# Clique-width of Graph Classes Defined by Two Forbidden Induced Subgraphs*

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**Abstract.** If a graph has no induced subgraph isomorphic to any graph in a finite family $\{H_1, \ldots, H_p\}$, it is said to be $(H_1, \ldots, H_p)$-free. The class of $H$-free graphs has bounded clique-width if and only if $H$ is an induced subgraph of the 4-vertex path $P_4$. We study the (un)boundedness of the clique-width of graph classes defined by two forbidden induced subgraphs $H_1$ and $H_2$. Prior to our study it was not known whether the number of open cases was finite. We provide a positive answer to this question. To reduce the number of open cases we determine new graph classes of bounded clique-width and new graph classes of unbounded clique-width. For obtaining the latter results we first present a new, generic construction for graph classes of unbounded clique-width. Our results settle the boundedness or unboundedness of the clique-width of the class of $(H_1, H_2)$-free graphs

(i) for all pairs $(H_1, H_2)$, both of which are connected, except two non-equivalent cases, and
(ii) for all pairs $(H_1, H_2)$, at least one of which is not connected, except 11 non-equivalent cases.

We also consider classes characterized by forbidding a finite family of graphs $\{H_1, \ldots, H_p\}$ as subgraphs, minors and topological minors, respectively, and completely determine which of these classes have bounded clique-width. Finally, we show algorithmic consequences of our results for the graph colouring problem restricted to $(H_1, H_2)$-free graphs.

**Keywords:** clique-width, forbidden induced subgraph, graph class

## 1 Introduction

Clique-width is a well-known graph parameter studied both in a structural and in an algorithmic context; we refer to the surveys of Gurski [24] and Kamiński, Lozin and Milanič [27] for an in-depth study of the properties of clique-width. We are interested in determining whether the clique-width of some given class of graphs is *bounded*, that is, is there a constant $c$ such that every graph from the class has

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clique-width at most $c$. For this purpose we study classes of graphs in which one or more specified graphs are forbidden as a "pattern". In particular, we consider classes of graphs that contain no graph from some specified family $\{H_1, \ldots, H_p\}$ as an *induced subgraph*; such classes are said to be $(H_1, \ldots, H_p)$-free. Our research is well embedded in the literature, as there are many papers that determine the clique-width of graph classes characterized by one or more forbidden induced subgraphs; see e.g. [1,2,3,4,5,6,7,8,9,10,14,15,16,17,23,31,32,33,34].

As we show later, it is not difficult to verify that the class of $H$-free graphs has bounded clique-width if and only if $H$ is an induced subgraph of the 4-vertex path $P_4$. Hence, it is natural to consider the following problem:

*For which pairs $(H_1, H_2)$ does the class of $(H_1, H_2)$-free graphs have bounded clique-width?*

In this paper we address this question by narrowing the gap between the known and open cases significantly; in particular we show that the number of open cases is finite. We emphasise that the *underlying* research question is: what kind of properties of a graph class ensure that its clique-width is bounded? Our paper is to be interpreted as a further step towards this direction, and in our research project (see also [3,15,17]) we aim to develop general techniques for attacking a number of the open cases simultaneously.

**Algorithmic Motivation.** For problems that are NP-complete in general, one naturally seeks to find subclasses of graphs on which they are tractable, and graph classes of bounded clique-width have been studied extensively for this purpose, as we discuss below.

Courcelle, Makowsky and Rotics [13] showed that all $\mathrm{MSO}_1$ graph problems, which are problems definable in Monadic Second Order Logic using quantifiers on vertices but not on edges, can be solved in linear time on graphs with clique-width at most $c$, provided that a $c$-expression of the input graph is given. Later, Espelage, Gurski and Wanke [19], Kobler and Rotics [28] and Rao [43] proved the same result for many non-$\mathrm{MSO}_1$ graph problems. Although computing the clique-width of a given graph is NP-hard, as shown by Fellows, Rosamond, Rotics and Szeider [20], it is possible to find an $(8^c - 1)$-expression for any $n$-vertex graph with clique-width at most $c$ in cubic time. This is a result of Oum [38] after a similar result (with a worse bound and running time) had already been shown by Oum and Seymour [39]. Hence, the NP-complete problems considered in the aforementioned papers [13,19,28,43] are all polynomial-time solvable on any graph class of bounded clique-width even if no $c$-expression of the input graph is given.

