An extended isogeometric boundary element method for two-dimensional wave scattering problems

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Abstract

Isogeometric analysis, using the same basis functions that describe a geometry in CAD software to approximate unknown fields in numerical simulations, is a topic of considerable interest. Until recently, much of the research in this field has concentrated on using this approach for finite element analysis (FEA); however, now, there is an increased focus on boundary integral methods. In contrast to FEA, the boundary element method (BEM) requires only the bounding surfaces of a domain to be meshed. Non-uniform, rational basis splines (NURBS), used commonly in CAD software, describe only the boundary of geometries; hence, NURBS would appear to be a natural tool for the BEM and isogeometric analysis. The partition of unity method has provided significant benefits over conventional, piecewise boundary element methods; here, a partition-of-unity extended, isogeometric BEM is presented and numerical results of an acoustic wave scattering problem given.

Keywords: Isogeometric analysis, boundary element method, partition of unity, NURBS

1. Introduction

Creating a suitable mesh can be one of the most significant stages in numerical analysis. Procedures that improve mesh quality or reduce the time required to create such a mesh are of interest to both academics and industry. To this end, Hughes *et al.* [1] introduced the concept of isogeometric analysis (IGA): using the basis functions used to describe a geometry in computer-aided design (CAD) to construct exact geometries for numerical analysis. The basis functions used in most IGA papers are non-uniform rational B-splines (NURBS); however, more recently, other functions have been investigated such as T-splines [2].

While [1] concentrates on using the finite element method (FEM) for analysis, the current paper focuses on the use of the boundary element method (BEM). The BEM provides benefits over the FEM for Helmholtz problems such as the wave scattering in infinite domains considered here. This is because, unlike the FEM, the BEM only requires the boundaries of scattering surfaces to be meshed; there is no truncation of infinite domains and no artificial boundary conditions are set to model infinity. NURBS, used commonly in CAD software, also only describe the boundaries of geometries; hence, IGA and BEM would appear to be a natural combination. Some examples of this combination can be found in Simpson *et al.* [3], who coined the pairing IGABEM, and Politis *et al.* [4]; Takahashi and Matsumoto [5] have also applied the fast multipole method to IGABEM for the Laplace equation.

Like the FEM, conventional BEM schemes require a mesh to be refined as the wavenumber, k, of a problem increases. This puts a practical limit, given a fixed computational resource, on the size of wavenumber that can be considered for a specified problem. A number of approaches have been developed to increase this limit on boundary integral approaches [6, 7, 8, 9, 10]; in this paper, the partition of unity method (PUM), developed by Melenk and Babuška [11], is used. The PUM, in which the character of the wave propagation is included in the approximating function, was first used with the BEM for acoustic wave scattering by de la Bourdonnaye [12] under the name 'microlocal discretization'. Later, Perrey-Debain *et al.* [13] showed that the number of degrees of freedom for a given problem could be dramatically reduced using this method.

This paper develops a collocation, isogeometric BEM employing PUM; we call this the extended isogeometric BEM (XIBEM). The scheme is developed for two-dimensional wave scattering problems and it is shown that this advancement on IGABEM improves the accuracy of simulations and increases the wavenumber limit on the problems that can be solved numerically.

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2. Formulation of IGABEM for Helmholtz

2.1. Boundary integral equation

Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain containing a smooth scatterer of boundary $\Gamma := \partial\Omega$. Assuming $e^{-i\omega t}$ time dependence, the wave equation can be reduced to the well-known Helmholtz equation:

$$\nabla^2\phi(\mathbf{q}) + k^2\phi(\mathbf{q}) = 0, \quad \phi \in \mathbb{C}, \mathbf{q} \in \Omega, \tag{1}$$

where $\nabla^2(\cdot)$ is the Laplacian operator, $\phi(\mathbf{q})$ is the unknown wave potential at point $\mathbf{q}$, and k is wavenumber – related to the wavelength, λ, by $k = 2\pi/\lambda$. The scatterer is impinged by an incident, plane wave,

$$\phi^{\mathrm{I}}(\mathbf{q}) = A^{\mathrm{I}} \exp\left(ik\mathbf{d}^{\mathrm{I}} \cdot \mathbf{q}\right), \quad \left|\mathbf{d}^{\mathrm{I}}\right| = 1, \tag{2}$$

where A^{I} is the incident wave amplitude and $\mathbf{d}^{\mathrm{I}}$ is the direction of propagation.

