Bounding Clique-width via Perfect Graphs ⋆

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Abstract. Given two graphs H_1 and H_2, a graph G is (H_1, H_2)-free if it contains no subgraph isomorphic to H_1 or H_2. We continue a recent study into the clique-width of (H_1, H_2)-free graphs and present three new classes of (H_1, H_2)-free graphs that have bounded clique-width. We also show the implications of our results for the computational complexity of the Colouring problem restricted to (H_1, H_2)-free graphs. The three new graph classes have in common that one of their two forbidden induced subgraphs is the diamond (the graph obtained from a clique on four vertices by deleting one edge). To prove boundedness of their clique-width we develop a technique based on bounding clique covering number in combination with reduction to subclasses of perfect graphs.

Keywords: clique-width, forbidden induced subgraphs, graph class

1 Introduction

Clique-width is a well-known graph parameter and its properties are well studied; see for example the surveys of Gurski [20] and Kamiński, Lozin and Milanič [22]. Computing the clique-width of a given graph is NP-hard, as shown by Fellows, Rosamond, Rotics and Szeider [18]. Nevertheless, many NP-complete graph problems are solvable in polynomial time on graph classes of *bounded* clique-width, that is, classes in which the clique-width of each of its graphs is at most c for some constant c. This follows by combining the fact that if a graph G has clique-width at most c then a so-called $(8^c - 1)$-expression for G can be found in polynomial time [28] together with a number of results [13,23,30], which show that if a q-expression is provided for some fixed q then certain classes of problems can be solved in polynomial time. A well-known example of such a problem is

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the Colouring problem, which is that of testing whether the vertices of a graph can be coloured with at most k colours such that no two adjacent vertices are coloured alike. Due to these algorithmic implications, it is natural to research whether the clique-width of a given graph class is bounded.

It should be noted that having bounded clique-width is a more general property than having bounded tree-width, that is, every graph class of bounded treewidth has bounded clique-width but the reverse is not true [11]. Clique-width is also closely related to other graph width parameters, e.g. for any class, having bounded clique-width is equivalent to having bounded rank-width [29] and also equivalent to having bounded NLC-width [21]. Moreover, clique-width has been studied in relation to graph operations, such as edge or vertex deletions, edge subdivisions and edge contractions. For instance, a recent result of Courcelle [12] solved an open problem of Gurski [20] by proving that if $\mathcal{G}$ is the class of graphs of clique-width 3 and $\mathcal{G}'$ is the class of graphs obtained from graphs in $\mathcal{G}$ by applying one or more edge contraction operations then $\mathcal{G}'$ has unbounded clique-width.

The classes that we consider in this paper consist of graphs that can be characterized by a family $\{H_1,\ldots,H_p\}$ of forbidden induced subgraphs (such graphs are said to be $(H_1,\ldots,H_p)$-free). The clique-width of such graph classes has been extensively studied in the literature (e.g. [1,2,3,4,5,6,7,8,9,14,16,19,24,25,26,27]). It is straightforward to verify that the class of H-free graphs has bounded clique-width if and only if H is an induced subgraph of the 4-vertex path P_4 (see also [17]). Hence, Dabrowski and Paulusma [17] investigated for which pairs (H_1,H_2) the class of (H_1,H_2)-free graphs has bounded clique-width. In this paper we solve a number of the open cases. The underlying research question is:

What kind of properties of a graph class ensure that its clique-width is bounded?

As such, our paper is to be interpreted as a further step towards this direction. Rather than coming up with ad hoc techniques for solving specific cases, we aim to develop more general techniques for attacking a number of the open cases simultaneously. Our technique in this paper is obtained by generalizing an approach followed in the literature. In order to illustrate this approach with some examples, we first need to introduce some notation (see Section 2 for all other terminology).

Notation. The disjoint union $(V(G)\cup V(H), E(G)\cup E(H))$ of two vertex-disjoint graphs G and H is denoted by $G+H$ and the disjoint union of r copies of a graph G is denoted by rG. The complement of a graph G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and only if these vertices are not adjacent in G. The graphs C_r, K_r and P_r denote the cycle, complete graph and path on r vertices, respectively. The graph $\overline{2P_1+P_2}$ is called the *diamond*. The graph $K_{1,3}$ is the 4-vertex star, also called the *claw*. For $1 \leq h \leq i \leq j$, let $S_{h,i,j}$ be the *subdivided claw* whose three edges are subdivided $h-1$, $i-1$ and $j-1$ times, respectively; note that $S_{1,1,1} = K_{1,3}$.