As a consequence of the above, when solving an NP-complete problem on some graph class $\mathcal{G}$, it is natural to try to determine *first* whether the clique-width of $\mathcal{G}$ is bounded. In particular this is the case if we aim to determine the computational complexity of some NP-complete problem when restricted to graph classes characterized by some common type of property. This property may be the absence of a family of forbidden induced subgraphs $H_1, \ldots, H_p$ and we may want to classify for which families of graphs $H_1, \ldots, H_p$ the problem is still NP-hard and for which ones it becomes polynomial-time solvable (in order to increase our understanding of the hardness of the problem in general). We give examples later.

**Our Results.** In Section 2 we state a number of basic results on clique-width and two results on $H$-free bipartite graphs that we showed in a very recent paper [17]; we need these results for proving our new results. We then identify a number of new classes of $(H_1, H_2)$-free graphs of bounded clique-width (Section 3) and unbounded clique-width (Section 4). In particular, the new unbounded cases are obtained from a new, general construction for graph classes of unbounded clique-width. In Section 5, we first observe for which graphs $H_1$ the class of $H_1$-free graphs has bounded clique-width. We then present our main theorem that gives a summary of our current knowledge of those pairs $(H_1, H_2)$ for which the class of $(H_1, H_2)$-free graphs has bounded clique-width and unbounded clique-width, respectively.[1] In this way we are able to narrow the gap to 13 open cases (up to some equivalence relation, which we explain later); when we only consider pairs $(H_1, H_2)$ of connected graphs the number of non-equivalent open cases is only two. In order to present our summary, we will need several results from the papers listed above. We also consider graph classes characterized by forbidding a finite family of graphs $\{H_1, \ldots, H_p\}$ as subgraphs, minors and topological minors, respectively. For these containment relations we are able to completely determine which of these classes have bounded clique-width.

**Algorithmic Consequences.** Our results are of interest for any NP-complete problem that is solvable in polynomial time on graph classes of bounded clique-width. In Section 6 we give a concrete application of our results by considering the well-known Colouring problem, which is that of testing whether a graph can be coloured with at most $k$ colours for some given integer $k$ and which is solvable in polynomial time on any graph class of bounded clique-width [28]. The complexity of Colouring has been studied extensively for $(H_1, H_2)$-free graphs [14,16,22,29,35,44], but a full classification is still far from being settled. Many of the polynomial-time results follow directly from bounding the clique-width in such classes. As such this forms a direct motivation for our research.

**Related Work.** We finish this section by briefly discussing one related result. A graph class $\mathcal{G}$ has power-bounded clique-width if there is a constant $r$ so that the class consisting of all $r$-th powers of all graphs from $\mathcal{G}$ has bounded clique-width. Recently, Bonomo, Grippo, Milanič and Safe [2] determined all pairs of connected graphs $H_1, H_2$ for which the class of $(H_1, H_2)$-free graphs has power-bounded clique-width. If a graph class has bounded clique-width, it has power-bounded clique-width. However, the reverse implication does not hold in general. The latter can be seen as follows. Bonomo et al. [2] showed that the class of $H$-free graphs

[1] Before finding the combinatorial proof of our main theorem we first obtained a computer-assisted proof using Sage [46] and the Information System on Graph Classes and their Inclusions [18] (which keeps a record of classes for which boundedness or unboundedness of clique-width is known). In particular, we would like to thank Nathann Cohen and Ernst de Ridder for their help.

has power-bounded clique-width if and only if $H$ is a linear forest (recall that such a class has bounded clique-width if and only if $H$ is an induced subgraph of $P_4$). Their classification for connected graphs $H_1, H_2$ is the following. Let $S_{1,i,j}$ be the graph obtained from a 4-vertex star by subdividing one leg $i-1$ times and another leg $j-1$ times. Let $T_{1,i,j}$ be the line graph of $S_{1,i,j}$. Then the class of $(H_1, H_2)$-free graphs has power-bounded clique-width if and only if one of the following two cases applies: (i) one of $H_1, H_2$ is a path or (ii) one of $H_1, H_2$ is isomorphic to $S_{1,i,j}$ for some $i, j \geq 1$ and the other one is isomorphic to $T_{1,i',j'}$ for some $i', j' \geq 1$. In particular, the classes of power-unbounded clique-width were already known to have unbounded clique-width.

