A Reconfigurations Analogue of Brooks' Theorem

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Abstract. Let G be a simple undirected graph on n vertices with maximum degree Δ. Brooks' Theorem states that G has a Δ-colouring unless G is a complete graph, or a cycle with an odd number of vertices. To *recolour* G is to obtain a new proper colouring by changing the colour of one vertex. We show that from a k-colouring, $k > \Delta$, a Δ-colouring of G can be obtained by a sequence of $O(n^2)$ recolourings using only the original k colours unless

– G is a complete graph or a cycle with an odd number of vertices, or
– $k = \Delta + 1$, G is Δ-regular and, for each vertex v in G, no two neighbours of v are coloured alike.

We use this result to study the *reconfiguration graph* $R_k(G)$ of the k-colourings of G. The vertex set of $R_k(G)$ is the set of all possible k-colourings of G and two colourings are adjacent if they differ on exactly one vertex. It is known that

– if $k \leq \Delta(G)$, then $R_k(G)$ might not be connected and it is possible that its connected components have superpolynomial diameter,
– if $k \geq \Delta(G) + 2$, then $R_k(G)$ is connected and has diameter $O(n^2)$.

We complete this structural classification by settling the missing case:

– if $k = \Delta(G) + 1$, then $R_k(G)$ consists of isolated vertices and at most one further component which has diameter $O(n^2)$.

We also describe completely the computational complexity classification of the problem of deciding whether two k-colourings of a graph G of maximum degree Δ belong to the same component of $R_k(G)$ by settling the case $k = \Delta(G) + 1$. The problem is

– $O(n^2)$ time solvable for $k = 3$,
– PSPACE-complete for $4 \leq k \leq \Delta(G)$,
– $O(n)$ time solvable for $k = \Delta(G) + 1$,
– $O(1)$ time solvable for $k \geq \Delta(G) + 2$ (the answer is always yes).

1 Introduction

Definitions and Background Let $G = (V, E)$ denote a simple undirected graph and let k be a positive integer. A *k-colouring* of G is a function $\gamma : V \to \{1, 2, \ldots, k\}$ such that if $uv \in E$, $\gamma(u) \neq \gamma(v)$. The *k-colouring reconfiguration graph* of G has as its vertex set all possible k-colourings of G, and

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two k-colourings γ_1 and γ_2 are joined by an edge if, for some vertex $u \in V$, $\gamma_1(u) \neq \gamma_2(u)$, and, for all $v \in V \setminus \{u\}$, $\gamma_1(v) = \gamma_2(v)$; that is, if γ_1 and γ_2 *disagree* on exactly one vertex. The reconfiguration graph is denoted by $R_k(G)$.

The study of reconfiguration graphs of colourings began in [10,11]. The problem of deciding whether two 3-colourings of a graph G are in the same component of $R_3(G)$ was shown to be solvable in time $O(n^2)$ in [12]; it was also proved that the diameter of any component of $R_3(G)$ is $O(n^2)$. In contrast, in [5] the analagous problem for k-colourings, $k \geq 4$, was shown to be PSPACE-complete, and examples of reconfiguration graphs with components of superpolynomial diameter were given. In [2], reconfiguration graphs of k-colourings of chordal graphs were shown to be connected with diameter $O(n^2)$ whenever k is more than the size of the largest clique (and an infinite class of chordal graphs was described whose reconfiguration graphs have diameter $\Omega(n^2)$). In [1] this was generalized to show that if k is at least two greater than the treewidth $tw(G)$ then, again, $R_k(G)$ is connected with diameter $O(n^2)$. (Notice that if $k = tw(G) + 1$, then $R_k(G)$ might not be connected since, for example, G might be a complete graph on $tw(G) + 1$ vertices and then $R_k(G)$ contains no edges.)

Our Results We study reconfigurations of colourings for graphs of bounded maximum degree. The celebrated theorem of Brooks [8] states that a graph G with maximum degree Δ has a Δ-colouring unless it is the complete graph on $\Delta + 1$ vertices or a cycle with an odd number n of vertices; we denote these two graphs by $K_{\Delta+1}$ and C_n respectively. The question we address is: given a k-colouring γ of G, is there a path from γ to a Δ-colouring in $R_k(G)$? (Note that we are abusing our terminology. When we are working with $R_k(G)$, by a Δ-colouring we mean a k-colouring in which only Δ colours appear on the vertices.) Our first result provides a complete answer to this question. We require two definitions. A k-colouring γ of a graph is *frozen* if, for every vertex v, every colour except $\gamma(v)$ is used on a neighbour of v. Notice that a frozen colouring is an isolated vertex in $R_k(G)$. The length of a shortest path between colourings α and β in $R_k(G)$ is denoted by $d_k(\alpha, \beta)$. We state our results for connected graphs as other graphs can be considered component-wise.