We start with the conventional boundary integral equation for the Helmholtz equation [14],

$$\frac{1}{2}\phi(\mathbf{p}) = \int_{\Gamma}\left[\frac{\partial\phi(\mathbf{q})}{\partial n}G(\mathbf{p}, \mathbf{q}) - \phi(\mathbf{q})\frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n}\right]\mathrm{d}\Gamma(\mathbf{q}) + \phi^{\mathrm{I}}(\mathbf{p}), \quad \mathbf{p}, \mathbf{q} \in \Gamma, \tag{3}$$

where $\mathbf{p}$ is an evaluation point and n is the outward-pointing, unit normal at integration point $\mathbf{q}$. Further, $G(\mathbf{p}, \mathbf{q})$ is the fundamental solution (Green's function), representing the field experienced at $\mathbf{q}$ due to a unit source radiating at $\mathbf{p}$ (or *vice versa*).

For compact presentation, only one boundary condition is considered: the case of a perfectly reflecting ("sound-hard") cylinder. A solution to (1) is sought, subject to this boundary condition,

$$\frac{\partial\phi(\mathbf{q})}{\partial n} = 0. \tag{4}$$

(3) may be then be reformulated as

$$\frac{1}{2}\phi(\mathbf{p}) + \int_{\Gamma}\frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n}\phi(\mathbf{q})\,\mathrm{d}\Gamma(\mathbf{q}) = \phi^{\mathrm{I}}(\mathbf{p}). \tag{5}$$

2.2. NURBS

To discretise (5), we employ NURBS. An important component of both non-rational and rational B-splines is the knot vector, Ξ; this is a sequence of nondecreasing, real numbers (knots). We assume the curve has a knot vector of the form,

$$\Xi = \{\underbrace{0, \ldots, 0}_{p+1}, \xi_{p+1}, \ldots, \xi_{s-p+1}, \underbrace{1, \ldots, 1}_{p+1}\},$$

where p is the degree of the curve, there are $s + 1$ knots, and $\xi_j \leq \xi_{j+1}$ for $j = 0, \ldots, s - 1$. This particular form of knot vector is called an open knot vector because the first and last knots are repeated $p + 1$ times.

The jth non-rational, B-spline basis function of pth degree, $N_{j,p}$, is defined for $p = 0$ as

$$N_{j,0}(\xi) = \begin{cases} 1 & \text{if } \xi_j \leq \xi \leq \xi_{j+1} \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

while for $p = 1, 2, 3, \ldots$,

$$N_{j,p}(\xi) = \frac{\xi - \xi_j}{\xi_{j+p} - \xi_j}N_{j,p-1}(\xi) + \frac{\xi_{j+p+1} - \xi}{\xi_{j+p+1} - \xi_{j+1}}N_{j+1,p-1}(\xi). \tag{7}$$

The number of basis functions, $J+1$, is related to the degree of the basis and length of the knot vector through $J = s - p - 1$. For NURBS, each basis function is given an individual weighting, w_j; the jth NURBS basis function of pth degree, $R_{j,p}$, is then defined as

$$R_{j,p}(\xi) = \frac{N_{j,p}(\xi)w_j}{\sum\limits_{i=0}^{J} N_{i,p}(\xi)w_i}. \tag{8}$$

