_A\rangle}{\langle \mathbf{r}_B, \mathbf{a}_B\rangle\langle \mathbf{r}_A, \mathbf{a}_A\rangle}, \tag{4.6}$$

$$\mathbb{X}_2(A,B) = [a_B, r_A, a_A, r_B] = \frac{\langle \mathbf{a}_A, \mathbf{a}_B\rangle\langle \mathbf{r}_B, \mathbf{r}_A\rangle}{\langle \mathbf{r}_B, \mathbf{a}_B\rangle\langle \mathbf{a}_A, \mathbf{r}_A\rangle}, \tag{4.7}$$

$$\mathbb{X}_3(A,B) = [a_A, r_A, a_B, r_B] = \frac{\langle \mathbf{a}_B, \mathbf{a}_A\rangle\langle \mathbf{r}_B, \mathbf{r}_A\rangle}{\langle \mathbf{r}_B, \mathbf{a}_A\rangle\langle \mathbf{a}_B, \mathbf{r}_A\rangle}. \tag{4.8}$$

We can parametrise Zariski dense subgroups of $\mathrm{SU}(2,1)$ generated by a pair of loxodromic maps by their traces and the cross-ratios of their fixed points. This is Theorem 7.1 of [23]:

Theorem 4.13: *Suppose that A and B are loxodromic elements of* $\mathrm{SU}(2,1)$ *with distinct fixed points. Suppose also that $\langle A, B\rangle$ does not preserve a complex line. Then the group $\langle A,\ B\rangle$ is determined up to conjugation in* $\mathrm{SU}(2,1)$ *by:* $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathbb{X}_1(A,B)$, $\mathbb{X}_2(A,B)$ *and* $\mathbb{X}_3(A,B)$.

Obviously, this result is asymmetrical in that it depends on a choice of two of the boundary curves. In order to get around this difficulty, Parker and Platis, Theorem 7.2 of [23], show that choosing a different pair of boundary coordinates amounts to a real change of coordinates.

Proposition 4.14: *Let A, B and C be loxodromic elements of* $\mathrm{SU}(2,1)$ *with $ABC = I$. Then* $\mathrm{tr}(C)$, $\mathbb{X}_1(A,C)$, $\mathbb{X}_2(A,C)$ *and* $\mathbb{X}_3(A,C)$ *may be expressed as real analytic functions of* $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathbb{X}_1(A,B)$, $\mathbb{X}_2(A,B)$ *and* $\mathbb{X}_3(A,B)$.

We conclude this section by showing how these mixed trace and cross-ratio coordinates are related to the trace coordinates we found in the previous section; see Propositions 6.4 and 7.6 of [23].

Proposition 4.15: *Let A and B be loxodromic maps in* $\mathrm{SU}(2,1)$ *with* $\mathrm{tr}(A) = \lambda + \overline{\lambda}\lambda^{-1} + \overline{\lambda}^{-1}$ *and* $\mathrm{tr}(B) = \mu + \overline{\mu}\mu^{-1} + \overline{\mu}^{-1}$. *Let* $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_3$

be the cross-ratios of their fixed points given by (4.6), (4.7) *and* (4.8). *Then the traces of* AB, $A^{-1}B$ *and* $[A,B]$ *are given by*

$$\begin{aligned}
&\mathrm{tr}(AB)\\
&= (\lambda+\overline{\lambda}^{-1})\overline{\mu}\mu^{-1} + \overline{\lambda}\lambda^{-1}(\mu+\overline{\mu}^{-1}) - \overline{\lambda}\lambda^{-1}\overline{\mu}\mu^{-1}\\
&\qquad +\mathbb{X}_1\,(\overline{\lambda}^{-1}-\overline{\lambda}\lambda^{-1})(\overline{\mu}^{-1}-\overline{\mu}\mu^{-1}) + \overline{\mathbb{X}}_1\,(\lambda-\overline{\lambda}\lambda^{-1})(\mu-\overline{\mu}\mu^{-1})\\
&\qquad +\mathbb{X}_2\,(\lambda-\overline{\lambda}\lambda^{-1})(\overline{\mu}^{-1}-\overline{\mu}\mu^{-1}) + \overline{\mathbb{X}}_2\,(\overline{\lambda}^{-1}-\overline{\lambda}\lambda^{-1})(\mu-\overline{\mu}\mu^{-1}),\\
&\mathrm{tr}(A^{-1}B)\\
&= (\lambda^{-1}+\overline{\lambda})\overline{\mu}\mu^{-1} + \lambda\overline{\lambda}^{-1}(\mu+\overline{\mu}^{-1}) - \lambda\overline{\lambda}^{-1}\overline{\mu}\mu^{-1}\\
&\qquad +\mathbb{X}_1\,(\overline{\lambda}-\lambda\overline{\lambda}^{-1})(\overline{\mu}^{-1}-\overline{\mu}\mu^{-1}) + \overline{\mathbb{X}}_1\,(\lambda^{-1}-\lambda\overline{\lambda}^{-1})(\mu-\overline{\mu}\mu^{-1})\\
&\qquad +\mathbb{X}_2\,(\lambda^{-1}-\lambda\overline{\lambda}^{-1})(\overline{\mu}^{-1}-\overline{\mu}\mu^{-1}) + \overline{\mathbb{X}}_2\,(\overline{\lambda}-\lambda\overline{\lambda}^{-1})(\mu-\overline{\mu}\mu^{-1})
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{tr}[A,B]\\
&= 3 - 2\Re\Big(\mathbb{X}_1\,(\overline{\lambda}-\lambda\overline{\lambda}^{-1})(\overline{\lambda}^{-1}-\overline{\lambda}\lambda^{-1})(\overline{\mu}-\mu\overline{\mu}^{-1})(\overline{\mu}^{-1}-\overline{\mu}\mu^{-1})\Big)\\
&\qquad -2\Re\Big(\mathbb{X}_2\,(\lambda-\overline{\lambda}\lambda^{-1})(\lambda^{-1}-\lambda\overline{\lambda}^{-1})(\overline{\mu}-\mu\overline{\mu}^{-1})(\overline{\mu}^{-1}-\overline{\mu}\mu^{-1})\Big)\\
&\qquad +\Big(1-2\Re(\mathbb{X}_1+\mathbb{X}_2)\Big)\\
&\qquad\quad \cdot\Big(\big|(\lambda-\overline{\lambda}\lambda^{-1})(\mu-\overline{\mu}\mu^{-1})\big|^2 + \big|(\lambda^{-1}-\lambda\overline{\lambda}^{-1})(\mu^{-1}-\mu\overline{\mu}^{-1})\big|^2\Big)\\
&\qquad +\left|\begin{array}{l}\mathbb{X}_1\,(\overline{\lambda}-\lambda\overline{\lambda}^{-1})(\overline{\mu}-\mu\overline{\mu}^{-1}) + \overline{\mathbb{X}}_1\,(\lambda^{-1}-\lambda\overline{\lambda}^{-1})(\mu^{-1}-\mu\overline{\mu}^{-1})\\ +\mathbb{X}_2\,(\lambda^{-1}-\lambda\overline{\lambda}^{-1})(\overline{\mu}-\mu\overline{\mu}^{-1}) + \overline{\mathbb{X}}_2\,(\overline{\lambda}-\lambda\overline{\lambda}^{-1})(\mu^{-1}-\mu\overline{\mu}^{-1})\end{array}\right|^2\\
&\qquad +\Big(|\mathbb{X}_2|^2-|\mathbb{X}_1|^2\mathbb{X}_3\Big)\\
&\qquad\quad \cdot\Big(\big|\lambda-\overline{\lambda}\lambda^{-1}\big|^2-\big|\lambda^{-1}-\lambda\overline{\lambda}^{-1}\big|^2\Big)\Big(\big|\mu-\overline{\mu}\mu^{-1}\big|^2-\big|\mu^{-1}-\mu\overline{\mu}^{-1}\big|^2\Big).
\end{aligned}$$

We can use the expressions from Proposition 4.15 to express $\mathbb{X}_1$ and $\mathbb{X}_2$ in terms of λ, μ, $\mathrm{tr}(AB)$ and $\mathrm{tr}(A^{-1}B)$. This means that these coordinates are "equivalent" to the trace coordinates we found earlier. Furthermore, the real part and modulus of $\mathrm{tr}[A,B]$ may be expressed in terms of λ, μ, $\mathbb{X}_1$ and $\mathbb{X}_2$ and the ambiguity in the sign of $\Im\big(\mathrm{tr}[A,B]\big)$ is exactly the same as the ambiguity in the sign of $\Im(\mathbb{X}_3)$. In Example 4.12 we have $\mathbb{X}_1=\mathbb{X}_2=-1/(2-2\cos(\theta))$ and $\mathbb{X}_3=e^{-2i\theta}$.

4.6. *Twist-bend parameters*

Let Σ be a surface of genus $g \geq 2$. We may decompose Σ into three holed spheres and use this decomposition to define complex hyperbolic Fenchel-Nielsen coordinates. The trace coordinates from Proposition 4.11 give the Fenchel-Nielsen complex lengths via Proposition 3.10. In this section we give a brief sketch of how to define Fenchel-Nielsen twist-bends. Details may be found in Parker and Platis [23].