Theorem 1. *Let G be a connected graph on n vertices with maximum degree Δ, and let $k \geq \Delta + 1$. Let α be a k-colouring of G. If α is not frozen and G is not $K_{\Delta+1}$ or, if n is odd, C_n, then there exists a Δ-colouring γ of G such that $d_k(\alpha, \gamma)$ is $O(n^2)$.*

Note that α can only be frozen if $k = \Delta + 1$, and only if G is Δ-regular. Let us briefly note that such colourings do exist: for example a 3-colouring of C_6 in which each colour appears exactly twice on vertices at distance 3, or a 4-colouring of the cube in which diagonally opposite vertices are coloured alike.

As we will see, the case $k = \Delta + 1$ is the only cause of difficulty in the proof of our first result. Using Theorem 1, however, we can, with the aid of one further lemma, give a characterization of $R_{\Delta+1}(G)$ which is our next result.

Theorem 2. *Let G be a connected graph on n vertices with maximum degree $\Delta \geq 3$. Let α and β be $(\Delta+1)$-colourings of G. If α and β are not frozen colourings, then $d_{\Delta+1}(\alpha, \beta)$ is $O(n^2)$. Moreover, it is possible to decide in time $O(n)$ whether or not there is a path between α and β in $R_{\Delta+1}(G)$.*

Theorem 2 implies that $R_{\Delta+1}(G)$ contains a number of isolated vertices (representing frozen colourings) plus, possibly, one further component. It is possible that the number of isolated vertices is zero (that is, there are no frozen $(\Delta+1)$-colourings; for example, consider 4-colourings of $K_{3,3}$), or that there are only isolated vertices (consider $R_4(K_4)$ for instance; and Brooks' theorem tells us that complete graphs are the only graphs for which $R_{\Delta+1}(G)$ is edgeless since other graphs have colourings in which only Δ colours are used and by recolouring any vertex with the unused colour we find a neighbouring colouring). We observe that the requirement that $\Delta \geq 3$ is necessary since, for example $R_3(C_n)$, n odd, has more than one component [10, 11].

Consequences of Our Results Our theorems complete both structural and algorithmic classifications for reconfigurations of colourings of graphs of bounded maximum degree.

In [9] it was noted that if $k \geq \Delta(G)+2$, $R_k(G)$ is connected with diameter $O(n^2)$. Combined with the results for general graphs noted above, and Theorem 1, we have the following summary of the structure of reconfiguration graphs:

- if $k \leq \Delta(G)$ then $R_k(G)$ might not be connected and it is possible that its connected components have superpolynomial diameter
- if $k = \Delta(G)+1$ then $R_k(G)$ consists of zero or more isolated vertices and at most one further component which has diameter $O(n^2)$ (if it exists).
- if $k \geq \Delta(G)+2$ then $R_k(G)$ is connected and has diameter $O(n^2)$.

And we summarise what is known about the computational complexity of the problem of deciding, given a graph and two k-colourings, whether the two colourings belong to the same connected component of $R_k(G)$ using Theorem 2 for the previously missing third case.

- $O(n^2)$ time solvable for $k = 3$,
- PSPACE-complete for $4 \leq k \leq \Delta(G)$,
- $O(n)$ time solvable for $k = \Delta(G)+1$,
- $O(1)$ time solvable for $k \geq \Delta(G)+2$ (the answer is always yes).

Related Work We note that reconfiguration graphs can be defined for any search problem: vertices correspond to solutions and edges join solutions that are "close" to one another; that is, solutions that differ as little as possible (for a given problem, there might be more than one way to define an edge relation). Reconfiguration graphs have been studied for a number of combinatorial problems; the questions asked are typically (as we have seen for colouring) is the graph connected?, what is the diameter of the graph (or of its connected components)?, how difficult is it to decide whether there is a path between a pair of given solutions? Problems studied include boolean satisfiability [13, 21],clique and vertex cover [16], independent set [6, 20], list edge colouring [17],shortest path [3, 4], and subset sum [15] (see also a recent survey [14]). Recent work has included looking at finding the *shortest* path in the reconfiguration graph between given solutions [19], and studying the fixed-parameter-tractability of these problems [7, 18, 23, 24].