## 2 Preliminaries

Below we define the graph terminology used throughout our paper. Let $G$ be a graph. The set $N(u) = \{v \in V(G) \mid uv \in E(G)\}$ is the *(open) neighbourhood* of $u \in V(G)$ and $N[u] = N(u) \cup \{u\}$ is the *closed neighbourhood* of $u \in V(G)$. The *degree* of a vertex in a graph is the size of its neighbourhood. The *maximum degree* of a graph is the maximum vertex degree. For a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of $G$ *induced* by $S$, which has vertex set $S$ and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. If $S = \{s_1, \ldots, s_r\}$ then, to simplify notation, we may also write $G[s_1, \ldots, s_r]$ instead of $G[\{s_1, \ldots, s_r\}]$. Let $H$ be another graph. We write $H \subseteq_i G$ to indicate that $H$ is an induced subgraph of $G$.

Let $\{H_1, \ldots, H_p\}$ be a set of graphs. We say that a graph $G$ is $(H_1, \ldots, H_p)$-*free* if $G$ has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$. If $p = 1$, we may write $H_1$-free instead of $(H_1)$-free. The *disjoint union* $G + H$ of two vertex-disjoint graphs $G$ and $H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We denote the disjoint union of $r$ copies of $G$ by $rG$.

For positive integers $s$ and $t$, the *Ramsey number* $R(s, t)$ is the smallest number $n$ such that all graphs on $n$ vertices contain an independent set of size $s$ or a clique of size $t$. Ramsey's Theorem [40] states that such a number exists for all positive integers $s$ and $t$.

The *clique-width* of a graph $G$, denoted $\mathrm{cw}(G)$, is the minimum number of labels needed to construct $G$ by using the following four operations:

1. creating a new graph consisting of a single vertex $v$ with label $i$ (denoted by $i(v)$);
2. taking the disjoint union of two labelled graphs $G_1$ and $G_2$ (denoted by $G_1 \oplus G_2$);
3. joining each vertex with label $i$ to each vertex with label $j$ ($i \neq j$, denoted by $\eta_{i,j}$);
4. renaming label $i$ to $j$ (denoted by $\rho_{i \to j}$).

An algebraic term that represents such a construction of $G$ and uses at most $k$ labels is said to be a *$k$-expression* of $G$ (i.e. the clique-width of $G$ is the minimum $k$ for which $G$ has a $k$-expression). For instance, an induced path on four consecutive vertices $a, b, c, d$ has clique-width equal to 3, and the following 3-expression can be used to construct it:

$$\eta_{3,2}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))).$$

Alternatively, any $k$-expression for a graph $G$ can be represented by a rooted tree, where the leaves correspond to the operations of vertex creation and the internal nodes correspond to the other three operations. The rooted tree representing the above $k$-expression is depicted in Fig. 1. A class of graphs $\mathcal{G}$ has *bounded* clique-width if there is a constant $c$ such that the clique-width of every graph in $\mathcal{G}$ is at most $c$; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

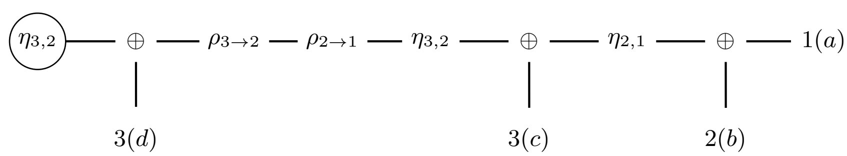


Fig. 1: The rooted tree representing a 3-expression for $P_4$.

Let $G$ be a graph. The *complement* of $G$, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and only if these vertices are not adjacent in $G$.