For hyperbolic surfaces, a Fenchel-Nielsen *twist* about a simple, closed, oriented curve α involves cutting the surface along α and then re-attaching so that points on one side are moved a hyperbolic distance k relative to the other side. It is often useful to think about doing this with the lift $\tilde{\alpha}$ of α to the hyperbolic plane. If $\tilde{\alpha}$ is the geodesic in the upper halfplane with endpoints 0 and ∞, then this process involves applying the dilation $K : z \longmapsto e^k z$ to the part of the hyperbolic plane on one side of $\tilde{\alpha}$, say the part with $\mathrm{Re}(z) > 0$. We allow k to be negative and this corresponds to moving in the opposite direction relative to the orientation of α. If we twist by a hyperbolic distance k equal to the length of α then the Fenchel-Nielsen twist is the same as a Dehn twist.

We can generalise the definition of Fenchel-Nielsen twist to twist-bends in hyperbolic 3-space $\mathbf{H}^2_{\mathbb{R}}$, following Tan [30] and Kourouniotis [12]. The easiest way to describe this is to suppose that the universal cover of the surface is a hyperbolic plane inside $\mathbf{H}^3_{\mathbb{R}}$ and that $\tilde{\alpha}$ is again the geodesic with endpoints 0 and ∞. As well as applying a Fenchel-Nielsen twist, we may also rotate through an angle θ in the plane normal to $\tilde{\alpha}$. This is called a *bend* and corresponds to applying the rotation $K : z \longmapsto e^{i\theta} z$. Doing a twist through distance k and a bend through angle θ gives a *twist-bend* with parameter $k + i\theta$. It corresponds to applying the loxodromic map $K : z \longmapsto e^{k+i\theta} z$. Doing this process to a hyperbolic surface results in a pleated surface where the bending locus is the geodesic α. The relationship between traces and bending for such surfaces is explored by Parker and Series in [25].

This description can be extended to the case of complex hyperbolic representations of surface groups even though there is no longer a hyperbolic surface. However, the local picture is the same. Namely, a twist will be a hyperbolic translation of distance k along a geodesic α (or $\tilde{\alpha}$) and a bend through angle θ in the plane normal to the complex line containing $\tilde{\alpha}$.

We can describe twist-bends on the level of the fundamental group. This works for all of the cases described above, but we only give details in the case of SU(2,1). There are two cases, (a) when the parts of the surface on either side of α are in different three holed spheres and (b) when they are in the same three holed sphere. In terms group theory, (a) corresponds to a free product with amalgamation and (b) to an HNN extension. The definition of twist bends in each case are similar but not quite the same.

Let Y be a three holed sphere with oriented boundary geodesics α, β, γ and let $\rho_0 : \pi_1 \longmapsto \mathrm{SU}(2,1)$ be a corresponding representation with $\rho_0([\alpha]) = A$, $\rho_0([\beta]) = B$ and $\rho_0([\gamma]) = C$ where $ABC = I$. Suppose that Y' is another such surface. We have α', β', γ', ρ_0', A', B', C' all as above. We must decide when we can attach Y and Y' along α and α'. We can do so when α' has the same length as α but the opposite orientation. Hence we must have A' being conjugate to A^{-1}.

There are two cases to consider. First we must investigate what happens when we attach two distinct three holed spheres along a common boundary; see Section 8.2 of [23]. For our initial configuration, we suppose $A' = A^{-1}$. Attaching Y and Y' is then the same as taking the free product of $\langle A, B\rangle$ and $\langle A', B'\rangle$ with amalgamation along the common subgroup $\langle A\rangle = \langle A'\rangle$. The resulting group is then

$$\langle A, B\rangle *_{\langle A\rangle} \langle A', B'\rangle = \langle A, B, A', B' \mid A' = A^{-1}\rangle = \langle A, B, B'\rangle.$$

In this case a twist-bend consists of fixing the surface corresponding to $\langle A, B\rangle$ and moving the surface corresponding to $\langle A', B'\rangle$ by a hyperbolic translation along the axis of A (the twist) and a rotation around the complex axis of A (the bend). In other words, we take a map K that commutes with $A' = A^{-1}$ and we conjugate $\langle A', B'\rangle$ by K. Thus the new group is

$$\begin{aligned}\langle A, B\rangle *_{\langle A\rangle} K\langle A', B'\rangle K^{-1} &= \langle A, B\rangle *_{\langle A\rangle} \langle KA'K^{-1}, KB'K^{-1}\rangle \\ &= \langle A, B, KA'K^{-1}, KB'K^{-1} \mid KA'K^{-1} = A^{-1}\rangle \\ &= \langle A, B, KB'K^{-1}\rangle.\end{aligned}$$

Note that if we swap the roles of Y and Y' then the same process yields a twist-bend associated to the matrix K^{-1}. That is, the new group is $\langle A', B', K^{-1}BK\rangle$ which is conjugate, via K^{-1}, to $\langle A, B, KB'K^{-1}\rangle$.

Secondly, we must consider the case where we close a handle; see Section 8.3 of [23]. In this case we consider Y and we want to glue two of its boundary components. Suppose one of them is represented by A then the other must be conjugate to A^{-1}, say it is $BA^{-1}B^{-1}$. Note that if A and $BA^{-1}B^{-1}$ correspond to boundary components of the same three holed sphere, this means the third boundary component is $C = (BA^{-1}B^{-1})^{-1}A^{-1} = [B, A]$. Then in order to close the handle we take HNN extension associated to the isomorphism $\phi : \langle A \rangle \longrightarrow \langle BAB^{-1} \rangle$ given by $\phi(A) = BA^{-1}B^{-1}$. It is clear that we may do this by adjoining the stable letter B to obtain

$$\langle A, BA^{-1}B^{-1} \rangle *_\phi = \langle A, BA^{-1}B^{-1}, B \mid BA^{-1}B^{-1} = \phi(A) \rangle = \langle A, B \rangle.$$

Because we want to keep track of the stable letter, we will write this extension as

$$\langle A, BA^{-1}B^{-1} \rangle *_\phi (B)$$

If K is a map that commutes with A then we also have $\phi(A) = (BK)A^{-1}(BK)^{-1}$. Therefore we can take an isomorphic HNN extension by adding the stable letter BK instead of B:

$$\langle A, (BK)A^{-1}(BK)^{-1} \rangle *_\phi (BK) = \langle A, BK \rangle.$$

Thus, in the case of closing a handle performing a complex twist associated to a map K that commutes with A involves changing the stable letter of the HNN extension from B to BK.

In both cases, the geometry of the complex twist is recorded by $\mathrm{tr}(K)$ in exactly the same way that $\mathrm{tr}(A)$ is related to $\ell(\alpha) + i\phi(\alpha)$ as described in Proposition 3.10. Therefore if K corresponds to a twist through distance $k \in \mathbb{R}$ and bend through angle $\theta \in (-\pi, \pi]$ then we have

$$\mathrm{tr}(K) = 2\cosh(k/2)e^{-i\theta/3} + e^{2i\theta/3}.$$

We note that there are subtleties about the direction of twist and the sign of $\Re(\kappa)$; see [23].

We need to find a conjugation invariant way of measuring the twist-bend parameter κ from data associated to the representation of the fundamental group π_1. In [23] this was done using cross-ratios. We will not discuss this here.

5. Traces for triangle groups

5.1. *Introduction*

In this section we consider groups generated by three complex reflections. Such groups are important because of their connection with Deligne-Mostow groups. In [17] Mostow showed that certain groups generated by three complex reflections are non-arithmetic lattices. Later Livné, Mostow, Deligne, Thurston and Deraux constructed more examples of lattices that are generated by three complex reflections. See [20] for a survey of this area, including a discussion of arithmeticity. Recently, Deraux, Parker and Paupert [4] and Thompson [31] have found new candidates for non-arithmetic lattices from triangle groups.

Suppose that Δ is a group generated by three complex reflections in $\mathrm{SU}(2,1)$ all with the same angle. The goal of this section is to give combinatorial formulae for the traces of elements of Δ. These formulae are due to Pratoussevitch [27] generalising earlier work of Sandler [28]. These formulae are useful when deciding arithmeticity and commensurability in the case where Δ is a lattice; see Paupert [26] for more details. In particular, one may use this technique for the groups constructed by Mostow in [17].

5.2. *Reflections*

Consider $\mathbb{R}^{n+1}$ with the standard inner product. Let Π be a hyperplane through the origin and let $\mathbf{n}$ be a normal vector to Π. Thus the orthogonal complement $\Pi^\perp$ of Π is spanned by $\mathbf{n}$. We may decompose a vector $\mathbf{x} \in \mathbb{R}^{n+1}$ into a component in Π and a component in $\Pi^\perp$. Reflection R in Π is obtained by multiplying the component in $\Pi^\perp$ by -1. Specifically,

$$\mathbf{x} = \left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\,\mathbf{n}\right) + \frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\,\mathbf{n},$$

where

$$\left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\,\mathbf{n}\right) \in \Pi, \qquad \frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\,\mathbf{n} \in \Pi^\perp.$$

Then

$$R(\mathbf{x}) = \left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\,\mathbf{n}\right) + (-1)\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\,\mathbf{n} = \mathbf{x} - 2\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\,\mathbf{n}.$$

Then R is given by a matrix in $\mathrm{O}(n+1)$ with determinant -1. The unit sphere S^n in $\mathbb{R}^{n+1}$ is given by all vectors with norm 1. Any reflection in great hypersphere on S^n may be obtained by restricting the above formula to S^n.