Further preliminaries The degree of a vertex v is denoted by $\deg(v)$. A graph is k-degenerate if every induced subgraph has a vertex with degree at most k. It is well-known that a graph is k-degenerate if and only if there exists a *degeneracy ordering* $v_1, v_2, \ldots, v_n$ of its vertices such that v_i has at most k neighbours v_j with $j < i$. A graph is r-regular if for every vertex v, $\deg(v) = r$.

A remark about our proofs. A common aim is to find a path between a pair of colourings α and β in a reconfiguration graph. That is, to find a sequence of colourings $\gamma_0, \gamma_1, \ldots, \gamma_t$ with $\alpha = \gamma_0$, $\beta = \gamma_t$ such that adjacent colourings disagree on a single vertex. We think of this sequence as a *recolouring* sequence. If, for $1 \leq i \leq t$, v_i is the vertex on which γ_i and γ_{i-1} disagree, then we can think of β as being obtained from α by recolouring the vertices $v_1, \ldots, v_t$ in order. Therefore, rather than explicitly considering the reconfiguration graph, we will often seek to find a recolouring sequence; that is, to describe a sequence of vertices and to say which colour each vertex should be recoloured with. Then we will need to show that if we apply this recolouring sequence to α, we obtain β, and that all the intermediate colourings obtained are proper.

2 Proofs of Theorems

To prove our theorems, we need a number of lemmas that are mostly concerned with $(\Delta+1)$-colourings which, as we shall see, present the only real difficulty in proving Theorem 1. Some proofs are omitted for space reasons.

We define a number of terms we will use to describe vertices of G with respect to some $(\Delta+1)$-colouring. A vertex v is *locked* if Δ distinct colours appear on its neighbours. A vertex that is not locked is *free*. Clearly a vertex can be recoloured only if it is free. If v is locked and then one of its neighbour is recoloured and v becomes free, we say that v is *unlocked*. A vertex v is *superfree* if there is a colour $c \neq \Delta+1$ such that neither v nor any of its neighbours is coloured c. A vertex can only be recoloured with a colour other than $\Delta+1$ if it is superfree. Note there are $\Delta-1$ distinct colours that must appear on the Δ neighbours of v if it is not superfree. We say that G is in $(\Delta+1)$-*reduced form* if for every vertex v coloured with $(\Delta+1)$, v and each of its neighbours are locked. This implies that the distance between any pair of vertices coloured $(\Delta+1)$ is at least 3 as no vertex can have two neighbours coloured $(\Delta+1)$.

The key to proving Theorem 1 will be to show that from a $(\Delta+1)$-colouring one can recolour some of the vertices to arrive at a colouring in which colour $\Delta+1$ appears on fewer vertices. We begin by considering the case where the colour $\Delta+1$ appears on only one vertex. The following lemma is inspired by a proof of Brooks' theorem [22].

Lemma 1. *Let $G = (V, E)$ be a connected graph on n vertices with maximum degree $\varDelta \geq 3$, and let α be a $(\varDelta + 1)$-colouring of G with exactly one vertex v coloured $\varDelta + 1$. If G does not contain $K_{\varDelta+1}$ as a subgraph, then G can be recoloured to a $\varDelta$-colouring in $O(n)$ steps.*

Proof. We can assume that G is in $(\varDelta+1)$-reduced form since if v is not locked then we can immediately recolour it; if a neighbour of v is not locked then it can be recoloured and this will unlock v and allow us to recolour it.

Let us fix a labelling of the neighbours of v: let x_i be the neighbour such that $\alpha(x_i) = i$, $1 \leq i \leq \varDelta$. Our aim is to find a recolouring sequence that unlocks v. There is one recolouring sequence that we will use several times. Suppose that C is a connected component of a subgraph of G induced by two colours i and j, $\varDelta + 1 \notin \{i, j\}$, and no vertex coloured j in C is adjacent to v. First the vertices coloured j are recoloured with $\varDelta + 1$. Then the vertices coloured i are recoloured j, and finally the vertices initially coloured j are recoloured i. It is clear that all colourings are proper and the overall effect is to *swap* the colours i and j on C.

We say that any colouring γ where G is in $(\varDelta + 1)$-reduced form, only v is coloured $\varDelta + 1$ and $\gamma(x_i) = i$, $1 \leq i \leq \varDelta$, is *good.* For any good colouring γ, let G^{γ}_{ij} be the maximal connected component containing x_i of the subgraph of G induced by the vertices coloured i and j by γ.