A set of NURBS basis functions can be used to interpolate a set of control points $\mathbf{P}_j$, using the following relationship:

$$\mathbf{C}(\xi) = \sum_{j=0}^{J} R_{j,p}(\xi)\mathbf{P}_j. \tag{9}$$

It is this relationship that provides an analytical geometry given by

$$\Gamma = \{\mathbf{C}(\xi) : \xi \in [0, 1)\}, \tag{10}$$

where $\mathbf{C} : \mathbb{R} \to \mathbb{R}^2$. The mapping between $\mathbf{q} \in \Gamma$ and ξ is unique, hence it is assumed that any function $f(\mathbf{q})$ is equivalent to $f(\xi)$.

2.3. Isogeometric approximation with partition of unity

For isogeometric BEM, NURBS are used to represent the boundary, Γ, but also the variation of wave potential along Γ:

$$\phi(\xi) = \sum_{j=0}^{J} R_{j,p}(\xi)\phi_j, \tag{11}$$

where ϕ_j is the potential associated with each NURBS basis function. Partition of unity introduces a linear expansion of plane waves on each NURBS function; (11) is rewritten,

$$\phi(\xi) = \sum_{j=0}^{J} R_{j,p} \sum_{m=0}^{M} A_{jm} \exp\left(ik\mathbf{d}_{jm} \cdot \mathbf{q}\right), \quad |\mathbf{d}_{jm}| = 1, \tag{12}$$

where there are $M + 1$ plane waves relating to each NURBS function with prescribed direction of propagation, $\mathbf{d}_{jm} \in \mathbb{R}^2$, and unknown amplitudes, $A_{jm} \in \mathbb{C}$.

Substituting (12) into (5) gives

$$\frac{1}{2}\phi(\mathbf{p}) + \sum_{j=0}^{J} \sum_{m=0}^{M} \int_0^1 \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n} R_j(\xi) \exp(ik\,\mathbf{d}_{jm} \cdot \mathbf{q})|J^t(\xi)|\,\mathrm{d}\xi\, A_{jm} = \phi^{\mathrm{I}}(\mathbf{p}) \tag{13}$$

where $J^t(\xi)$ is the Jacobian of the mapping in (10). $\phi(\mathbf{p})$ can also be expanded in a similar fashion to $\phi(\mathbf{q})$ in (12).

To find the potential on Γ, (13) is collocated at a series of collocation points, $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{Z-1}$, equally-spaced on $\xi \in [0, 1)$; $Z = (J + 1) \times (M + 1)$ is the total number of unknown amplitudes, A_{jm}, that are sought. This yields a square system of linear equations,

$$[(1/2)\mathbf{P} + \mathbf{H}]\{\mathbf{x}\} = \{\mathbf{b}\}, \tag{14}$$

where the (usually sparse) square matrix $\mathbf{P}$ results from interpolations of the plane waves at $\mathbf{p}$; the right-hand side vector $\{\mathbf{b}\}$ contains the incident wave potentials at the collocation points; and the unknown vector $\{\mathbf{x}\}$ contains the amplitudes, A_{jm}. The square matrix $\mathbf{H}$ is fully populated with integrals,

$$h_{jm}^z = \int_0^1 \frac{\partial G(\mathbf{p}_z, \mathbf{q})}{\partial n} R_j(\xi) \exp(ik\,\mathbf{d}_{jm} \cdot \mathbf{q}) J^t(\xi)\,\mathrm{d}\xi, \tag{15}$$

and takes the form

$$\mathbf{H} = \begin{bmatrix} h_{0,0}^0 & h_{0,1}^0 & \cdots & h_{0,M}^0 & h_{1,0}^0 & h_{1,1}^0 & \cdots & h_{J,M}^0 \\ h_{0,0}^1 & h_{0,1}^1 & \cdots & h_{0,M}^1 & h_{1,0}^1 & h_{1,1}^1 & \cdots & h_{J,M}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{0,0}^{Z-1} & h_{0,1}^{Z-1} & \cdots & h_{0,M}^{Z-1} & h_{1,0}^{Z-1} & h_{1,1}^{Z-1} & \cdots & h_{J,M}^{Z-1} \end{bmatrix}.$$