Our technique. Dabrowski and Paulusma [16] determined all graphs H for which the class of H-free bipartite graphs has bounded clique-width. Such a classification turns out to also be useful for proving boundedness of the clique-width for other graph classes. For instance, in order to prove that $(\overline{P_1 + P_3}, P_1 + S_{1,1,2})$-free graphs have bounded clique-width, the given graphs were first reduced to $(P_1 + S_{1,1,2})$-free bipartite graphs [17]. In a similar way, Dabrowski, Lozin, Raman and Ries [15] proved that $(K_3, K_{1,3} + K_2)$-free graphs and $(K_3, S_{1,1,3})$-free have bounded clique-width by reducing to a subclass of bipartite graphs. Note that bipartite graphs are perfect graphs. This motivated us to develop a technique based on perfect graphs that are not necessarily bipartite. In order to so, we need to combine this approach with an additional tool. This tool is based on the following observation. If the vertex set of a graph can be partitioned into a small number of cliques and the edges between them are sufficiently sparse, then the clique-width is bounded (see also Lemma 10). Our technique can be summarized as follows.

1. Reduce the input graph to a graph that is in some subclass of perfect graphs;
2. While doing so, bound the clique covering number of the input graph.

Another well-known subclass of perfect graphs is the class of chordal graphs. We show that besides the class of bipartite graphs, the class of chordal graphs and the class of perfect graphs itself may be used for Step 1. We explain Steps 1-2 of our technique in detail in Section 3.

Our results. In this paper, we investigate whether our technique can be used to find new pairs (H_1, H_2) for which the clique-width of (H_1, H_2)-free graphs is bounded. We show that this is indeed the case. By applying our technique, we are able to present three new classes of (H_1, H_2)-free graphs of bounded clique-width.[1] Namely, it enables us to prove the following result, which we prove in Section 4.

Theorem 1. *The class of (H_1, H_2)-free graphs has bounded clique-width if*

(i) $H_1 = \overline{2P_1 + P_2}$ *and* $H_2 = 3P_1 + P_2$*;*
(ii) $H_1 = \overline{2P_1 + P_2}$ *and* $H_2 = 2P_1 + P_3$*;*
(iii) $H_1 = \overline{2P_1 + P_2}$ *and* $H_2 = P_2 + P_3$*.*

Structural consequences. Theorem 1 reduces the number of open cases in the classification of the boundedness of the clique-width for (H_1, H_2)-free graphs to 13 open cases, up to some equivalence relation, see also [17]. Note that the graph H_1 is the diamond in each of the three results in Theorem 1. Out of the 13 remaining cases, there are still three cases in which H_1 is the diamond, namely when $H_2 \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5\}$. However, for each of these graphs H_2, it is not even known whether the clique-width of the corresponding smaller subclasses of (K_3, H_2)-free graphs is bounded. Of particular note is the class of $(K_3, P_1 + 2P_2)$-free graphs, which is contained in all of the above open cases and for which the boundedness of clique-width is unknown. Settling this

[1] We do not specify our upper bounds as this would complicate our proofs for negligible gain. This is because in our proofs we apply graph operations that exponentially increase the upper bound of the clique-width, which means that the bounds that could be obtained from our proofs would be very large and far from being tight.

case is a natural next step in completing the classification. Note that for K_3-free graphs the clique covering number is proportional to the size of the graph. Another natural research direction is to determine whether the clique-width of $(\overline{P_1 + P_4}, H_2)$-free graphs is bounded for $H_2 = P_2 + P_3$ (the clique-width is known to be unbounded for $H_2 \in \{3P_1 + P_2, 2P_1 + P_3\}$).

Dabrowski, Golovach and Paulusma [14] showed that Colouring restricted to $(sP_1 + P_2, \overline{tP_1 + P_2})$-free graphs is polynomial-time solvable for all pairs of integers s, t. They justified their algorithm by proving that the clique-width of the class of $(sP_1, \overline{tP_1 + P_2})$-free graphs is bounded only for small values of s and t, namely only for $s \leq 2$ or $t \leq 1$ or $s + t \leq 6$. In the light of these two results it is natural to try to classify the clique-width of the class of $(sP_1 + P_2, \overline{tP_1 + P_2})$-free graphs for all pairs (s, t). Theorem 1, combined with the aforementioned classification of the clique-width of $(sP_1, \overline{tP_1 + P_2})$-free graphs and the fact that any class of (H_1, H_2)-free graphs has bounded clique-width if and only if the class of $(\overline{H_1}, \overline{H_2})$-free graphs has bounded clique-width, immediately enables us to do this.