We make some claims about good colourings. When we claim that v can be unlocked, it is implicit that colour $\varDelta + 1$ is not used on any other vertex in the graph so that unlocking v allows us to reach a colouring where $\varDelta+1$ is not used.

Claim 1: If γ is good and $x_j \notin G^{\gamma}_{ij}$, then v can be unlocked.

If $x_j \notin G^{\gamma}_{ij}$, then the only vertex adjacent to v in G^{γ}_{ij} is x_i. Thus the colours i and j can be swapped on G^{γ}_{ij}. Then v has two neighbours with colour j and is unlocked.

Claim 2: If γ is good and G^{γ}_{ij} is not a path from x_i to x_j, then v can be unlocked.

By Claim 1, we can assume that x_i and x_j are in G^{γ}_{ij}. They must have degree 1 in G_{ij} since, as G is in $(\varDelta + 1)$-reduced form, they are locked. Suppose that G^{γ}_{ij} is not a path and consider the shortest path in G^{γ}_{ij} from x_i to x_j, and the vertex w nearest to x_i on the path that has degree more than 2. Then w has at least three neighbours coloured alike in G and is superfree and can be recoloured with a colour other than i, j or $\varDelta+1$. Call this new colouring γ' and note that, by the choice of w, $G^{\gamma'}_{ij}$ does not contain x_j. Now Claim 1 implies Claim 2.

As G is $K_{\varDelta+1}$-free, v and its neighbours are not a clique so we can assume that x_1 and x_2 are not adjacent. Let u be the unique neighbour of x_1 coloured 2. For a good colouring γ, note that u is in G^{γ}_{12}, and let H^{γ}_{23} be the component of the subgraph of G induced by the vertices with colour 2 and 3 that contains u.

Claim 3: If γ is good and u has more than one neighbour in H^{γ}_{23}, then v can be unlocked.

If G^{γ}_{12} is not a path, then use Claim 2. Otherwise u has two neighbours coloured 1; if u has two neighbours in H^{γ}_{23}, then it also has two neighbours coloured 3 and is superfree. Recolour it and apply Claim 1.

Claim 4: If γ is good and H_{23}^{γ} is a path, then v can be unlocked.

By Claim 2 we can assume G_{23}^{γ} is a path. If $H_{23}^{\gamma} = G_{23}^{\gamma}$, then we can use Claim 3. So we assume $H_{23}^{\gamma} \neq G_{23}^{\gamma}$ and so $x_2, x_3 \notin H_{23}$ and H_{23} contains no neighbour of v. Let γ' be the colouring obtained by swapping the colours 2 and 3 on H_{23}^{γ}.

By Claim 3, u is an endvertex of H_{23}^{γ}. Let the other endvertex be w. (If $w = u$, then u has no neighbour coloured 3 and can be recoloured. Then use Claim 2.)

If $G_{12}^{\gamma'}$ is not a path from x_1 to x_2, we use Claim 2. If it is such a path, then let the unique neighbour of x_1 in $G_{12}^{\gamma'}$ be y and clearly $y \in H_{23}^{\gamma}$. From x_2 traverse $G_{12}^{\gamma'}$ until the last vertex z that is also in G_{12}^{γ} is reached. Let t be the next vertex along from z towards x_1 in $G_{12}^{\gamma'}$. Clearly t is also in H_{23}^{γ}. In fact, we can assume that $w = y = t$ since if y or t has degree 2 in H_{23} as well as in $G_{12}^{\gamma'}$ it has two neighbours coloured 1 and two neighbours coloured 3 in γ' and is superfree. It can be recoloured and then Claim 2 is used.

So x_1wz is coloured 131 in γ so is in G_{13}^{γ}. Then z is in both G_{13}^{γ} and G_{12}^{γ} so is superfree and can be recoloured so that Claim 2 can be used. This completes the proof of Claim 4.

To complete the proof: we know that the initial colouring α is good. If none of the four claims can be used, then consider H_{23}^{α}. We know that u has degree 1 in H_{23} but H_{23} is not a path. So traversing edges away from u in H_{23}^{α}, let s be the first vertex reached with degree 3. Then s is superfree and can be recoloured so that H_{23} becomes a path, and then Claim 4 can be used. □

In Lemma 3, we shall see how, for *regular* graphs, the number of vertices coloured $\Delta + 1$ can be reduced when more than one is present. First we need some definitions and a lemma. Let P be a path:

- P is *nearly* $(\Delta + 1)$*-locked* if its endvertices are locked and coloured $\Delta + 1$;
- P is $(\Delta + 1)$*-locked* if it is nearly $(\Delta + 1)$-locked and every vertex on the path is locked.