Let $G$ be a graph. We define the following five operations. The *contraction* of an edge $uv$ removes $u$ and $v$ from $G$, and replaces them by a new vertex made adjacent to precisely those vertices that were adjacent to $u$ or $v$ in $G$. By definition, edge contractions create neither self-loops nor multiple edges. The *subdivision* of an edge $uv$ replaces $uv$ by a new vertex $w$ with edges $uw$ and $vw$. Let $u \in V(G)$ be a vertex that has exactly two neighbours $v, w$, and moreover let $v$ and $w$ be non-adjacent. The *vertex dissolution* of $u$ removes $u$ and adds the edge $vw$. For an induced subgraph $G' \subseteq_i G$, the *subgraph complementation* operation (acting on $G$ with respect to $G'$) replaces every edge present in $G'$ by a non-edge, and vice versa. Similarly, for two disjoint vertex subsets $X$ and $Y$ in $G$, the *bipartite complementation* operation with respect to $X$ and $Y$ acts on $G$ by replacing every edge with one end-vertex in $X$ and the other one in $Y$ by a non-edge and vice versa.

We now state some useful facts for dealing with clique-width. We will use these facts throughout the paper. Let $k \geq 0$ be a constant and let $\gamma$ be some graph operation. We say that a graph class $\mathcal{G}'$ is $(k, \gamma)$*-obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

(i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing $\gamma$ at most $k$ times, and
(ii) for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from $G$ by performing $\gamma$ at most $k$ times.

If we do not impose a finite upper bound $k$ on the number of applications of $\gamma$ then we write that $\mathcal{G}'$ is $(\infty, \gamma)$-obtained from $\mathcal{G}$.

We say that $\gamma$ *preserves* boundedness of clique-width if for any finite constant $k$ and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is $(k, \gamma)$-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [31].

Fact 2. Subgraph complementation preserves boundedness of clique-width [27].

Fact 3. Bipartite complementation preserves boundedness of clique-width [27].

Fact 4. For a class of graphs $\mathcal{G}$ of *bounded* maximum degree, let $\mathcal{G}'$ be a class of graphs that is $(\infty, \texttt{es})$-obtained from $\mathcal{G}$, where $\texttt{es}$ is the edge subdivision operation. Then $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}'$ has bounded clique-width [27].

For $r \geq 1$, the graphs $C_r$, $K_r$, $P_r$ denote the cycle, complete graph and path on $r$ vertices, respectively, and the graph $K_{1,r}$ denotes the star on $r+1$ vertices. The graph $K_{1,3}$ is also called the *claw*. For $1 \leq h \leq i \leq j$, let $S_{i,j,k}$ denote the tree that has only one vertex $x$ of degree 3 and that has exactly three leaves, which are of distance $i$, $j$ and $k$ from $x$, respectively. Observe that $S_{1,1,1} = K_{1,3}$. A graph $S_{i,j,k}$ is said to be a *subdivided claw*. We let $\mathcal{S}$ be the class of graphs each connected component of which is either a subdivided claw or a path.

The following lemma is well known.

**Lemma 1 ([32]).** *Let $\{H_1, \ldots, H_p\}$ be a finite set of graphs. If $H_i \notin \mathcal{S}$ for $i = 1, \ldots, p$ then the class of $(H_1, \ldots, H_p)$-free graphs has unbounded clique-width.*

We say that $G$ is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets $B$ and $W$. We say that $(B, W)$ is a *bipartition* of $G$. Lozin and Volz [33] characterized all bipartite graphs $H$ for which the class of strongly $H$-free bipartite graphs has bounded clique-width (see [17] for the definition of strongly). Recently, we proved a similar characterization for $H$-free bipartite graphs; we will use this result in Section 5.

**Lemma 2 ([17]).** *Let $H$ be a graph. The class of $H$-free bipartite graphs has bounded clique-width if and only if one of the following cases holds: $H = sP_1$ for some $s \geq 1$, $H \subseteq_i K_{1,3} + 3P_1$, $H \subseteq_i K_{1,3} + P_2$, $H \subseteq_i P_1 + S_{1,1,3}$, or $H \subseteq_i S_{1,2,3}$.*

From the same paper we will also need the following lemma.