Corollary 2. *The class of $(sP_1 + P_2, \overline{tP_1 + P_2})$-free graphs has bounded clique-width if and only if $s \leq 1$ or $t \leq 1$ or $s + t \leq 5$.*

Algorithmic consequences. Our research was (partially) motivated by a study into the computational complexity of the Colouring problem for (H_1, H_2)-free graphs. As mentioned, Colouring is polynomial-time solvable on any graph class of bounded clique-width. Of the three classes for which we prove boundedness of clique-width in this paper, only the case of $(\overline{2P_1 + P_2}, 3P_1 + P_2)$-free (and equivalently $(2P_1 + P_2, \overline{3P_1 + P_2})$-free) graphs was previously known to be polynomial-time solvable [14]. Hence, Theorem 1 gives us four new pairs (H_1, H_2) with the property that Colouring is polynomial-time solvable when restricted to (H_1, H_2)-free graphs, namely if

- $H_1 = 2P_1 + P_2$ and $H_2 \in \{\overline{2P_1 + P_3}, \overline{P_2 + P_3}\}$;
- $H_1 = \overline{2P_1 + P_2}$ and $H_2 \in \{2P_1 + P_3, P_2 + P_3\}$.

As such, there are still 15 potential classes of (H_1, H_2)-free graphs left for which both the complexity of Colouring and the boundedness of their clique-width is unknown [17].

2 Preliminaries

Below we define some graph terminology used throughout our paper. Let G be a graph. For $u \in V(G)$, the set $N(u) = \{v \in V(G) \mid uv \in E(G)\}$ is the *neighbourhood* of u in G. The *degree* of a vertex in G is the size of its neighbourhood. The *maximum degree* of G is the maximum vertex degree. For a subset $S \subseteq V(G)$, we let $G[S]$ denote the *induced* subgraph of G, which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. If $S = \{s_1, \ldots, s_r\}$ then, to simplify notation, we may also write $G[s_1, \ldots, s_r]$ instead of $G[\{s_1, \ldots, s_r\}]$. Let H be another graph. We write $H \subseteq_i G$ to indicate that H is an induced subgraph of G. Let $X \subseteq V(G)$. We write $G \setminus X$ for the graph obtained from G after removing X. A set $M \subseteq E(G)$ is a *matching* if no two edges in M share an end-vertex. We say that two disjoint sets $S \subseteq V(G)$ and $T \subseteq V(G)$ are *complete* to each other if every vertex of S is adjacent to every vertex of T. If no vertex of S is joined to a vertex of T by an edge, then S and T are *anti-complete* to each other. Similarly, we say that a vertex u and a set S not containing u may be complete or anti-complete to each other. Let $\{H_1, \ldots, H_p\}$ be a set of graphs. Recall that G is $(H_1, \ldots, H_p)$*-free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we may write H_1-free instead of (H_1)-free.

The *clique-width* of a graph G, denoted by $\text{cw}(G)$, is the minimum number of labels needed to construct G by using the following four operations:

(i) creating a new graph consisting of a single vertex v with label i;
(ii) taking the disjoint union of two labelled graphs G_1 and G_2;
(iii) joining each vertex with label i to each vertex with label j ($i \neq j$);
(iv) renaming label i to j.

A class of graphs $\mathcal{G}$ has *bounded* clique-width if there is a constant c such that the clique-width of every graph in $\mathcal{G}$ is at most c; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

Let G be a graph. We say that G is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets B and W. We say that (B, W) is a *bipartition* of G.

Let G be a graph. We define the following two operations. For an induced subgraph $G' \subseteq_i G$, the *subgraph complementation* operation (acting on G with respect to G') replaces every edge present in G' by a non-edge, and vice versa. Similarly, for two disjoint vertex subsets X and Y in G, the *bipartite complementation* operation with respect to X and Y acts on G by replacing every edge with one end-vertex in X and the other one in Y by a non-edge and vice versa.

We now state some useful facts for dealing with clique-width. We will use these facts throughout the paper. Let $k \geq 0$ be a constant and let γ be some graph operation. We say that a graph class $\mathcal{G}'$ is (k, γ)*-obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

(i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and
(ii) for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times.

We say that γ *preserves* boundedness of clique-width if for any finite constant k and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is (k, γ)-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [24].

Fact 2. Subgraph complementation preserves boundedness of clique-width [22].

Fact 3. Bipartite complementation preserves boundedness of clique-width [22].

The following lemma is well-known and straightforward to check.

Lemma 3. *The clique-width of a forest is at most* 3.

Let G be a graph. The size of a largest independent set and a largest clique in G are denoted by $\alpha(G)$ and $\omega(G)$, respectively. The chromatic number of G is denoted by $\chi(G)$. We say that G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

We need the following well-known result, due to Chudnovsky, Robertson, Seymour and Thomas.

Theorem 4 (The Strong Perfect Graph Theorem [10]). *A graph is perfect if and only if it is* C_r*-free and* $\overline{C_r}$*-free for every odd* $r \geq 5$.