Lemma 2. *Let G be a graph in $(\Delta+1)$-reduced form. If G has a $(\Delta+1)$-locked path P, then each endvertex of P is an endvertex of an $(\Delta + 1)$-locked path of length 3.*

A path is *nice* if it is a nearly $(\Delta + 1)$-locked path, it contains free vertices and the endvertices and their neighbours are the only locked vertices. Notice that a nice path is not necessarily induced and, in particular, may contain a $(\Delta + 1)$-locked subpath.

Lemma 3. *Let G be a connected regular graph on n vertices with degree $\Delta \geq 3$, let α be a $(\Delta+1)$-colouring of G, and suppose that G is in $(\Delta+1)$-reduced form. If G has at least two $(\Delta + 1)$-locked vertices and is not frozen, then there exists a $(\Delta + 1)$-colouring γ of G, such that $d_{\Delta+1}(\alpha, \gamma) = O(n)$ and fewer vertices are coloured $\Delta + 1$ with γ than with α.*

Proof. We consider a number of cases.

Case 1: There exists a free vertex u adjacent to a $(\Delta+1)$-locked path P.

Let b be the vertex on the path adjacent to u. As b is locked it has a neighbour a coloured $\Delta+1$. Let c be a neighbour of b on P other than a. As c is locked it has a neighbour d coloured $\Delta+1$.

Since G is in $(\Delta+1)$-reduced form, u is not adjacent to a or d but might be adjacent to c. In each case, it is routine to verify that by recolouring u to $\Delta+1$, b and c can both be recoloured unlocking a and d and allowing them to be recoloured. Thus the number of vertices coloured $\Delta+1$ is reduced.

Case 2: G has a nice path.

Let P be a shortest nice path. Let the endpoints be v and w with neighbours s and t on P respectively. If s and t are adjacent, then the path $vstw$ is $(\Delta+1)$-locked and has a free vertex adjacent to s so use Case 1. Thus assume that P is induced since the presence of any other edge would imply either a shorter nice path could be found or that the graph was not in $(\Delta+1)$-reduced form.

We use induction on the number ℓ of free vertices in P to show that there is a sequence of recolourings that lead to a colouring that has fewer vertices coloured $\Delta+1$.

If $\ell=1$, let u be the free vertex in P. Recolour u to $\Delta+1$. Now s and t have two neighbours coloured $\Delta+1$ and can be recoloured. Then v and w are unlocked and can both be recoloured, and this leaves one vertex on P coloured $\Delta+1$ rather than two.

Suppose that $\ell=2$. Let $P=vsu_1u_2tw$ where u_1 and u_2 are free vertices.

Subcase 2.1: *u_1 and u_2 do not share a neighbour.* Let x_1 and x_2 be neighbours of u_1 and u_2 not in P. Clearly $x_1 \neq x_2$ and u_1x_2 and u_2x_1 are not edges.

Subcase 2.1.1: *x_1 is locked.* We know x_1 has a $(\Delta+1)$-locked neighbour, and this must be v (if it is some other vertex z, then vsu_1x_1z is a nice path that is shorter than P).

Suppose x_1s is not an edge. Recolour u_1 to $\Delta+1$. This unlocks x_1 which can be recoloured with $\alpha(u_1)$ which, in turn, unlocks v and allows us to recolour it with $\alpha(x_1)$. If u_1 is free, it can be recoloured and the number of vertices coloured $\Delta+1$ is reduced and we are done. If u_1 is locked, then note that s has been unlocked (as it no longer has a neighbour coloured $\alpha(u_1)$). Thus we can recolour s and then recolour u_1 with $\alpha(s)$ and again we have removed one instance of the colour $\Delta+1$.

Suppose instead that x_1s is an edge. Notice that $\alpha(s)$, $\alpha(u_1)$ and $\alpha(x_1)$ are distinct as the three vertices form a triangle. Recolour u_1 with $\Delta+1$ and then s with $\alpha(u_1)$. Now v is unlocked and can be recoloured with $\alpha(s)$. If u_1 is free, then recolour it and we are done. Otherwise this sequence of recolourings leaves u_1 locked (with $\alpha(u_1)$ and $\alpha(x_1)$ as the colours on s and x_1 respectively). So, from α, we do the following instead: again start by recolouring u_1 with $\Delta+1$, but then recolour x_1 with $\alpha(u_1)$ to unlock v. Now that $\alpha(x_1)$ is not used on a neighbour of u_1, u_1 is free and can be recoloured.