3. Numerical results

As in the derivation, the numerical examples here use the sound-hard boundary condition; the are no singular integrals in this case and, so, no regularisation or special integration schemes are required. The integral in (13) is evaluated using Gauss-Legendre quadrature, subdividing the boundary into cells of approximately $\lambda/4$ in length. To overcome the well-known nonuniqueness problem associated with the BEM, the authors use the CHIEF method [15]; the system is solved using singular value decomposition (SVD).

The test case considered is that of a unit-radius cylinder, centred at the origin, being impinged by a unit-amplitude, incident plane wave propagating in the direction $\mathbf{d}^{\mathrm{I}} = (1, 0)$. This problem has a known analytical solution [16]. The errors, $\mathcal{E}$, presented are calculated in a relative L_2 norm sense:

$$\mathcal{E} = \frac{\left\|\Phi - \Phi^{\mathrm{analyt}}\right\|_{L_2}}{\left\|\Phi^{\mathrm{analyt}}\right\|_{L_2}}, \tag{16}$$

where Φ is a vector of potentials along the boundary of the scatterer calculated from the numerical simulation; and Φ^{analyt} is a vector of potentials calculated analytically.

Figure 1 shows the errors in a conventional, piecewise polynomial BEM solution compared to those from a NURBS-based IGABEM. The variable, τ, is defined as the number of degrees of freedom per wavelength of the problem. Simulations over a range of wavenumbers are run, maintaining a value $\tau \approx 10$; this leads to the sawtooth effect seen at low wavenumbers as the addition of single elements (in the piecewise polynomial case) or knots (in the IGABEM) is made.

IGABEM clearly provides a greater accuracy of approximation; this is due to the integration points being mapped to the analytical surface of the cylinder by the NURBS and also as the potential approximation is mapped in the same way. As the wavenumber increases, the benefit is reduced as the shorter elements become a closer fit to the analytical curve of the cylinder.
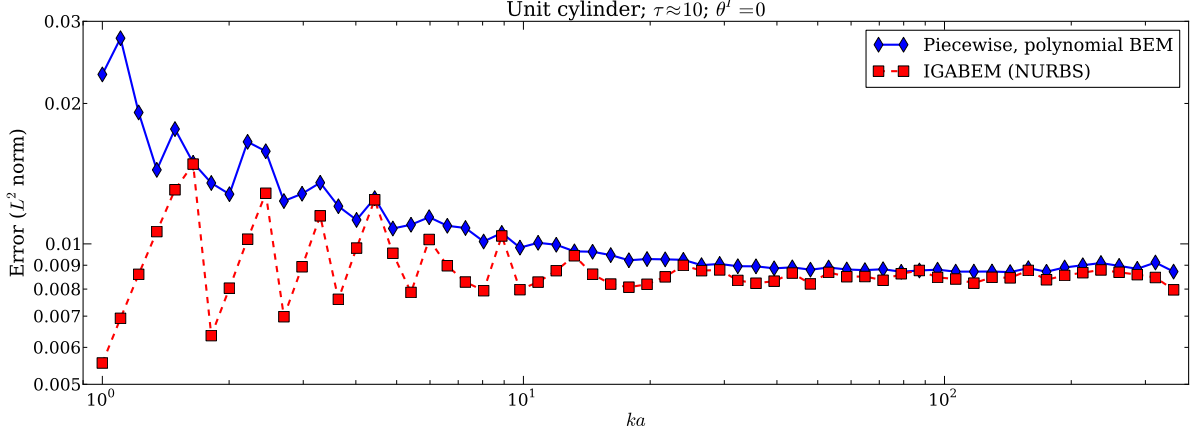

Figure 1: Comparison of errors for conventional, piecewise BEM and isogeometric BEM.