The *clique covering number* $\overline{\chi}(G)$ of a graph G is the smallest number of (mutually vertex-disjoint) cliques such that every vertex of G belongs to exactly one clique. If G is perfect, then $\overline{G}$ is also perfect (by Theorem 4). By definition, $\overline{G}$ can be partitioned into $\omega(\overline{G}) = \alpha(G)$ independent sets. This leads to the following well-known lemma.

Lemma 5. *Let* G *be any perfect graph. Then* $\overline{\chi}(G) = \alpha(G)$.

We say that a graph G is *chordal* if G contain no induced cycle on four or more vertices. Bipartite graphs and chordal graphs are perfect (by Theorem 4).

The following three lemmas give us a number of subclasses of perfect graphs with bounded clique-width. We will make use of these lemmas later on in the proofs as part of our technique.

Lemma 6 ([16]). *Let* H *be a graph. The class of* H*-free bipartite graphs has bounded clique-width if and only if* $H \subseteq_i K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}$ *or* $S_{1,2,3}$ *or* $H = sP_1$ *for some* $s \geq 1$.

Lemma 7 ([19]). *The class of chordal* $(\overline{2P_1 + P_2})$*-free graphs has clique-width at most* 3.

Lemma 8 ([15]). *The class of* $(K_3, K_{1,3} + P_2)$*-free graphs has bounded clique-width.*

Finally, we also need the following lemma, which corresponds to the first lemma of [14] by complementing the graphs under consideration.

Lemma 9 ([14]). *Let* $s \geq 0$ *and* $t \geq 0$*. Then every* $(\overline{sP_1 + P_2}, tP_1 + P_2)$*-free graph is* $(K_{s+1}, tP_1 + P_2)$*-free or* $(\overline{sP_1 + P_2}, (s^2(t-1)+2)P_1)$*-free.*

3 The Clique Covering Lemma

In Section 2 we stated several lemmas that can be used to bound the clique-width if we can manage to reduce to some specific graph class. As we shall see, such a reduction is not always sufficient and the following lemma forms a crucial part of our technique (we use it in the proofs of each of our main results). We omit the proof due to space restrictions.

Lemma 10. *Let $k \geq 1$ be a constant and let G be a $(\overline{2P_1+P_2}, 2P_2+P_4)$-free graph. If $\overline{\chi}(G) \leq k$ then $\mathrm{cw}(G) \leq f(k)$ for some function f that only depends on k.*

It is easy to see that for any fixed constant $s \geq 2$ we can generalize Lemma 10 to be valid for $(\overline{2P_1+P_2}, 2K_s+P_4)$-free graphs. By more complicated arguments it is also possible to generalize it to other graph classes, such as $(\overline{2P_1+P_2}, K_s+P_6)$-free graphs for any fixed $s \geq 0$. However, this is not necessary for the main results of this paper.

4 The Proof of Theorem 1

Theorem 1 (i). *The class of $(\overline{2P_1+P_2}, 3P_1+P_2)$-free graphs has bounded clique-width.*

To prove this theorem, suppose G is a $(\overline{2P_1+P_2}, 3P_1+P_2)$-free graph. Applying Lemma 9 we find that G is $(K_3, 3P_1+P_2)$-free or $(\overline{2P_1+P_2}, 10P_1)$-free. If G is $(K_3, 3P_1+P_2)$-free then it has bounded clique-width by Lemma 8, so we may assume it is $(\overline{2P_1+P_2}, 10P_1, 3P_1+P_2)$-free. We can then show that the vertex set of the graph can be partitioned into a bounded number of cliques, so the clique-width is bounded by Lemma 10. We omit the proof details.

We also omit the proof of our second main result.

Theorem 1 (ii). *The class of $(\overline{2P_1+P_2}, 2P_1+P_3)$-free graphs has bounded clique-width.*

We now prove the last of our three main results, namely that the class of $(\overline{2P_1+P_2}, P_2+P_3)$-free graphs has bounded clique-width. We first establish, via a series of lemmas, that we may restrict ourselves to graphs in this class that are also (C_4, C_5, C_6, K_5)-free. We omit the proofs for the first two of these lemmas.

Lemma 11. *The class of those $(\overline{2P_1+P_2}, P_2+P_3)$-free graphs that contain a K_5 has bounded clique-width.*

Lemma 12. *The class of those $(\overline{2P_1+P_2}, P_2+P_3, K_5)$-free graphs that contain an induced C_5 has bounded clique-width.*

Lemma 13. *The class of those $(\overline{2P_1+P_2}, P_2+P_3, K_5, C_5)$-free graphs that contain an induced C_4 has bounded clique-width.*

Proof. Suppose that G is a $(\overline{2P_1+P_2}, P_2+P_3, K_5, C_5)$-free graph containing a C_4, say on vertices v_1, v_2, v_3, v_4 in order. Let Y be the set of vertices adjacent to v_1 and v_2 (and possibly other vertices on the cycle). If $y_1, y_2 \in Y$ are non-adjacent then $G[v_1, v_2, y_1, y_2]$ would be a $\overline{2P_1+P_2}$. Therefore Y is a clique. Since G is K_5-free, there are at most four such vertices. Therefore by Fact 1 we may assume that no vertex in G has two consecutive neighbours on the cycle. For $i \in \{1, 2\}$ let V_i be the set of vertices outside the cycle adjacent to v_{i+1} and v_{i+3} (where $v_5 = v_1$). For $i \in \{1, 2, 3, 4\}$ let W_i be the set of vertices whose unique neighbour on the cycle is v_i. Let X be the set of vertices with no neighbours on the cycle.