We may repeat this construction for Minkowski space $\mathbb{R}^{n,1}$ and (real) hyperbolic n-space. Specifically, consider a non-degenerate bilinear form of signature $(n,1)$ given by the symmetric matrix B:

$$(\mathbf{x},\mathbf{y}) = \mathbf{y}^t B\mathbf{x}.$$

Suppose that Π is a real hyperplane through the origin with space-like normal vector $\mathbf{n}$. That is, for given $\mathbf{n} \in \mathbb{R}^{n,1}$ with $(\mathbf{n},\mathbf{n}) > 0$

$$\Pi = \left\{\mathbf{x} \in \mathbb{R}^{n,1} \ : \ (\mathbf{x},\mathbf{n}) = 0\right\}.$$

Also $\Pi^\perp$ is the line spanned by $\mathbf{n}$. We may generalise the formula for reflection in Π given above. That is we can write any $\mathbf{x}\in\mathbb{R}^{n,1}$ as

$$\mathbf{x}=\left(\mathbf{x}-\frac{(\mathbf{x},\mathbf{n})}{(\mathbf{n},\mathbf{n})}\,\mathbf{n}\right)+\frac{(\mathbf{x},\mathbf{n})}{(\mathbf{n},\mathbf{n})}\,\mathbf{n}.$$

Then

$$R(\mathbf{x})=\left(\mathbf{x}-\frac{(\mathbf{x},\mathbf{n})}{(\mathbf{n},\mathbf{n})}\,\mathbf{n}\right)+(-1)\frac{(\mathbf{x},\mathbf{n})}{(\mathbf{n},\mathbf{n})}\,\mathbf{n}=\mathbf{x}-2\frac{(\mathbf{x},\mathbf{n})}{(\mathbf{n},\mathbf{n})}\,\mathbf{n}.$$

This is represented by a matrix in $\mathrm{O}(n,1)$ with determinant -1. The hyperboloid model of (real) hyperbolic n-space is given by

$$\left\{\mathbf{x}\in\mathbb{R}^{n,1}\ :\ (\mathbf{x},\mathbf{x})=-1,\ x_{n+1}>0\right\}.$$

Then R maps this hyperboloid to itself. Alternatively, we may want to obtain the Klein-Beltrami model of $\mathbf{H}^n_{\mathbb{R}}$ by taking a projective action on the points with $x_{n+1}=1$.

We want to generalise this whole setup from the real world to the complex world. To do so, consider $\mathbb{C}^{n+1}$ with a Hermitian form H, which we assume to be non-degenerate but, at this stage, without restrictions on its signature. We use angle brackets to denote the Hermitian form associated to H as usual: $\mathbf{w}^*H\mathbf{z}=\langle\mathbf{z},\mathbf{w}\rangle$ for $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{C}^{n+1}$. Since we are interested in complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ we mainly think of the case where the form H has signature $(n,1)$ but this is not necessary for the definition of complex reflections.

Suppose that Π is a complex hyperplane in $\mathbb{C}^{n+1}$. That is $\Pi^\perp$ is spanned by $\mathbf{n}\in\mathbb{C}^{n+1}$ and so we have $\Pi=\left\{\mathbf{z}\in\mathbb{C}^{n+1}\ :\ \langle\mathbf{z},\mathbf{n}\rangle=0\right\}$. Any $\mathbf{z}\in\mathbb{C}^{n+1}$ may then be decomposed into components in Π and $\Pi^\perp$:

$$\mathbf{z}=\left(\mathbf{z}-\frac{\langle\mathbf{z},\mathbf{n}\rangle}{\langle\mathbf{n},\mathbf{n}\rangle}\,\mathbf{n}\right)+\frac{\langle\mathbf{z},\mathbf{n}\rangle}{\langle\mathbf{n},\mathbf{n}\rangle}\,\mathbf{n},$$

where

$$\left(\mathbf{z}-\frac{\langle\mathbf{z},\mathbf{n}\rangle}{\langle\mathbf{n},\mathbf{n}\rangle}\,\mathbf{n}\right)\in\Pi,\qquad\frac{\langle\mathbf{z},\mathbf{n}\rangle}{\langle\mathbf{n}\cdot\mathbf{n}\rangle}\,\mathbf{n}\in\Pi^\perp.$$

At this point we come to most crucial difference between real and complex reflections. Once again a complex reflection will preserve the decomposition of $\mathbf{z}$ into components in Π and $\Pi^\perp$, will pointwise fix the component in Π and will preserve the norm of $\mathbf{z}$. However, since $\Pi^\perp$ is a complex line we have greater freedom than we did before: we may multiply the component in

$\Pi^\perp$ by any complex number with modulus 1. Hence we define *the reflection in* Π *with angle* ψ to be the map $R(\mathbf{z})$ given by

$$R(\mathbf{z}) = \left(\mathbf{z} - \frac{\langle \mathbf{z}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle}\mathbf{n}\right) + e^{i\psi}\frac{\langle \mathbf{z}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle}\mathbf{n} = \mathbf{z} + (e^{i\psi} - 1)\frac{\langle \mathbf{z}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle}\mathbf{n}. \tag{5.1}$$

The map R is given by a matrix in $U(H)$ with n eigenvalues $+1$ and 1 eigenvalue $e^{i\psi}$. Hence its determinant is $e^{i\psi}$. In order to obtain a map in $\mathrm{SU}(H)$ we must multiply this matrix by $e^{-i\psi/(n+1)}$.

In what follows, we will mainly be interested in the case where $n = 2$, H has signature $(2, 1)$ and $\mathbf{n} \in V_+$. This means that, in terms of its action of $\mathbf{H}^2_{\mathbb{C}}$, the reflection R fixes a complex line $L = \mathbb{P}\Pi \cap \mathbf{H}^2_{\mathbb{C}}$. However, it will be useful to consider the space of groups generated by three complex reflections (all with the same angle) for a Hermitian form H and then consider the subspace where H has the correct signature.

We now give two examples in signature $(1, 1)$ that show that in this case complex reflections are just rotations, that is elliptic matrices.

Example 5.1:

(i) Consider $\mathbb{C}^{1,1}$ where the Hermitian form is given by H_0, as given in (2.1). Let Π be the complex line in $\mathbb{C}^{1,1}$ with polar vector $\mathbf{n}$ where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then using (5.1) the reflection in Π with angle ψ is given by

$$\begin{aligned} R(z) &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + (e^{i\psi} - 1)\frac{z_1}{1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{i\psi} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \end{aligned}$$

The matrix in the last line is in $U(1, 1)$. In order to obtain a matrix in $\mathrm{SU}(1, 1)$ we must multiply by $e^{-i\psi/2}$ to get

$$R = \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}.$$

(ii) Consider $\mathbb{C}^{1,1}$ where the Hermitian form is given by H_0'. Let Π be the complex line in $\mathbb{C}^{1,1}$ with polar vector $\mathbf{n}$ where

$$\mathbf{n} = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Then $\langle \mathbf{n}, \mathbf{n} \rangle = 2$. Then, arguing as before, the reflection in Π with angle ψ is

$$\begin{aligned} R(z) &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + (e^{i\psi} - 1)\frac{iz_1 + z_2}{2}\begin{pmatrix} -i \\ 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (e^{i\psi}+1)z_1 - i(e^{i\psi}-1)z_2 \\ i(e^{i\psi}-1)z_1 + (e^{i\psi}+1)z_2 \end{pmatrix} \\ &= \begin{pmatrix} e^{i\psi/2}\cos(\psi/2) & e^{i\psi/2}\sin(\psi) \\ -e^{i\psi/2}\sin(\psi/2) & e^{i\psi/2}\cos(\psi/2) \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \end{aligned}$$

In order to obtain a matrix with determinant 1 we must multiply by $e^{-i\psi/2}$ to get

$$R = \begin{pmatrix} \cos(\psi/2) & \sin(\psi/2) \\ -\sin(\psi/2) & \cos(\psi/2) \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

5.3. *Complex reflections in* SU(2, 1)

We now give more details for the case we are most interested in, namely where $n = 2$ and H has signature $(2, 1)$. Let Π be a complex hyperplane in $\mathbb{C}^{2,1}$ with normal vector $\mathbf{n} \in \mathbb{C}^{2,1}$ with $\langle \mathbf{n}, \mathbf{n} \rangle > 0$. The complex reflection with angle ψ fixing Π is given by (5.1). In order that R is represented by a matrix in $\mathrm{SU}(2, 1)$, we multiply this formula by $e^{-i\psi/3}$. This gives

$$R(\mathbf{z}) = e^{-i\psi/3}\left(\mathbf{z} + (e^{i\psi} - 1)\frac{\langle \mathbf{z}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle}\,\mathbf{n}\right) = e^{-i\psi/3}\mathbf{z} + (e^{2i\psi/3} - e^{-\psi/3})\frac{\langle \mathbf{z}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle}\,\mathbf{n}. \tag{5.2}$$

If R is as above, for any $A \in \mathrm{SU}(2, 1)$ we now relate $\mathrm{tr}(RA)$ and $\mathrm{tr}(A)$.