Subcase 2.1.2: x_1 *is free.* If x_2 is locked, we can, by symmetry, use the previous subcase, so we can assume that both x_1 and x_2 are free. Recolour u_2 to $\Delta + 1$. Then t is unlocked and can be recoloured which, in turn, unlocks w allowing us to recolour it too. If u_2 is free, we recolour it and are done. If u_1 is free, we recolour it and unlock u_2 and, again, recolour it.

If u_1 and u_2 are both locked, observe that x_1 is still free as it has no neighbour coloured $\Delta+1$ since u_2x_1 is not an edge. Recolour x_1 to $\Delta+1$, and then recolour u_1 to $\alpha(x_1)$. Note that now s has no neighbour coloured $\alpha(u_1)$ and is free and can be recoloured so that v is unlocked and can also be recoloured. By recolouring u_1, we also unlock u_2, so we recolour it and are done.

Subcase 2.2: u_1 *and* u_2 *share a neighbour.* Let x_1 be a neighbour of u_1 and u_2. Since P is induced, x_1 is not in P. If x_1 is locked, then let its neighbour coloured $\Delta + 1$ be y. Then vsu_1x_1y is a shorter nice path unless $y = v$. By an analagous argument we need $y = w$. This contradiction tells us that x_1 must be free.

If x_1 is joined to both s and t, then vsx_1tw is a shorter nice path. So, without loss of generality, assume that x_1t is not an edge. Thus as u_2 has a neighbour that is not adjacent to x_1, x_1 has a neighbour x_3 that is not adjacent to u_2.

Subcase 2.2.1: $x_3 = s$. Recolour u_1 with $\Delta+1$ and then s with $\alpha(u_1)$. Now v is unlocked and can be recoloured with $\alpha(s)$. If u_1 is free, then recolour it and we are done. If u_2 or x_1 is still free, then recolour one of them to unlock u_1, which in turn can be recoloured and are done. Otherwise this sequence of recolourings leaves u_1, u_2 and x_1 locked so x_1 is the only neighbour of u_2 coloured $\alpha(x_1)$. So, from α, we do the following instead: recolour x_1 with $\Delta+1$ to unlock s and then v. If x_1 can be recoloured, then we do so and are done. Otherwise notice that $\alpha(x_1)$ is not used on a neighbour of u_2. It is thus free and can be recoloured to unlock x_1 and allow us to recolour it.

Subcase 2.2.2: $x_3 \neq s$*, and* x_3 *is free.* First, suppose x_3s is an edge. Recolour u_2 to $\Delta + 1$, t to $\alpha(u_2)$ and w to $\alpha(t)$. If either u_2 or one of its neighbours is now free, u_2 can be recoloured and we are done. Otherwise u_1, u_2 and x_1 are all locked, but x_3 is still free since it has no neighbour coloured $\Delta + 1$. Recolour x_3 to $\Delta + 1$ to unlock x_1; then recolour x_1 to unlock and recolour u_2. As x_3s is an edge, s has two neighbours coloured $\Delta + 1$. Thus we recolour s to unlock v.

If x_3t is an edge we can use a similar argument. So suppose x_3s and x_3t are not edges. Recolour u_2 to $\Delta + 1$, to unlock and recolour first t and then w. It is possible to recolour u_2 unless it and all its neighbours are locked. This implies that u_1, x_1 and u_2 are locked. We consider two subcases.

Subcase 2.2.2.1: u_1x_3 *is not an edge.* We recolour x_3 to $\Delta + 1$ to unlock and recolour x_1 and then u_2. Notice that u_1 is now free since it has no neighbour coloured $\Delta + 1$. Recoloured u_1 unlocks s, so we recolour it, which in turn unlocks v. Observe that x_1 now has two neighbours u_1 and x_3 with colour $\Delta+1$ so is free. If u_1 or u_3 is free, we can recolour at least one of them directly and we are done. Otherwise, we recolour x_1 so that x_3 and u_1 can now be recoloured.

Subcase 2.2.2.2: u_1x_3 *is an edge.* Recolour u_3 to $\Delta + 1$, then recolour u_1, s and v. Observe that x_1 now has two neighbours u_2 and u_3 with colour $\Delta + 1$. If

u_2 or u_3 are free, we are done. Otherwise, recolour x_1, then recolour u_2 and x_3, and we are done.