We first prove the following properties:

(i) V_i are independent sets for $i = 1, 2$.
(ii) W_i are independent sets for $i = 1, 2, 3, 4$.
(iii) X is an independent set.
(iv) X is anti-complete to W_i for $i = 1, 2, 3, 4$.
(v) Without loss of generality $W_3 = \emptyset$ and $W_4 = \emptyset$.
(vi) Without loss of generality W_1 is anti-complete to W_2.

To prove Property (i), if $x, y \in V_i$ are adjacent then $G[x, y, v_{i+1}, v_{i+3}]$ is a $\overline{2P_1 + P_2}$. For $i = 1, \ldots, 4$, the set $W_i \cup X$ must also be independent, since if $x, y \in W_1 \cup X$ were adjacent then $G[x, y, v_2, v_3, v_4]$ would be a $P_2 + P_3$. This proves Properties (ii)–(iv).

To prove Property (v), suppose that $x \in W_1$ and $y \in W_3$ are adjacent. In that case $G[v_1, v_2, v_3, y, x]$ would be a C_5. This contradiction means that no vertex of W_1 is adjacent to a vertex of W_3. Now suppose that $x, x' \in W_1$ and $y \in W_3$. Then $G[y, v_3, x, v_1, x']$ would be a $P_2 + P_3$ by Property (ii). Therefore, if both W_1 and W_3 are non-empty, then they each contain at most one vertex and we can delete these vertices by Fact 1. Without loss of generality we may therefore assume that W_3 is empty. Similarly, we may assume W_4 is empty. Hence we have shown Property (v).

We are left to prove Property (vi). Suppose that $x \in W_1$ is adjacent to $y \in W_2$. Then x cannot have a neighbour in V_2. Indeed, suppose for contradiction that x has a neighbour $z \in V_2$. Then $G[x, z, y, v_1]$ is a $\overline{2P_1 + P_2}$ if y and z are adjacent, and $G[x, y, v_2, v_3, z]$ is a C_5 if y and z are not adjacent. By symmetry, y cannot have a neighbour in V_1. Now y must be complete to V_2. Indeed, if y has a non-neighbour $z \in V_2$ then $G[x, y, z, v_3, v_4]$ is a $P_2 + P_3$. By symmetry, x is complete to V_1. Recall that $W_1 \cup X$ is an independent set by Properties (ii)–(iv). We conclude that any vertex in W_1 with a neighbour in W_2 is complete to V_1 and anti-complete to $V_2 \cup X$. Similarly, any vertex in W_2 with a neighbour in W_1 is complete to V_2 and anti-complete to $V_1 \cup X$.

Let W_1^* (respectively W_2^*) be the set of vertices in W_1 (respectively W_2) that have a neighbour in W_2 (respectively W_1). Then, by Fact 3, we may apply two bipartite complementations, one between W_1^* and $V_1 \cup \{v_1\}$ and the other between W_2^* and $V_2 \cup \{v_2\}$. After these operations, G will be split into two disjoint parts, $G[W_1^* \cup W_2^*]$ and $G \setminus (W_1^* \cup W_2^*)$, both of which are induced subgraphs of G. The first of these is a bipartite $(P_2 + P_3)$-free graph and therefore has bounded clique-width by Lemma 6. We therefore only need to consider the second graph $G \setminus (W_1^* \cup W_2^*)$. In other words, we may assume without loss of generality that W_1 is anti-complete to W_2. This proves Property (vi).

If a vertex in X has no neighbours in $V_1 \cup V_2$ then it is an isolated vertex by Property (iv) and the definition of the set X. In this case we may delete it without affecting the clique-width. Hence, we may assume without loss of generality that every vertex in X has at least one neighbour in $V_1 \cup V_2$. We partition X into three sets X_0, X_1, X_2 as follows. Let X_1 (respectively X_2) denote the set of vertices in X with at least one neighbour in V_1 (respectively V_2), but no neighbours in V_2 (respectively V_1). Let X_0 denote the set of vertices in X adjacent to at least one vertex of V_1 and at least one vertex of V_2.