Lemma 5.2: *Let R be complex reflection in the hyperplane orthogonal to $\mathbf{n}$ with angle ψ given by (5.2). Let A be any element of* $\mathrm{SU}(2, 1)$. *Then*

$$\mathrm{tr}(RA) = e^{-i\psi/3}\mathrm{tr}(A) + (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle A\mathbf{n}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle}.$$

Proof: We have

$$\begin{aligned} RA(\mathbf{z}) &= e^{-i\psi/3}A\mathbf{z} + \frac{(e^{2i\psi/3} - e^{-\psi/3})}{\langle \mathbf{n}, \mathbf{n} \rangle}\mathbf{n}\langle A\mathbf{z}, \mathbf{n} \rangle \\ &= e^{-i\psi/3}A\mathbf{z} + \frac{(e^{2i\psi/3} - e^{-\psi/3})}{\langle \mathbf{n}, \mathbf{n} \rangle}\mathbf{n}\mathbf{n}^*HA\mathbf{z}. \end{aligned}$$

Therefore, the matrix of RA is

$$e^{-i\psi/3}A + \frac{(e^{2i\psi/3} - e^{-\psi/3})}{\langle \mathbf{n}, \mathbf{n} \rangle}\mathbf{n}\mathbf{n}^*HA.$$

Now if a matrix can be written in the form $\mathbf{u}\mathbf{v}^*$ for column vectors $\mathbf{u}$ and $\mathbf{v}$, then its trace is just $\mathbf{v}^*\mathbf{u}$. Thus

$$\mathrm{tr}(\mathbf{n}\mathbf{n}^*HA) = \mathrm{tr}\big(\mathbf{n}(A^*H\mathbf{n})^*\big) = (A^*H\mathbf{n})^*\mathbf{n} = \mathbf{n}^*HA\mathbf{n} = \langle A(\mathbf{n}), \mathbf{n}\rangle.$$

Hence

$$\begin{aligned}\mathrm{tr}(RA) &= e^{-i\psi/3}\mathrm{tr}(A) + \frac{(e^{2i\psi/3} - e^{-\psi/3})}{\langle \mathbf{n}, \mathbf{n}\rangle}\mathrm{tr}(\mathbf{n}\mathbf{n}^*H) \\ &= e^{-i\psi/3}\mathrm{tr}(A) + (e^{2i\psi/3} - e^{-\psi/3})\frac{\langle A\mathbf{n}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle}. \qquad \square\end{aligned}$$

Setting A to be the identity matrix, we see the fact (which we already knew from our consideration of eigenvalues, Proposition 3.7) that $\mathrm{tr}(R) = 3e^{-i\psi/3} + (e^{2i\psi/3} - e^{-i\psi/3}) = e^{2i\psi/3} + 2e^{-i\psi/3}$.

5.4. *Equilateral triangle groups*

Suppose that we are given three complex lines L_1, L_2 and L_3 in $\mathbf{H}^2_{\mathbb{C}}$. These correspond to hyperplanes Π_1, Π_2 and Π_3 in $\mathbb{C}^{2,1}$ with normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ with $\langle \mathbf{n}_j, \mathbf{n}_j\rangle > 0$. For $j = 1, 2, 3$, consider complex reflections R_j with angle ψ about complex lines with polar vectors $\mathbf{n}_j$. Using (5.2) that is

$$R_j(\mathbf{z}) = e^{-i\psi/3}\mathbf{z} + (e^{2i\psi/3} - e^{-\psi/3})\frac{\langle \mathbf{z}, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}\,\mathbf{n}_j. \tag{5.3}$$

Note that this formula is preserved if $\mathbf{n}_j$ is sent to $\lambda\mathbf{n}_j$ for any $\lambda \in \mathbb{C} - \{0\}$.

Suppose first that the three complex lines L_1, L_2, L_3 form an equilateral triangle. That is, there is a J map of order 3 cyclically permuting them. In other words $J \in \mathrm{SU}(2,1)$ satisfies $\Pi_2 = J\Pi_1$, $\Pi_3 = J\Pi_2 = J^{-1}\Pi_1$ and $\mathbf{n}_2 = J\mathbf{n}_1$, $\mathbf{n}_3 = J\mathbf{n}_2 = J^{-1}\mathbf{n}_1$. Thus

$$\langle \mathbf{n}_1, \mathbf{n}_1\rangle = \langle \mathbf{n}_2, \mathbf{n}_2\rangle = \langle \mathbf{n}_3, \mathbf{n}_3\rangle, \quad \langle \mathbf{n}_2, \mathbf{n}_1\rangle = \langle \mathbf{n}_3, \mathbf{n}_2\rangle = \langle \mathbf{n}_1, \mathbf{n}_3\rangle.$$

Note that if ω is a cube root of unity, all these formulae remain valid if, for $j = 1, 2, 3$, we send $\mathbf{n}_j$ to $\omega^j\mathbf{n}_j$.

The map J will have eigenvalues 1, ω and $\overline{\omega}$ and so $\mathrm{tr}(J) = 0$. Using this fact, the following result is an easy corollary of Lemma 5.2.

Lemma 5.3: *Let R be a complex reflection with angle ψ fixing a complex line L with polar vector $\mathbf{n}$. Let $J \in \mathrm{SU}(2,1)$ be a regular elliptic map of order 3. Then*

$$\mathrm{tr}(RJ) = (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle J\mathbf{n}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle}.$$

Motivated by Lemma 5.3, we define the variable τ to be this trace (where indices are taken cyclically):

$$\tau = \mathrm{tr}(R_j J) = (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle J\mathbf{n}_j, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle} = (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle \mathbf{n}_{j+1}, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}. \tag{5.4}$$

Sending $\mathbf{n}_j$ to $\omega^j\mathbf{n}_j$ means that τ is multiplied by ω. Therefore given R_1, R_2 and R_3 the map τ is only defined up to multiplication by a cube root of unity.

Furthermore, following Section 3.3.2 of Goldman [8], if L_j and L_{j+1} meet with angle θ (by symmetry this is the same for all three pairs of lines) then

$$\cos(\theta) = \frac{\left|\langle \mathbf{n}_{j+1}, \mathbf{n}_j\rangle\right|}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle} = \frac{|\tau|}{2\sin(\psi/2)}. \tag{5.5}$$

This shows that once again the traces lead to geometrical information about the group.

All of this has been defined without reference to any particular Hermitian form. Following Mostow [17], we choose $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ to be the standard basis vectors of $\mathbb{C}^{2,1}$; see Parker and Paupert [22]. Thus

$$\mathbf{n}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \mathbf{n}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \mathbf{n}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}. \tag{5.6}$$

An immediate consequence is that the permuting matrix is

$$J = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \tag{5.7}$$

Sending $\mathbf{n}_j$ to $\omega^j\mathbf{n}_j$ means that J is multiplied by ω. Therefore given complex reflections R_1, R_2, R_3 the symmetry map J is only defined up to multiplication by a cube root of unity.

The matrix defining the Hermitian form is then (a multiple of) the Gram matrix. This makes sense only when these three vectors are linearly independent or equivalently the group generated by R_1, R_2 and R_3 does not preserve any lower dimensional complex subspace, that is it is Zariski dense (see Remark 10 of Will [34]). Moreover, (5.6) determines the Hermitian form H up to a real multiple. In order to avoid denominators, we choose

$$\langle \mathbf{n}_j, \mathbf{n}_j\rangle = |e^{2i\psi/3} - e^{-i\psi/3}|^2 = 2 - e^{i\psi} - e^{-i\psi}.$$

This means that

$$\langle \mathbf{n}_{j+1}, \mathbf{n}_j \rangle = \frac{\langle \mathbf{n}_j, \mathbf{n}_j \rangle}{e^{2i\psi/3} - e^{-i\psi/3}}\, \tau = (e^{-2i\psi/3} - e^{i\psi/3})\tau,$$

and so

$$H = \begin{bmatrix} 2 - e^{i\psi} - e^{-i\psi} & (e^{-2i\psi/3} - e^{i\psi/3})\tau & (e^{2i\psi/3} - e^{-i\psi/3})\overline{\tau} \\ (e^{2i\psi/3} - e^{-i\psi/3})\overline{\tau} & 2 - e^{i\psi} - e^{-i\psi} & (e^{-2i\psi/3} - e^{i\psi/3})\tau \\ (e^{-2i\psi/3} - e^{i\psi/3})\tau & (e^{2i\psi/3} - e^{-i\psi/3})\overline{\tau} & 2 - e^{i\psi} - e^{-i\psi} \end{bmatrix}. \quad (5.8)$$

As we indicated in Section 5.2 the construction of R_1, R_2 and R_2 in terms of $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ works whatever the signature of H. It is only when H has signature $(2,1)$ that these reflections act on complex hyperbolic space as complex reflections in complex lines L_1, L_2 and L_3. We now discuss the geometry when H has other signatures.