**Lemma 3 ([17]).** *Let $H \in \mathcal{S}$. Then $H$ is $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3)$-free if and only if $H = sP_1$ for some integer $s \geq 1$ or $H$ is an induced subgraph of one of the graphs in $\{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$.*

We say that a graph $G$ is *complete multipartite* if $V(G)$ can be partitioned into $k$ independent sets $V_1, \ldots, V_k$ for some integer $k$, such that two vertices are adjacent if and only if they belong to two different sets $V_i$ and $V_j$. The next result is due to Olariu [37] (the graph $\overline{P_1 + P_3}$ is also called the *paw*).

**Lemma 4 ([37]).** *Every connected* $(\overline{P_1 + P_3})$*-free graph is either complete multipartite or* $K_3$*-free.*

Every complete multipartite graph has clique-width at most 2. Also, the definition of clique-width directly implies that the clique-width of any graph is equal to the maximum clique-width of its connected components. Hence, Lemma 4 immediately implies the following (well-known) result.

**Lemma 5.** *For any graph* $H$*, the class of* $(\overline{P_1 + P_3}, H)$*-free graphs has bounded clique-width if and only if the class of* $(K_3, H)$*-free graphs has bounded clique-width.*

Kratsch and Schweitzer [30] proved that the Graph Isomorphism problem is graph-isomorphism-complete for the class of $(K_4, P_1 + P_4)$-free graphs. It is a straightforward exercise to simplify their construction and use analogous arguments to prove that the class of $(K_4, P_1 + P_4)$-free graphs has unbounded clique-width. Very recently, Schweitzer [45] showed that any graph class that allows a so-called simple path encoding has unbounded clique-width, implying this result as a direct consequence.

**Lemma 6 ([45]).** *The class of* $(K_4, P_1 + P_4)$*-free graphs has unbounded clique-width.*

## 3 New Classes of Bounded Clique-width

In this section we identify two new graph classes that have bounded clique-width, namely the classes of $(\overline{P_1 + P_3}, P_1 + S_{1,1,2})$-free graphs and $(\overline{P_1 + P_3}, K_{1,3} + 3P_1)$-free graphs. We omit the proofs of both these results. The proof of the first result uses a similar approach to that used by Dabrowski, Lozin, Raman and Ries [16] to prove that the classes of $(K_3, S_{1,1,3})$-free and $(K_3, K_{1,3} + P_2)$-free graphs have bounded clique-width.

**Theorem 1.** *The class of* $(\overline{P_1 + P_3}, P_1 + S_{1,1,2})$*-free graphs has bounded clique-width.*

**Theorem 2.** *The class of* $(\overline{P_1 + P_3}, K_{1,3} + 3P_1)$*-free graphs has bounded clique-width.*

## 4 New Classes of Unbounded Clique-width

In order to prove our results, we first present a general construction for obtaining graph classes of unbounded clique-width. We then use our construction to obtain two new classes of unbounded clique-width. Our construction generalizes the constructions used by Golumbic and Rotics [23],[2] Brandstädt et al. [4] and Lozin

[2] The class of (square) grids was first shown to have unbounded clique-width by Makowsky and Rotics [34]. The construction of [23] determines the exact clique-width of square grids and narrows the clique-width of non-square grids to two values.

and Volz [33] to prove that the classes of square grids, $K_4$-free co-chordal graphs and $2P_3$-free graphs, respectively, have unbounded clique-width.

**Theorem 3.** *For $m \geq 0$ and $n > m+1$ the clique-width of a graph $G$ is at least $\lfloor \frac{n-1}{m+1} \rfloor + 1$ if $V(G)$ has a partition into sets $V_{i,j} (i, j \in \{0, \ldots, n\})$ with the following properties:*