Let $G^* = G[V_1 \cup V_2 \cup W_1 \cup W_2 \cup X_1 \cup X_2]$. We prove the following additional properties:

(vii) G^* is bipartite.
(viii) Without loss of generality $X_0 \neq \emptyset$.
(ix) Every vertex in V_1 that has a neighbour in X is complete to V_2.
(x) Every vertex in V_2 that has a neighbour in X is complete to V_1.
(xi) Every vertex in X_0 has exactly one neighbour in V_1 and exactly one neighbour in V_2.
(xii) Without loss of generality, every vertex in $V_1 \cup V_2$ has at most one neighbour in X_0.
(xiii) Without loss of generality, V_1 is anti-complete to W_2.
(xiv) Without loss of generality, V_2 is anti-complete to W_1.

Property (vii) can be seen has follows. Because G is $(P_2 + P_3, C_5)$-free, G^* has no induced odd cycles of length at least 5. Suppose, for contradiction, that G^* is not bipartite. Then it must contain an induced C_3. Now V_1, V_2, W_1, W_2, X_1 and X_2 are independent sets, so at most one vertex of the C_3 can be in any one of these sets. The set X_1 is anti-complete to V_2, W_1, W_2 and X_2 (by definition of V_2 and Properties (iii) and (iv)). Hence no vertex of the C_3 can be in X_1. Similarly, no vertex of the C_3 be be in X_2. The sets W_1 and W_2 are anti-complete to each other by Property (vi), so the C_3 must therefore consist of one vertex from each of V_1 and V_2, along with one vertex from either W_1 or W_2. However, in this case, these three vertices, along with either v_1 or v_2, respectively would induce a $\overline{2P_1 + P_2}$ in G, which would be a contradiction. Hence we have proven Property (vii).

We now prove Property (viii). Suppose X_0 is empty. Then, since G^* is $(P_2 + P_3)$-free and bipartite (by Property (vii)), it has bounded clique-width by Lemma 6. Hence, G has bounded clique-width by Fact 1, since we may delete v_1, v_2, v_3 and v_4 to obtain G^*. This proves Property (viii).

We now prove Property (ix). Let $y_1 \in V_1$ have a neighbour $x \in X$. Suppose, for contradiction, that y_1 has a non-neighbour $y_2 \in V_2$. Then $G[x, y_2, v_1, v_2, y_1]$ is a C_5 if x is adjacent to y_2 and $G[x, y_1, v_1, y_2, v_3]$ is a $P_2 + P_3$ if x is non-adjacent to y_2, a contradiction. This proves Property (ix). By symmetry, Property (x) holds.

We now prove Property (xi). By definition, every vertex in X_0 has at least one neighbour in V_1 and at least one neighbour in V_2. Suppose, for contradiction, that a vertex $x \in X_0$ has two neighbours $y, y' \in V_1$. By definition, x must also have a neighbour $z \in V_2$. Then z must be adjacent to both y and y' by Property (x). However, then $G[x, z, y, y']$ is a $\overline{2P_1 + P_2}$ by Property (i), a contradiction.This proves Property (xi).

We now prove Property (xii). Suppose a vertex $y \in V_1$ has two neighbours $x, x' \in X_0$. If there is another vertex $z \in X_0$ then z must have a unique neighbour z' in V_1. If z' is a different vertex from y then $G[z, z', x, y, x']$ would be a $P_2 + P_3$ by Properties (i) and (iii). Thus $z' = y$, that is, every vertex in X_0 must be adjacent to y and to no other vertex of V_1. By Fact 1, we may delete y. In the resulting graph no vertex of X would have neighbours in both V_1 and V_2. So X_0 would become empty, in which case we can argue as in the proof of Property (viii). This proves Property (xii).

We now prove Property (xiii). First, for $i \in \{1,2\}$, suppose that a vertex $y \in V_i$ is adjacent to a vertex $x \in X$. Then y can have at most one non-neighbour in W_i. Indeed, suppose for contradiction that $z, z' \in W_i$ are non-neighbours of y. Then $G[x, y, z, v_i, z']$ is a $P_2 + P_3$ by Properties (ii) and (vi), a contradiction. We claim that at most one vertex of W_2 has a neighbour in V_1. Suppose, for contradiction, that W_2 contains two vertices w and w' adjacent to (not necessarily distinct) vertices z and z' in V_1, respectively. Since $X_0 \neq \emptyset$ by Property (viii), there must be a vertex $y \in V_2$ with a neighbour in X_0. As we just showed that such a vertex y can have at most one non-neighbour in W_2, we may assume without loss of generality that y is adjacent to w. Since y has a neighbour in X, it must also be adjacent to z by Property (x). Now $G[w, z, y, v_2]$ is a $\overline{2P_1 + P_2}$, which is a contradiction. Therefore at most one vertex of W_2 has a neighbour in V_1 and similarly, at most one vertex of W_1 has a neighbour in V_2. By Fact 1, we may delete these vertices if they exist. This proves Properties (xiii) and (xiv).