Since the trace of H is positive, we see that it must have at least one positive eigenvalue. If H has signature $(3,0)$ then our triangle lies on $\mathbb{CP}^2$ and $\langle R_1, R_2, R_3 \rangle$ is a subgroup of SU(3). If H has signature $(2,1)$, which is the case we are interested in, then $\langle R_1, R_2, R_3 \rangle$ is generated by reflections in complex lines in complex hyperbolic space. If H has signature $(1,2)$ then $\langle R_1, R_2, R_3 \rangle$ is generated by reflections in points in complex hyperbolic space, generalising Theorem 6.8.12 of Davis [3]. We now give a criterion for determining when H has signature $(2,1)$.

Lemma 5.4: *The signature of the matrix H given by (5.8) is $(2,1)$ if and only if*

$$\begin{aligned} 0 < 3(2 - e^{i\psi} - e^{-i\psi})|\tau|^2 - (1 - e^{-i\psi})\tau^3 - (1 - e^{i\psi})\overline{\tau}^3 \\ -(2 - e^{i\psi} - e^{-i\psi})^2. \end{aligned}$$

Proof: We must find when H has two eigenvalues that are positive and one that is negative. Since the sum of the eigenvalues of H is

$$\mathrm{tr}(H) = 3(2 - e^{i\psi} - e^{-i\psi}) > 0,$$

it is easy to see that all three eigenvalues cannot be negative simultaneously. This means we only need to check when H has negative determinant. Hence

$$\begin{aligned} 0 &> \det(H) \\ &= (2 - e^{i\psi} - e^{-i\psi})^3 + (e^{-2i\psi/3} - e^{i\psi/3})^3\tau^3 + (e^{2i\psi/3} - e^{-i\psi/3})^3\overline{\tau}^3 \\ &\quad -3(2 - e^{i\psi} - e^{-i\psi})(e^{-2i\psi/3} - e^{i\psi/3})(e^{2i\psi/3} - e^{-i\psi/3})^3|\tau|^2 \\ &= (2 - e^{i\psi} - e^{-i\psi})^3 \\ &\quad +(2 - e^{i\psi} - e^{-i\psi})(1 - e^{-i\psi})\tau^3 + (2 - e^{i\psi} - e^{-i\psi})(1 - e^{i\psi})\overline{\tau}^3 \\ &\quad -3(2 - e^{i\psi} - e^{-i\psi})^2|\tau|^2. \end{aligned}$$

The result follows since $2 - e^{i\psi} - e^{-i\psi} > 0$ □

Corollary 5.5: *Suppose that the matrix H given by (5.8) has signature $(2,1)$. For $j = 1, 2, 3$ let $\mathbf{n}_j$ be given by (5.6) and let L_j be the complex line with polar vector $\mathbf{n}_j$. If L_j and L_{j+1} intersect in $\mathbf{H}^2_{\mathbb{C}}$ then they meet at an angle of less than $\pi/3$.*

Proof: Since H has signature $(2,1)$ Lemma 5.4 implies

$$\begin{aligned} 0 &< 3(2 - e^{i\psi} - e^{-i\psi})|\tau|^2 - (1 - e^{-i\psi})\tau^3 - (1 - e^{i\psi})\overline{\tau}^3 \\ &\quad -(2 - e^{i\psi} - e^{-i\psi})^2 \\ &\le 4\sin(\psi/2)|\tau|^3 + 12\sin^2(\psi/2)|\tau|^2 - 16\sin^4(\psi/2) \\ &= 4\sin(\psi/2)\big(|\tau| - \sin(\psi/2)\big)\big(|\tau| + 2\sin(\psi/2)\big)^2. \end{aligned}$$

This implies that $|\tau| > \sin(\psi/2)$. Note that the converse of this inequality is not necessarily true, since in the second line we used $2\mathrm{Re}\big(-(1 - e^{-i\psi})\tau^3\big) \le 2|1 - e^{-i\psi}|\,|\tau^3| = 4\sin(\psi/2)|\tau|^3$.

If L_j and L_{j+1} intersect in $\mathbf{H}^2_{\mathbb{C}}$ then, from (5.5), the angle θ between L_j and L_{j+1} is given by:

$$\cos(\theta) = \frac{\big|\langle \mathbf{n}_{j+1}, \mathbf{n}_j\rangle\big|}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle} = \frac{|\tau|}{2\sin(\psi/2)} > \frac{1}{2}.$$

Therefore $\theta < \pi/3$ as claimed. □

Using H and the formula (5.3) we have

$$R_1 = \begin{bmatrix} e^{2i\psi/3} & \tau & -e^{i\psi/3}\overline{\tau} \\ 0 & e^{-i\psi/3} & 0 \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix}, \tag{5.9}$$

$$R_2 = JR_1J^{-1} = \begin{bmatrix} e^{-i\psi/3} & 0 & 0 \\ -e^{i\psi/3}\overline{\tau} & e^{2i\psi/3} & \tau \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix}, \tag{5.10}$$

$$R_3 = J^{-1}R_1J = \begin{bmatrix} e^{-i\psi/3} & 0 & 0 \\ 0 & e^{-i\psi/3} & 0 \\ \tau & -e^{i\psi/3}\overline{\tau} & e^{2i\psi/3} \end{bmatrix}. \tag{5.11}$$

The parameter τ completely determines the group $\langle R_1, R_2, R_3\rangle$, or equivalently the normal subgroup $\langle R_1, J\rangle$, up to conjugation. Therefore, in principle, the trace of any element of $\langle R_1, R_2, R_3\rangle$ may be given as a function of τ. Pratoussevitch [27] has given some beautiful combinatorial formulae for these traces. This generalises earlier work of Sandler [28] for the case where $\psi/3 = \pi$ and so R_j has order 2. With a little more work, Pratoussevitch gives similar formulae for the non-equilateral case. We derive these formulae in the Section 5.6.

5.5. *General triangle groups*

In this section we consider three complex lines in general position and the group generated by complex reflections of angle ψ in their sides. Let L_1, L_2 and L_3 be three complex lines in $\mathbf{H}^2_{\mathbb{C}}$ with normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. Suppose that $\langle \mathbf{n}_1, \mathbf{n}_1 \rangle = \langle \mathbf{n}_2, \mathbf{n}_2 \rangle = \langle \mathbf{n}_3, \mathbf{n}_3 \rangle > 0$. Define

$$\rho = (e^{2i\psi/3} - e^{-i\psi/3}) \frac{\langle \mathbf{n}_2, \mathbf{n}_1 \rangle}{\langle \mathbf{n}_1, \mathbf{n}_1 \rangle},$$
$$\sigma = (e^{2i\psi/3} - e^{-i\psi/3}) \frac{\langle \mathbf{n}_3, \mathbf{n}_2 \rangle}{\langle \mathbf{n}_2, \mathbf{n}_2 \rangle},$$
$$\tau = (e^{2i\psi/3} - e^{-i\psi/3}) \frac{\langle \mathbf{n}_1, \mathbf{n}_3 \rangle}{\langle \mathbf{n}_3, \mathbf{n}_3 \rangle}.$$

These formulae generalise (5.4) but now, since we no longer have the symmetry J, they are not the trace of any group elements. Using Proposition 5.10 below, we will be able to relate them to other traces.

As before, using Section 3.3.2 of Goldman [8], if L_j and L_k meet with angle θ_{jk} then

$$\cos(\theta_{12}) = \frac{|\rho|}{2\sin(\psi/2)}, \quad \cos(\theta_{23}) = \frac{|\sigma|}{2\sin(\psi/2)}, \quad \cos(\theta_{31}) = \frac{|\tau|}{2\sin(\psi/2)}. \tag{5.12}$$

We can use ρ, σ, τ to define a Hermitian form. Once again we normalise so that $\langle \mathbf{n}_j, \mathbf{n}_j \rangle = 2 - e^{i\psi} - e^{-i\psi} > 0$. Then

$$H = \begin{bmatrix} 2 - e^{i\psi} - e^{-i\psi} & (e^{-2i\psi/3} - e^{i\psi/3})\rho & (e^{2i\psi/3} - e^{-i\psi/3})\overline{\tau} \\ (e^{2i\psi/3} - e^{-i\psi/3})\overline{\rho} & 2 - e^{i\psi} - e^{-i\psi} & (e^{-2i\psi/3} - e^{i\psi/3})\sigma \\ (e^{-2i\psi/3} - e^{i\psi/3})\tau & (e^{2i\psi/3} - e^{-i\psi/3})\overline{\sigma} & 2 - e^{i\psi} - e^{-i\psi} \end{bmatrix}. \tag{5.13}$$

We require that H should have signature $(2,1)$. Since its trace is positive, the same argument we used before shows that this is equivalent to $\det(H) < 0$. Doing so and then arguing in a similar fashion to the proof of Lemma 5.4, we find:

Lemma 5.6: *The matrix H given by* (5.13) *has signature* $(2,1)$ *if and only* *if:*

$$0 < (2 - e^{i\psi} - e^{-i\psi})\big(|\rho|^2 + |\sigma|^2 + |\tau|^2\big) \\ -(1 - e^{-i\psi})\rho\sigma\tau - (1 - e^{i\psi})\overline{\rho}\,\overline{\sigma}\,\overline{\tau} - (2 - e^{i\psi} - e^{-i\psi})^2.$$