Figure 2 introduces XIBEM. NURBS curves can be considered as a set of piecewise rational Bézier curves; elements are defined between unique values contained within the knot vector, Ξ. Two meshes are used: one with four elements (five unique knot values), one with three elements (four unique knot values). Figure 2 shows the accuracy, for a fixed wavenumber but varying τ, obtained with these two meshes with partition-of-unity compared to the IGABEM with no enrichment. It is observed that, for a fixed τ (thus system size), accuracies in excess of three orders of magnitude of improvement are obtained using XIBEM.
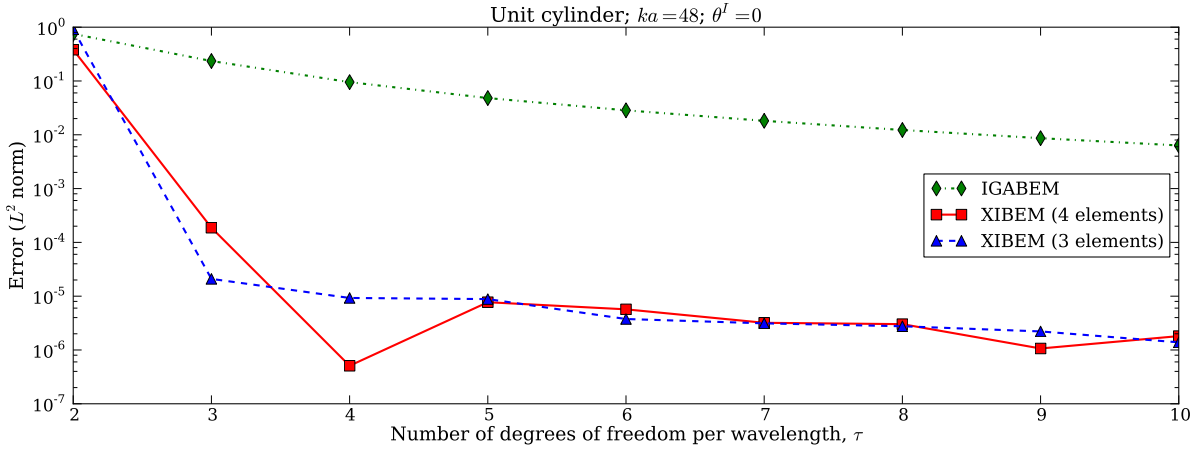

Figure 2: Comparison of error reduction with increased degrees of freedom: IGABEM versus XIBEM

Figure 3 also compares XIBEM to IGABEM. In this case, $\tau \approx 3$ for XIBEM simulations and $\tau \approx 10$ for IGABEM simulations. The IGABEM simulations achieve a consistent accuracy of $\sim 1\%$ while the XIBEM simulations, with a system matrix nine times smaller than that of the IGABEM simulations, achieve significantly greater accuracy.