For $i = 1, 2$ let V_i' be the set of vertices in V_i that have a neighbour in X_0. We show two more properties:

(xv) Every vertex in $W_1 \cup X_1$ is adjacent to either none, precisely one or all vertices of V_1'.
(xvi) Every vertex of $W_2 \cup X_2$ is adjacent to either none, precisely one or all vertices of V_2'.

We prove Property (xv) as follows. Suppose a vertex $x \in X_1 \cup W_1$ has at least two neighbours in $z, z' \in V_1$. We claim that x must be complete to V_1'. Suppose, for contradiction, that x is not adjacent to $y \in V_1'$. By definition, y has a neighbour $y' \in X_0$. Then $G[y, y', z, x, z']$ is a $P_2 + P_3$ by Properties (i), (iii) and (iv), a contradiction. This proves Property (xv). Property (xvi) follows by symmetry.

Let W_i' and X_i' be the sets of vertices in W_i and X_i respectively that are adjacent to precisely one vertex of V_i'. We delete v_1, v_2, v_3 and v_4, which we may do by Fact 1. We do a bipartite complementation between V_1' and those vertices in $W_1 \cup X_1$ that are complete to V_1'. We also do this between V_2' and those vertices in $W_2 \cup X_2$ that are complete to V_2'. Finally, we perform a bipartite complementation between V_1' and $V_2 \setminus V_2'$ and also between V_2' and $V_1 \setminus V_1'$. We may do all of this by Fact 3. Afterwards, Properties (i)–(vi), (ix), (x), (xiii)–(xvi) and the definitions of V_1', V_2', W_1', W_2', X_1, X_2 imply that there are no edges between the following two vertex-disjoint graphs:

1. $G[W_1' \cup W_2' \cup X_1' \cup X_2' \cup V_1' \cup V_2' \cup X_0]$ and
2. $G \setminus (W_1' \cup W_2' \cup X_1' \cup X_2' \cup V_1' \cup V_2' \cup X_0 \cup \{v_1, v_2, v_3, v_4\})$

Both of these graphs are induced subgraphs of G. The second of these graphs does not contain any vertices of X_0. So it is bipartite by Property (vii) and therefore has bounded clique-width, as argued before (in the proof of Property (viii)).

Now consider the first graph, which is $G[W_1' \cup W_2' \cup X_1' \cup X_2' \cup V_1' \cup V_2' \cup X_0]$. By Fact 3, we may complement the edges between V_1' and V_2'. This yields a new graph G'. By definition of V_1', V_2' and Properties (ix) and (x), we find that V_1' is anti-complete to V_2' in G'. Hence, by definition of V_1', V_2' and Properties (i), (iii), (xi) and (xii), we find that $G'[V_1' \cup V_2' \cup X_0]$ is a disjoint union of P_3's. For $i \in \{1, 2\}$, every vertex in $W_i' \cup X_i'$ is adjacent to precisely one vertex in V_i' by definition. As the last bipartite complementation operation did not affect these sets, this is still the case in G'. By Properties (ii)–(iv) and (vi), we find that $W_1' \cup W_2' \cup X_0 \cup X_1' \cup X_2'$ is an independent set. Then, by also using Properties (xiii) and (xiv) together with the definitions of X_1 and X_2, we find that no vertex in $W_i' \cup X_i'$ has any other neighbour in G' besides its neighbour in V_i'. Therefore G' is a disjoint union of trees and thus has bounded clique-width by Lemma 3. We conclude that G has bounded clique-width. This completes the proof of Lemma 13. □

We omit the proof of the next lemma.

Lemma 14. *The class of those $(\overline{2P_1 + P_2}, P_2 + P_3, K_5, C_5, C_4)$-free graphs that contain an induced C_6 has bounded clique-width.*

We now use Lemmas 11–14 and the fact that $(\overline{2P_1 + P_2}, P_2 + P_3, C_4, C_5, C_6)$-free graphs are chordal graphs, and so have bounded clique-width by Lemma 7, to obtain:

Theorem 1 (iii). *The class of $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs has bounded clique-width.*