This criterion is equivalent to the one given by Pratoussevitch in Proposition 1 of [27]. A simple geometric consequence of this inequality, generalising Corollary 5.5, is:

Corollary 5.7: *The angles θ_{jk} from* (5.12) *satisfy $\theta_{12} + \theta_{23} + \theta_{31} < \pi$.*

Proof: Using the inequality from Lemma 5.6 we find:

$$\begin{aligned} 0 &< (2 - e^{i\psi} - e^{-i\psi})\big(|\rho|^2 + |\sigma|^2 + |\tau|^2\big) \\ &\quad -(1 - e^{-i\psi})\rho\sigma\tau - (1 - e^{i\psi})\overline{\rho}\,\overline{\sigma}\,\overline{\tau} - (2 - e^{i\psi} - e^{-i\psi})^2 \\ &\le 4\sin^2(\psi/2)\big(|\rho|^2 + |\sigma|^2 + |\tau|^2\big) \\ &\quad +4\sin(\psi/2)|\rho||\sigma||\tau| - 16\sin^4(\psi/2) \\ &= 16\sin^4(\psi/2)\big(\cos^2(\theta_{12}) + \cos^2(\theta_{23}) + \cos^2(\theta_{31})\big) \\ &\quad +32\sin^4(\psi/2)\cos(\theta_{12})\cos(\theta_{23})\cos(\theta_{31}) - 16\sin^4(\psi/2) \\ &= 16\sin^4(\psi/2)\big(\cos(\theta_{12})\cos(\theta_{23}) + \cos(\theta_{31})\big)^2 \\ &\quad -16\sin^4(\psi/2)\sin^2(\theta_{12})\sin^2(\theta_{23}) \\ &= 16\sin^4(\psi/2)\big(\cos(\theta_{12} - \theta_{23}) + \cos(\theta_{31})\big)\big(\cos(\theta_{12} + \theta_{23}) + \cos(\theta_{31})\big). \end{aligned}$$

Since $\theta_{jk} \in (0, \pi/2)$ we see that $\cos(\theta_{12} - \theta_{23})$ and $\cos(\theta_{31})$ are both positive. Thus we must have

$$\cos(\theta_{31}) > -\cos(\theta_{12} + \theta_{23}) = \cos(\pi - \theta_{12} - \theta_{23}).$$

Hence $\theta_{31} < \pi - \theta_{12} - \theta_{23}$ as required. □

Matrices for the reflections R_1, R_2, R_3 can the be obtained by using H in the formula (5.3):

$$R_1 = \begin{bmatrix} e^{2i\psi/3} & \rho & -e^{i\psi/3}\overline{\tau} \\ 0 & e^{-i\psi/3} & 0 \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix}, \tag{5.14}$$

$$R_2 = \begin{bmatrix} e^{-i\psi/3} & 0 & 0 \\ -e^{i\psi/3}\overline{\rho} & e^{2i\psi/3} & \sigma \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix}, \tag{5.15}$$

$$R_3 = \begin{bmatrix} e^{-i\psi/3} & 0 & 0 \\ 0 & e^{-i\psi/3} & 0 \\ \tau & -e^{i\psi/3}\overline{\sigma} & e^{2i\psi/3} \end{bmatrix}. \tag{5.16}$$

5.6. *Traces in general triangle groups*

Let R_1, R_2 and R_3 be as in (5.14), (5.15) and (5.16). We will be interested in finding a formula for the trace of each element of $\Delta = \langle R_1,\, R_2,\, R_3\rangle$, written as a word in R_1, R_2, R_3 and their inverses. Since cyclic permutation does not affect the trace it will be easier for us to consider cyclic words. Consider an element $R_{a_1}^{\epsilon_1}\ldots R_{a_r}^{\epsilon_r}$ of $\langle R_1, R_2, R_3\rangle$ where $a_j \in \{1,2,3\}$ and $\epsilon_j \in \{1,-1\}$. For ease of notation, we make the canonical identification between this word and the sequences $a = (a_1 \ldots a_r)$ and $\epsilon = (\epsilon_1 \ldots \epsilon_r)$. We shall regard these indices as being defined cyclically, that is $a_{r+1} = a_1$ and $\epsilon_{r+1} = \epsilon_1$.

We need to introduce some notation. For the sequence $a = (a_1 \ldots a_r)$ as above and for $j = 1, 2, 3$ taken cyclically (so when $j = 3$ we have $j+1 = 1$) we define

$$z_j(a) = \#\Big\{k \in \{1, \ldots, r\} \;:\; a_{k+1} = a_k = j\Big\}, \tag{5.17}$$

$$p_j(a) = \#\Big\{k \in \{1, \ldots, r\} \;:\; a_{k+1} = j+1,\; a_k = j\Big\}, \tag{5.18}$$

$$n_j(a) = \#\Big\{k \in \{1, \ldots, r\} \;:\; a_{k+1} = j,\; a_k = j+1\Big\}. \tag{5.19}$$

It is easy to see that

$$\#\Big\{k \in \{1, \ldots, r\} \;:\; a_k = j\Big\} = z_j(a) + p_j(a) + n_{j-1}(a),$$

$$\#\Big\{k \in \{1, \ldots, r\} \;:\; a_{k+1} = j\Big\} = z_j(a) + p_{j-1}(a) + n_j(a).$$

By relabelling the sequence a, it is clear that these numbers must be the same. That is $z_j(a) + p_j(a) + n_{j-1}(a) = z_j(a) + p_{j-1}(a) + n_j(a)$. Therefore we have

$$p_1(a) - n_1(a) = p_2(a) - n_2(a) = p_3(a) - n_3(a).$$

Following Sandler [28], see also Pratoussevitch [27], we define the *winding number* $w(a)$ of the sequence $a = (a_1 \ldots a_r)$ to be

$$w(a) = p_j(a) - n_j(a). \tag{5.20}$$

Similarly, for $\epsilon = (\epsilon_1 \ldots \epsilon_r)$ define

$$m_+(\epsilon) = \#\Big\{k \in \{1, \ldots, r\} \;:\; \epsilon_k = 1\Big\}, \tag{5.21}$$

$$m_-(\epsilon) = \#\Big\{k \in \{1, \ldots, r\} \;:\; \epsilon_k = -1\Big\}. \tag{5.22}$$

We now give the main result for computing traces which is due to Pratoussevitch, see Theorems 4 and 10 of [27].

Proposition 5.8: *Let $a = (a_1 \dots a_r)$ be a cyclic word with $a_k \in \{1, 2, 3\}$. Let $\epsilon = (\epsilon_1 \dots \epsilon_r)$ with $\epsilon_k \in \{1, -1\}$. Let $E = \sum_{j=1}^{n} \epsilon_j$. Then*

$$\begin{aligned}
&\mathrm{tr}(R_{a_1}^{\epsilon_1} \cdots R_{a_r}^{\epsilon_r}) \\
&= (e^{i\psi})^{-E/3} \left(3 + \sum_S \frac{(e^{i\psi}-1)^z(-e^{i\psi})^n(e^{i\psi})^w}{(-e^{i\psi})^{m_-}} \rho^{p_1}\overline{\rho}^{n_1}\sigma^{p_2}\overline{\sigma}^{n_2}\tau^{p_3}\overline{\tau}^{n_3} \right)
\end{aligned}$$

where the sum is taken over all non-empty subsets $S = \{k_1, \dots, k_m\}$ of the set $\{1, \dots, r\}$. Such a subset determines a subset $a_S = (a_{k_1} \dots a_{k_m})$ of a and $\epsilon_S = (\epsilon_{k_1} \dots \epsilon_{k_m})$ of ϵ. The numbers p_j, n_j, $w = p_j - n_j$, $z = z_1 + z_2 + z_3$, $n = n_1 + n_2 + n_3$ are determined from a_S by (5.17), (5.18), (5.19) and (5.20). Finally, m_- is determined from ϵ_S by (5.22).

Proof: Let $S = \{k_1, \dots, k_m\}$ be a non-empty subset of $\{1, \dots, r\}$ and denote the corresponding subsets of a and ϵ by $a_S = (a_{k_1} \dots a_{k_m})$ and $\epsilon_S = (\epsilon_{k_1} \dots \epsilon_{k_m})$. Write $a_{k_l} = b_l$ for $l = 1, \dots, m$.