Subcase 2.2.3: *$x_3 \neq s$, and x_3 is locked.* Then x_3 has a $(\Delta+1)$-locked neighbour y. If $y = v$, the path $H = vx_3x_1u_2tw$ is nice with two free vertices x_1 and u_2. Furthermore, u_1 is free and a neighbour of x_1 and u_2, in which case H satisfies the previous subcase unless x_3 and t are adjacent in which case use Subcase 2.1. A similar argument can be made if $y = w$ or $y \notin \{v, w\}$.

This completes the case $\ell = 2$.

Now suppose that for all $i < \ell$, if there is a nice path containing i free vertices, the number of vertices coloured $\Delta + 1$ can be reduced. Suppose that the shortest such path is $P = vsu_1u_2 \ldots u_\ell tw$ where $\ell \geq 3$. We recolour u_ℓ to $\Delta + 1$, then t and then w. If u_ℓ or one of its neighbours is free, then u_ℓ can be recoloured and we are done. Otherwise, u_ℓ and $u_{\ell-1}$ are locked. Consider the path $P' = vsu_1 \ldots u_{\ell-2}u_{\ell-1}u_\ell$. By our inductive hypothesis, the number of colour $\Delta + 1$ vertices in P' can be reduced. Case 2 is complete.

After Cases 1 and 2 we are left with:

Case 3: There does not exist a free vertex adjacent to a $(\Delta+1)$-locked path and G has no nice path.

As G contains more than one $(\Delta+1)$-locked vertex, it contains a nearly $(\Delta+1)$-locked path; let P be the shortest and let v and w be its endvertices. As G is in $(\Delta+1)$-reduced form, v, w and their neighbours are locked. If P contains no other vertices, it is $(\Delta+1)$-locked. Otherwise, since there are no nice paths, P contains another locked vertex u. Let y be the neighbour of u coloured $\Delta+1$. If y is on P, then we can assume, without loss of generality, that it is not between v and u. Then, whether or not y is on P, the subpath from v to u plus the edge uy is a shorter nearly $(\Delta+1)$-locked path. This contradiction proves that G must contain a $(\Delta+1)$-locked path.

As G is not frozen, it contains a free vertex. Let Q be the shortest path in G that joins a free vertex to a $(\Delta+1)$-locked vertex. Let v be the $(\Delta+1)$-locked endvertex. So v is an endpoint of a $(\Delta+1)$-locked path R, and, by Lemma 2, we can assume that R has length 3.

Let u be the endvertex of Q that is free. By the minimality of Q, u is the only free vertex in Q. Let a be the neighbour of u in Q. As a is locked it has a $(\Delta+1)$-locked neighbour z. Thus we must have $z = v$ and $Q = vau$.

Let $R = wtsv$. Observe that us, ut, uv and uw cannot be edges as no locked path has a free neighbour. Thus the vertices of R and Q other than v are distinct. Consider the (not necessarily induced) path $M = wtsvau$. Notice also that at is not an edge else the free vertex u is adjacent to the $(\Delta+1)$-locked path $vatw$.

Suppose M is an induced path. Recolour u with $\Delta+1$ to unlock and recolour a and then v. If u is not locked, then recolour and we are done. Else notice that the vertices v and s are free, and the vertices u, a, t, w are locked. Consequently, we have that M is a nice path, and by Case 2 we are done.

The only edge that might be present among the vertices of M is as so suppose this exists. Recolour u with $\Delta+1$ to unlock and recolour first a and then v. If u

or any of its neighbours are free, u can be recoloured and we are done. Otherwise note that recoloured v unlocks s. It follows that the path $H = uastw$ is nice, and we can use Case 2. This completes Case 3.

As each vertex is recoloured a constant number of times, the lemma follows. □

We need one final lemma before we prove Theorem 1.

Lemma 4. *Let $G = (V, E)$ be a connected graph on n vertices with maximum degree $\Delta \geq 1$ and degeneracy $\Delta - 1$. Let α be a $(\Delta + 1)$-colouring of G. Then there exists a Δ-colouring γ of G such that $d_{\Delta+1}(\alpha, \gamma) \leq n^2$.*

Proof (of Theorem 1). If $k > \Delta + 1$, then, by Brooks' Theorem, a Δ-colouring γ exists in $R_k(G)$ unless G is complete or an odd cycle. We know that, in this case, $R_k(G)$ is connected and has diameter $O(n^2)$ so certainly $d_k(\alpha, \gamma)$ is $O(n^2)$.