Finally, Figure 4 compares XIBEM simulations to piecewise quadratic, partition-of-unity BEM (PU-BEM) simulations (such as those use in [13]). The errors obtained are found to be broadly similar. However, for the piecewise quadratic PU-BEM simulations, collocation and integration points have to be artificially mapped to the analytical surface of scatterer to achieve these accuracies; for complex geometries, the mapping of these points becomes a complicated task. For XIBEM simulations, this mapping is inherent and no extra mapping is required. It is, therefore, possible to use meshes directly from CAD packages with XIBEM.
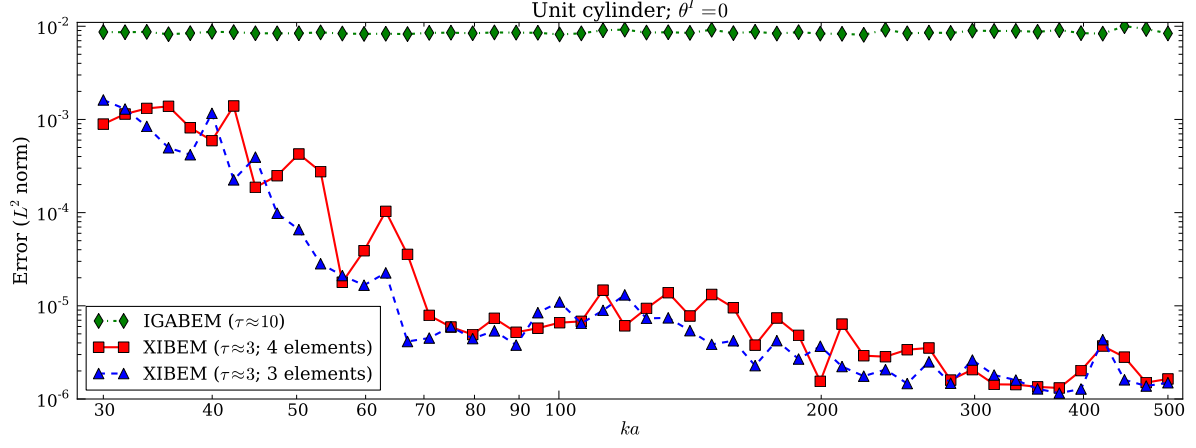
4

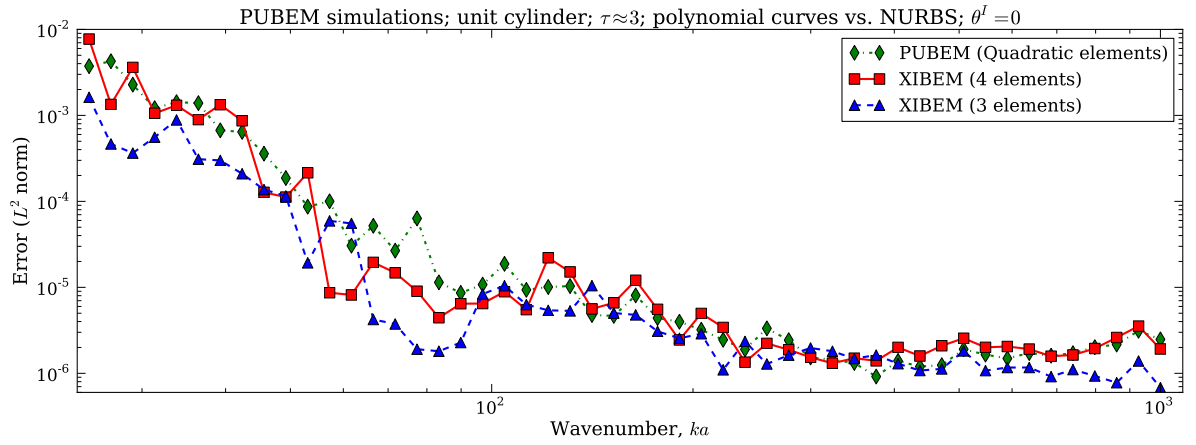Figure 3: Comparison of errors over a range of wavenumbers: IGABEM versus XIBEM

Figure 4: Comparison of errors over a range of wavenumbers: piecewise PUBEM versus XIBEM

4. Conclusions

Isogeometric analysis has significant potential for use with boundary elements methods. This paper has demonstrated this for both conventional and enriched methods. XIBEM provides a clear and significant improvement over standard IGABEM simulations. These improvements are the reduced system size and increased accuracy; this also extends the bandwidth of frequencies over which the method can be considered feasible.

Further developments may come in the form of adaptive schemes for the choice of the number of plane waves in the linear expansion. Moreover, in three-dimensional analysis, techniques to reduce the runtime further, such as the Fast Multipole Method or Adaptive Cross Approximation, could be used.

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