Using the expression for R_j given in equation (5.3), we have

$$\begin{aligned}
e^{i\psi/3}R_j\mathbf{z} &= \mathbf{z} + (e^{i\psi}-1)\frac{\langle \mathbf{z}, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}\mathbf{n}_j = \left(I + (e^{i\psi}-1)\frac{\mathbf{n}_j\mathbf{n}_j^*H}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle} \right)\mathbf{z}, \\
e^{-i\psi/3}R_j^{-1}\mathbf{z} &= \mathbf{z} + (e^{-i\psi}-1)\frac{\langle \mathbf{z}, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle}\mathbf{n}_j = \left(I + (e^{-i\psi}-1)\frac{\mathbf{n}_j\mathbf{n}_j^*H}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle} \right)\mathbf{z}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(e^{i\psi/3})^{\epsilon_1}R_{a_1}^{\epsilon_1} \cdots (e^{i\psi/3})^{\epsilon_r}R_{a_r}^{\epsilon_r} \\
&= \left(I + (e^{\epsilon_1 i\psi}-1)\frac{\mathbf{n}_{a_1}\mathbf{n}_{a_1}^*H}{\langle \mathbf{n}_{a_1}, \mathbf{n}_{a_1}\rangle} \right) \cdots \left(I + (e^{\epsilon_r i\psi}-1)\frac{\mathbf{n}_{a_r}\mathbf{n}_{a_r}^*H}{\langle \mathbf{n}_{a_r}, \mathbf{n}_{a_r}\rangle} \right) \\
&= I + \sum_{S\neq\emptyset} (e^{i\psi}-1)^{m_+}(e^{-i\psi}-1)^{m_-}\frac{\mathbf{n}_{b_1}\mathbf{n}_{b_1}^*H\mathbf{n}_{b_2}\cdots\mathbf{n}_{b_{m-1}}^*H\mathbf{n}_{b_m}\mathbf{n}_{b_m}^*H}{\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_1}\rangle\cdots\langle \mathbf{n}_{b_m}, \mathbf{n}_{b_m}\rangle} \\
&= I + \sum_{S\neq\emptyset} \frac{(e^{i\psi}-1)^{|S|}}{(-e^{i\psi})^{m_-}}\frac{\mathbf{n}_{b_1}\langle \mathbf{n}_{b_2}, \mathbf{n}_{b_1}\rangle\cdots\langle \mathbf{n}_{b_m}, \mathbf{n}_{b_{m-1}}\rangle\mathbf{n}_{b_m}^*H}{\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_1}\rangle\cdots\langle \mathbf{n}_{b_m}, \mathbf{n}_{b_m}\rangle}.
\end{aligned}$$

We can gather together the powers of $e^{i\psi}$ on the left hand side to obtain $(e^{i\psi/3})^E = (e^{i\psi})^{E/3}$. Arguing as in the proof of Lemma 5.2, we have

$\mathrm{tr}\big(\mathbf{n}_{b_1}\mathbf{n}_{b_m}^* H\big) = \langle \mathbf{n}_{b_1}, \mathbf{n}_{b_m}\rangle$. Hence

$$\begin{aligned}
&(e^{i\psi})^{E/3}\mathrm{tr}(R_{a_1}^{\epsilon_1}\cdots R_{a_r}^{\epsilon_r}) \\
&= 3 + \sum_{S\neq\emptyset} \frac{(e^{i\psi}-1)^{|S|}}{(-e^{i\psi})^{m_-}} \frac{\langle \mathbf{n}_{b_2}, \mathbf{n}_{b_1}\rangle}{\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_1}\rangle}\cdots \frac{\langle \mathbf{n}_{b_m}, \mathbf{n}_{b_{m-1}}\rangle}{\langle \mathbf{n}_{b_{m-1}}, \mathbf{n}_{b_{m-1}}\rangle}\cdot\frac{\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_m}\rangle}{\langle \mathbf{n}_{b_m}, \mathbf{n}_{b_m}\rangle} \\
&= 3 + \sum_{S\neq\emptyset} (-e^{-i\psi})^{m_-}\frac{(e^{i\psi}-1)\langle \mathbf{n}_{b_2}, \mathbf{n}_{b_1}\rangle}{\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_1}\rangle}\cdots\frac{(e^{i\psi}-1)\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_m}\rangle}{\langle \mathbf{n}_{b_m}, \mathbf{n}_{b_m}\rangle}.
\end{aligned}$$

From the definitions of ρ, σ and τ we have

$$\frac{(e^{i\psi}-1)\langle \mathbf{n}_{b_{k+1}}, \mathbf{n}_{b_k}\rangle}{\langle \mathbf{n}_{b_k}, \mathbf{n}_{b_k}\rangle} = \begin{cases} e^{i\psi}-1 & \text{if } b_{k+1}=b_k; \\ e^{i\psi/3}\rho & \text{if } b_{k+1}=2,\ b_k=1; \\ (-e^{i\psi})e^{-i\psi/3}\overline{\rho} & \text{if } b_{k+1}=1,\ b_k=2; \\ e^{i\psi/3}\sigma & \text{if } b_{k+1}=3,\ b_k=2; \\ (-e^{i\psi})e^{-i\psi/3}\overline{\sigma} & \text{if } b_{k+1}=2,\ b_k=3; \\ e^{i\psi/3}\tau & \text{if } b_{k+1}=1,\ b_k=3; \\ (-e^{i\psi})e^{-i\psi/3}\overline{\tau} & \text{if } b_{k+1}=3,\ b_k=1. \end{cases}$$

Thus for each sum $S\neq\emptyset$ we have:

$$\begin{aligned}
&(-e^{-i\psi})^{m_-}\frac{(e^{i\psi}-1)\langle \mathbf{n}_{b_2}, \mathbf{n}_{b_1}\rangle}{\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_1}\rangle}\cdots\frac{(e^{i\psi}-1)\langle \mathbf{n}_{b_1}, \mathbf{n}_{b_m}\rangle}{\langle \mathbf{n}_{b_m}, \mathbf{n}_{b_m}\rangle} \\
&= (-e^{-i\psi})^{m_-}\Big(e^{i\psi}-1\Big)^{z_1}\Big(e^{i\psi/3}\rho\Big)^{p_1}\Big((-e^{i\psi})e^{-i\psi/3}\overline{\rho}\Big)^{n_1} \\
&\quad\cdot\Big(e^{i\psi}-1\Big)^{z_2}\Big(e^{i\psi/3}\sigma\Big)^{p_2}\Big((-e^{i\psi})e^{-i\psi/3}\overline{\sigma}\Big)^{n_2} \\
&\quad\cdot\Big(e^{i\psi}-1\Big)^{z_3}\Big(e^{i\psi/3}\tau\Big)^{p_3}\Big((-e^{i\psi})e^{-i\psi/3}\overline{\tau}\Big)^{n_3} \\
&= \big(-e^{-i\psi}\big)^{m_-}\big(e^{i\psi}-1\big)^{z_1+z_2+z_3}\big(-e^{i\psi}\big)^{n_1+n_2+n_3} \\
&\quad\cdot\big(e^{i\psi/3}\big)^{p_1-n_1}\rho^{p_1}\overline{\rho}^{n_1}\big(e^{i\psi/3}\big)^{p_2-n_2}\sigma^{p_2}\overline{\sigma}^{n_2}\big(e^{i\psi/3}\big)^{p_3-n_3}\tau^{p_3}\overline{\tau}^{n_3} \\
&= \big(-e^{-i\psi}\big)^{m_-}\big(e^{i\psi}-1\big)^{z}\big(-e^{i\psi}\big)^{n}\big(e^{i\psi}\big)^{w}\rho^{p_1}\overline{\rho}^{n_1}\sigma^{p_2}\overline{\sigma}^{n_2}\tau^{p_3}\overline{\tau}^{n_3}
\end{aligned}$$

where, in the last line, we have used $w = p_1 - n_1 = p_2 - n_2 = p_3 - n_2$, $z = z_1 + z_2 + z_3$ and $n = n_1 + n_2 + n_3$. □

Note $|S| = m = m_+ + m_- = z + p + n$ where $z = z_1 + z_2 + z_3$, $p = p_1 + p_2 + p_3$ and $n = n_1 + n_2 + n_3$. This means that if we consider

$$(R_{a_1}^{\epsilon_1}\cdots R_{a_r}^{\epsilon_r})^{-1} = R_{a_r}^{-\epsilon_r}\cdots R_{a_1}^{-\epsilon_1}$$

then we must send E to $-E$ and swap m_+ and m_-; p_j and n_j. Using the formula of Proposition 5.8 we can deduce

$$\mathrm{tr}\big((R_{a_1}^{\epsilon_1}\cdots R_{a_r}^{\epsilon_r})^{-1}\big)=\overline{\mathrm{tr}(R_{a_1}^{\epsilon_1}\cdots R_{a_r}^{\epsilon_r})}.$$

This is a good check that the formula might be correct.

An immediately consequence of Proposition 5.8 is the following result, which enables us find the trace field of a triangle group; see Pratoussevitch [27] Theorem 9 for the case where the generators are involutions.