References

1. Boliac, R., Lozin, V.V.: On the clique-width of graphs in hereditary classes. Proc. ISAAC 2002, LNCS 2518, 44–54 (2002)
2. Bonomo, F., Grippo, L.N., Milanič, M., Safe, M.D.: Graphs of power-bounded clique-width. arXiv abs/1402.2135 (2014)
3. Brandstädt, A., Engelfriet, J., Le, H.O., Lozin, V.V.: Clique-width for 4-vertex forbidden subgraphs. Theory of Computing Systems 39(4), 561–590 (2006)
4. Brandstädt, A., Klembt, T., Mahfud, S.: P_6- and triangle-free graphs revisited: structure and bounded clique-width. Discrete Mathematics and Theoretical Computer Science 8(1), 173–188 (2006)
5. Brandstädt, A., Kratsch, D.: On the structure of (P_5,gem)-free graphs. Discrete Applied Mathematics 145(2), 155–166 (2005)
6. Brandstädt, A., Le, H.O., Mosca, R.: Gem- and co-gem-free graphs have bounded clique-width. International Journal of Foundations of Computer Science 15(1), 163–185 (2004)
7. Brandstädt, A., Le, H.O., Mosca, R.: Chordal co-gem-free and (P_5,gem)-free graphs have bounded clique-width. Discrete Applied Mathematics 145(2), 232–241 (2005)
8. Brandstädt, A., Mahfud, S.: Maximum weight stable set on graphs without claw and co-claw (and similar graph classes) can be solved in linear time. Information Processing Letters 84(5), 251–259 (2002)

9. Brandstädt, A., Mosca, R.: On variations of P_4-sparse graphs. Discrete Applied Mathematics 129(2–3), 521–532 (2003)
10. Chudnovsky, M., Robertson, N., Seymour, P., Thomas, R.: The strong perfect graph theorem. Annals of Mathematics 164, 51–229 (2006)
11. Corneil, D.G., Rotics, U.: On the relationship between clique-width and treewidth. SIAM Journal on Computing 34, 825–847 (2005)
12. Courcelle, B.: Clique-width and edge contraction. Information Processing Letters 114(1–2), 42–44 (2014)
13. Courcelle, B., Makowsky, J.A., Rotics, U.: Linear time solvable optimization problems on graphs of bounded clique-width. Theory of Computing Systems 33(2), 125–150 (2000)
14. Dabrowski, K.K., Golovach, P.A., Paulusma, D.: Colouring of graphs with Ramsey-type forbidden subgraphs. Theoretical Computer Science 522, 34–43 (2014)
15. Dabrowski, K.K., Lozin, V.V., Raman, R., Ries, B.: Colouring vertices of triangle-free graphs without forests. Discrete Mathematics 312(7), 1372–1385 (2012)
16. Dabrowski, K.K., Paulusma, D.: Classifying the clique-width of H-free bipartite graphs. Proc. COCOON 2014, LNCS 8591, 489–500 (2014)
17. Dabrowski, K.K., Paulusma, D.: Clique-width of graph classes defined by two forbidden induced subgraphs. CoRR abs/1405.7092 (2014)
18. Fellows, M.R., Rosamond, F.A., Rotics, U., Szeider, S.: Clique-width is NP-Complete. SIAM Journal on Discrete Mathematics 23(2), 909–939 (2009)
19. Golumbic, M.C., Rotics, U.: On the clique-width of some perfect graph classes. International Journal of Foundations of Computer Science 11(03), 423–443 (2000)
20. Gurski, F.: Graph operations on clique-width bounded graphs. CoRR abs/cs/0701185 (2007)
21. Johansson, Ö.: Clique-decomposition, NLC-decomposition, and modular decomposition - relationships and results for random graphs. Congressus Numerantium 132, 39–60 (1998)
22. Kamiński, M., Lozin, V.V., Milanič, M.: Recent developments on graphs of bounded clique-width. Discrete Applied Mathematics 157(12), 2747–2761 (2009)
23. Kobler, D., Rotics, U.: Edge dominating set and colorings on graphs with fixed clique-width. Discrete Applied Mathematics 126(2–3), 197–221 (2003)
24. Lozin, V.V., Rautenbach, D.: On the band-, tree-, and clique-width of graphs with bounded vertex degree. SIAM Journal on Discrete Mathematics 18(1), 195–206 (2004)
25. Lozin, V.V., Rautenbach, D.: The tree- and clique-width of bipartite graphs in special classes. Australasian Journal of Combinatorics 34, 57–67 (2006)
26. Lozin, V.V., Volz, J.: The clique-width of bipartite graphs in monogenic classes. International Journal of Foundations of Computer Science 19(02), 477–494 (2008)
27. Makowsky, J.A., Rotics, U.: On the clique-width of graphs with few P_4's. International Journal of Foundations of Computer Science 10(03), 329–348 (1999)
28. Oum, S.I.: Approximating rank-width and clique-width quickly. ACM Transactions on Algorithms 5(1), 10 (2008)
29. Oum, S.I., Seymour, P.D.: Approximating clique-width and branch-width. Journal of Combinatorial Theory, Series B 96(4), 514–528 (2006)
30. Rao, M.: MSOL partitioning problems on graphs of bounded treewidth and clique-width. Theoretical Computer Science 377(1–3), 260–267 (2007)