Suppose that $k = \Delta + 1$. If G is $(\Delta - 1)$-degenerate, the result follows from Lemma 4. We claim that the only graphs with maximum degree Δ that are not $(\Delta - 1)$-degenerate are Δ-regular graphs. To see this, consider a smallest possible counterexample G that has degeneracy and maximum degree Δ and contains a vertex v with $\deg(v) < \Delta$. Suppose $G - v$ has degeneracy Δ. Then, by the minimality of G, we find that $G - v$ is Δ-regular. This would mean that every neighbor of v in G has more than Δ neighbours, which is not possible. Hence, $G - v$ must have degeneracy $\Delta - 1$. But every induced subgraph of G is either an induced subgraph of $G - v$ or contains v, and, in either case, must contain a vertex of degree less than Δ contradicting the claim that G has degeneracy Δ.

So we can suppose now that G is Δ-regular and in $(\Delta+1)$-reduced form with α: if not, we try to recolour each vertex with colour $\Delta + 1$ either directly or by first recolouring one of its neighbours. Repeatedly applying Lemma 3 starting from α, we obtain a $(\Delta + 1)$-colouring γ' in $O(n^2)$ steps such that at most one vertex is coloured $(\Delta + 1)$ with γ'. Lemma 1 can now be applied to obtain a Δ-colouring γ from γ' in $O(n)$ steps. Consequently $d_{\Delta+1}(\alpha, \gamma) \leq O(n^2)$ as required. □

We finish the section by considering Theorem 2. First we need:

Lemma 5. *Let $G = (V, E)$ be a connected graph on n vertices with maximum degree $\Delta \geq 3$. Let γ_1 and γ_2 be Δ-colourings of G. Then $d_{\Delta+1}(\gamma_1, \gamma_2)$ is $O(n^2)$.*

The lemma says that there is a path between any pair of Δ-colourings, but, because we are working with $R_{\Delta+1}(G)$, the intermediate colourings might use $\Delta + 1$ colours.

Proof (of Theorem 2). Theorem 1 implies that from each of α and β there is a path in $R_{\Delta+1}$ to a Δ-colouring; Lemma 5 implies that there is a path between these two Δ-colourings that completes the path from α to β. Consequently, it is possible to decide in $O(n)$ time whether or not there is a path between α and β in $R_{\Delta+1}(G)$: it is necessary only to check for each vertex v in G, for each of α and β, whether v and its neighbours use every colour in $\{1, 2, \ldots, \Delta + 1\}$. If they do not, neither colouring is frozen so there is a path between them. □

3 Conclusions

We have completed the study of reconfiguration graphs of graphs of bounded degree by considering the case where the number of colours is one more than the maximum degree. In Theorem 2, we showed that the reconfiguration graph contains isolated vertices and one further component. As it is easy to recognize a frozen colouring, this also means that we can decide in polynomial time whether a given pair of colourings belong to the same component. We make two additional observations about when the reconfiguration graph can have isolated vertices.

Corollary 1. *Let G be a connected regular graph on n vertices with maximum degree $\Delta \geq 3$. If $n \not\equiv 0 \mod (\Delta+1)$ then $R_{\Delta+1}(G)$ has diameter $O(n^2)$.*

Proof. Let γ be a frozen colouring of G. Let $V_1, V_2, \dots, V_{\Delta+1}$ be the colour classes of γ. Suppose there exist integers i, j such that $|V_i| > |V_j|$. Because γ is a frozen colouring each $v \in V_i$ has a neighbour in V_j. Hence there is a vertex $u \in V_j$ with at least two neighbours in V_i. Since u has Δ neighbours, it follows that u is free and can thus be recoloured, a contradiction. Therefore $|V_1| = \dots = |V_{\Delta+1}|$. We have proved that whenever G has a frozen colouring, $n \equiv 0 \mod (\Delta+1)$, and by Theorem 2 if there is no frozen colouring, $R_{\Delta+1}(G)$ is connected. □

Corollary 2. *Let G be a connected graph with maximum degree $\Delta \geq 3$ and degeneracy $(\Delta-1)$. Then $R_{\Delta+1}(G)$ is connected with diameter $O(n^2)$.*

Proof. The result follows immediately from Theorem 2 by observing that a $(\Delta-1)$-degenerate graph has a vertex with at most $\Delta-1$ neighbours and is thus free in any $(\Delta+1)$-colouring of G. □

Cereceda [9] conjectured that the diameter of the reconfiguration graph on $(k+2)$-colourings of a k-degenerate graph on n vertices is $O(n^2)$. This conjecture has been answered in the positive for values of $k \in \{1, \Delta\}$ [9]. By the previous corollary, we further confirm this conjecture for the value $k = \Delta - 1$.

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