Corollary 5.9: *The trace of any element of Δ may be written as a power of $e^{i\psi/3}$ times a polynomial in $|\rho|^2$, $|\sigma|^2$, $|\tau|^2$, $\rho\sigma\tau$ and $\overline{\rho}\,\overline{\sigma}\,\overline{\tau}$ with coefficients in $\mathbb{Z}[e^{i\psi},e^{-i\psi}]$. In particular, when ψ is a rational multiple of π then the coefficients may be written in $\mathbb{Z}[e^{i\psi}]$.*

Proof: We examine the term coming from $S\neq\emptyset$ as in the proof of Proposition 5.8. First we have $p_j-n_j=w$ and so when $w\geq 0$ we have $p_j\geq n_j$. Thus writing $p_j=w+n_j$ we have

$$\begin{aligned}\rho^{p_1}\overline{\rho}^{n_1}=\rho^{w+n_1}\overline{\rho}^{n_1}&=\rho^{w}(|\rho|^2)^{n_1(s)},\\ \sigma^{p_2}\overline{\sigma}^{n_2}=\sigma^{w+n_2}\overline{\sigma}^{n_2}&=\sigma^{w}(|\sigma|^2)^{n_2(s)},\\ \tau^{p_3}\overline{\tau}^{n_3}=\tau^{w+n_3}\overline{\tau}^{n_3}&=\tau^{w}(|\tau|^2)^{n_3(s)}\end{aligned}$$

and so

$$\rho^{p_1}\overline{\rho}^{n_1}\sigma^{p_2}\overline{\sigma}^{n_2}\tau^{p_3}\overline{\tau}^{n_3}=(|\rho|^2)^{n_1}(|\sigma|^2)^{n_2}(|\tau|^2)^{n_3}(\rho\sigma\tau)^{|w|}.$$

Likewise, when $w\leq 0$, writing $n_j=p_j-w_j$ we have

$$\rho^{p_1}\overline{\rho}^{n_1}\sigma^{p_2}\overline{\sigma}^{n_2}\tau^{p_3}\overline{\tau}^{n_3}=(|\rho|^2)^{p_1}(|\sigma|^2)^{p_2}(|\tau|^2)^{p_3}(\overline{\rho}\,\overline{\sigma}\,\overline{\tau})^{|w|}.$$

In each case this is a monomial in $|\rho|^2$, $|\sigma|^2$, $|\tau|^2$, $\rho\sigma\tau$ and $\overline{\rho}\,\overline{\sigma}\,\overline{\tau}$. □

To make this construction explicit, we now go through it with some particularly important traces; see also Section 8 of Pratoussevitch [27].

Proposition 5.10: *Let R_1, R_2 and R_3 be as above. Then for any distinct j, k, $l\in\{1,2,3\}$ we have*

$$\begin{aligned}\mathrm{tr}(R_1R_2)&=e^{i\psi/3}\big(2-|\rho|^2\big)+e^{-2i\psi/3},\\ \mathrm{tr}(R_1R_2^{-1})&=1+2\cos(\psi)+|\rho|^2,\\ \mathrm{tr}(R_1R_2R_3)&=3-|\rho|^2-|\sigma|^2-|\tau|^2+\rho\sigma\tau,\\ \mathrm{tr}(R_3R_2R_1)&=3-|\rho|^2-|\sigma|^2-|\tau|^2-e^{i\psi}\overline{\rho}\,\overline{\sigma}\,\overline{\tau},\\ \mathrm{tr}(R_1R_2R_3R_2^{-1})&=e^{i\psi/3}\big(2-|\rho\sigma-\overline{\tau}|^2\big)+e^{-2i\psi/3}.\end{aligned}$$

Proof: First consider R_1R_2. We now enumerate all non-empty subsets, their index and winding number, and the contribution they make to the trace. For R_1R_2 the terms are given in the following table:

a_S	ϵ_S	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{1,2\}$	$\{+,+\}$	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\lvert\rho\rvert^2$

From this we see that

$$\mathrm{tr}(R_1R_2) = e^{-2i\psi/3}\big(3+e^{i\psi}-1+e^{i\psi}-1-e^{i\psi}|\rho|^2\big) = e^{i\psi/3}\big(2-|\rho|^2\big)+e^{-2i\psi/3}.$$

For $R_1R_2^{-1}$ this table becomes:

a_S	ϵ_S	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
$\{1,2\}$	$\{+,-\}$	1	0	1	1	0	0	0	0	0	$\lvert\rho\rvert^2$

From this we see that

$$\mathrm{tr}(R_1R_2^{-1}) = 3 + (e^{i\psi}-1) - e^{-i\psi}(e^{-i\psi}-1) + |\rho|^2 = 1 + 2\cos(\psi) + |\rho|^2.$$

Likewise, the table for $R_1R_2R_3$ is:

a_S	ϵ_S	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{3\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{1,2\}$	$\{+,+\}$	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\lvert\rho\rvert^2$
$\{2,3\}$	$\{+,+\}$	0	0	0	0	1	1	0	0	0	$-e^{i\psi}\lvert\sigma\rvert^2$
$\{1,3\}$	$\{+,+\}$	0	0	0	0	0	0	1	1	0	$-e^{i\psi}\lvert\tau\rvert^2$
$\{1,2,3\}$	$\{+,+,+\}$	0	0	1	0	1	0	1	0	1	$e^{i\psi}\rho\sigma\tau$

Thus

$$\begin{aligned}\mathrm{tr}(R_1R_2R_3) = e^{-i\psi}\big(&3 + e^{\psi} - 1 + e^{i\psi} - 1 + e^{i\psi} - 1 \\ &-e^{i\psi}|\rho|^2 - e^{i\psi}|\sigma|^2 - e^{i\psi}|\tau|^2 + e^{i\psi}\rho\sigma\tau\big) \\ = 3 &- |\rho|^2 - |\sigma|^2 - |\tau|^2 + \rho\sigma\tau.\end{aligned}$$

Next, we do the same thing with $R_3R_2R_1$.

a_S	ϵ_S	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{3\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2,1\}$	$\{+,+\}$	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\lvert\rho\rvert^2$
$\{3,2\}$	$\{+,+\}$	0	0	0	0	1	1	0	0	0	$-e^{i\psi}\lvert\sigma\rvert^2$
$\{3,1\}$	$\{+,+\}$	0	0	0	0	0	0	1	1	0	$-e^{i\psi}\lvert\tau\rvert^2$
$\{3,2,1\}$	$\{+,+,+\}$	0	0	0	1	0	1	0	1	-1	$(-e^{-i\psi})^{-3}(e^{i\psi})^{-1}\overline{\rho}\,\overline{\sigma}\,\overline{\tau}$

Thus

$$\begin{aligned}\mathrm{tr}(R_3R_2R_1) &= e^{-i\psi}\big(3+e^{\psi}-1+e^{i\psi}-1+e^{i\psi}-1\\ &\qquad -e^{i\psi}|\rho|^2-e^{i\psi}|\sigma|^2-e^{i\psi}|\tau|^2-e^{2i\psi}\overline{\rho}\,\overline{\sigma}\,\overline{\tau}\big)\\ &= 3-|\rho|^2-|\sigma|^2-|\tau|^2-e^{i\psi}\overline{\rho}\,\overline{\sigma}\,\overline{\tau}.\end{aligned}$$

Finally, we do the same thing for $R_1R_2R_3R_2^{-1}$.

a_S	ϵ_S	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{3\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$(-e^{-i\psi})(e^{i\psi}-1)$
$\{1,2\}$	$\{+,+\}$	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\lvert\rho\rvert^2$
$\{1,3\}$	$\{+,+\}$	0	0	0	0	0	0	1	1	0	$-e^{i\psi}\lvert\tau\rvert^2$
$\{1,2\}$	$\{+,-\}$	1	0	1	1	0	0	0	0	0	$\lvert\rho\rvert^2$
$\{2,3\}$	$\{+,+\}$	0	0	0	0	1	1	0	0	0	$-e^{i\psi}\lvert\sigma\rvert^2$
$\{2,2\}$	$\{+,-\}$	1	2	0	0	0	0	0	0	0	$(-e^{-i\psi})(e^{i\psi}-1)^2$
$\{3,2\}$	$\{+,-\}$	1	0	0	0	1	1	0	0	0	$\lvert\sigma\rvert^2$
$\{1,2,3\}$	$\{+.+,+\}$	0	0	1	0	1	0	1	0	1	$e^{i\psi}\rho\sigma\tau$
$\{1,2,2\}$	$\{+,+,-\}$	1	1	1	1	0	0	0	0	0	$(e^{i\psi}-1)\lvert\rho\rvert^2$
$\{1,3,2\}$	$\{+,+,-\}$	1	0	0	1	0	1	0	1	-1	$(-e^{i\psi})^2(e^{-i\psi})\overline{\rho}\,\overline{\sigma}\,\overline{\tau}$
$\{2,3,2\}$	$\{+,+,-\}$	1	1	0	0	1	1	0	0	0	$(e^{i\psi}-1)\lvert\sigma\rvert^2$
$\{1,2,3,2\}$	$\{+,+,+,-\}$	1	0	1	1	1	1	0	0	0	$(-e^{i\psi})\lvert\rho\rvert^2\lvert\sigma\rvert^2$

Thus

$$\begin{aligned}
&\mathrm{tr}(R_1R_2R_3R_2^{-1})\\
&= e^{-2i\psi/3}\big(3 + e^{i\psi} - 1 + e^{i\psi} - 1 + e^{i\psi} - 1 + e^{-i\psi} - 1\\
&\qquad -e^{i\psi}|\rho|^2 - e^{i\psi}|\tau|^2 + |\rho|^2 - e^{i\psi}|\sigma|^2 + e^{-i\psi} - 1 + |\sigma|^2\\
&\qquad +e^{i\psi}\rho\sigma\tau + (e^{i\psi} - 1)|\rho|^2 + e^{i\psi}\overline{\rho}\,\overline{\sigma}\,\overline{\tau} + (e^{i\psi} - 1)|\sigma|^2 - e^{i\psi}|\rho|^2|\sigma|^2\big)\\
&= e^{i\psi/3}\big(2 - |\rho\sigma - \overline{\tau}|^2\big) + e^{-2i\psi/3}. \qquad \square
\end{aligned}$$

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