1. $|V_{i,0}| \leq 1$ *for all* $i \geq 1$.
2. $|V_{0,j}| \leq 1$ *for all* $j \geq 1$.
3. $|V_{i,j}| \geq 1$ *for all* $i, j \geq 1$.
4. $G[\cup_{j=0}^{n} V_{i,j}]$ *is connected for all* $i \geq 1$.
5. $G[\cup_{i=0}^{n} V_{i,j}]$ *is connected for all* $j \geq 1$.
6. *For* $i, j, k \geq 1$, *if a vertex of* $V_{k,0}$ *is adjacent to a vertex of* $V_{i,j}$ *then* $i \leq k$.
7. *For* $i, j, k \geq 1$, *if a vertex of* $V_{0,k}$ *is adjacent to a vertex of* $V_{i,j}$ *then* $j \leq k$.
8. *For* $i, j, k, \ell \geq 1$, *if a vertex of* $V_{i,j}$ *is adjacent to a vertex of* $V_{k,\ell}$ *then* $|k - i| \leq m$ *and* $|\ell - j| \leq m$.

*Proof.* Fix integers $n, m$ with $m \geq 0$ and $n > m+1$, and let $G$ be a graph with a partition as described above. For $i > 0$ we let $R_i = \cup_{j=0}^{n} V_{i,j}$ be a *row* of $G$ and for $j > 0$ we let $C_j = \cup_{i=0}^{n} V_{i,j}$ be a *column* of $G$. Note that $G[R_i]$ and $G[C_j]$ are non-empty by Property 3. They are connected graphs by Properties 4 and 5, respectively.

Consider a $k$-expression for $G$. We will show that $k \geq \lfloor \frac{n-1}{m+1} \rfloor + 1$. As stated in Section 2, this $k$-expression can be represented by a rooted tree $T$, whose leaves correspond to the operations of vertex creation and whose internal nodes correspond to the other three operations (see Fig. 1 for an example). We denote the subgraph of $G$ that corresponds to the subtree of $T$ rooted at node $x$ by $G(x)$. Note that $G(x)$ may not be an induced subgraph of $G$ as missing edges can be added by operations corresponding to $\eta_{i,j}$ nodes higher up in $T$.

Let $x$ be a deepest (i.e. furthest from the root) $\oplus$ node in $T$ such that $G(x)$ contains an entire row or an entire column of $G$ (the node $x$ may not be unique). Let $y$ and $z$ be the children of $x$ in $T$. Colour all vertices in $G(y)$ blue and all vertices in $G(z)$ red. Colour all remaining vertices of $G$ yellow. Note that a vertex of $G$ appears in $G(x)$ if and only if it is coloured either red or blue and that there is no edge in $G(x)$ between a red and a blue vertex. Due to our choice of $x$, $G$ contains a row or a column none of whose vertices are yellow, but no row or column of $G$ is entirely blue or entirely red. Without loss of generality, assume that $G$ contains a non-yellow column.

Because $G$ contains a non-yellow column, each row of $G$ contains a non-yellow vertex, by Property 3. Since no row is entirely red or entirely blue, every row of $G$ is therefore coloured with at least two colours. Let $R_i$ be an arbitrary row. Since $G[R_i]$ is connected, there must be two adjacent vertices $v_i, w_i \in R_i$ in $G$, such that $v_i$ is either red or blue and $w_i$ has a different colour than $v_i$. Note that $v_i$ and $w_i$ are therefore not adjacent in $G(x)$ (recall that if $w_i$ is yellow then it is not even present as a vertex of $G(x)$).

Now consider indices $i, k \geq 1$ with $k > i + m$. By Properties 6 and 8, no vertex of $R_i$ is adjacent to a vertex of $R_k \setminus V_{k,0}$ in $G$. Therefore, since $|V_{k,0}| \leq 1$ by Property 1, we conclude that either $v_i$ and $w_i$ are not adjacent to $v_k$ in $G$, or $v_i$ and $w_i$ are not adjacent to $w_k$ in $G$. In particular, this implies that $w_i$ is not adjacent to $v_k$ in $G$ or that $w_k$ is not adjacent to $v_i$ in $G$. Recall that $v_i$ and $w_i$ are adjacent in $G$ but not in $G(x)$, and the same holds for $v_k$ and $w_k$. Hence, a $\eta_{i,j}$ node higher up in the tree, makes $w_i$ adjacent to $v_i$ but not to $v_k$, or makes $w_k$ adjacent to $v_k$ but not to $v_i$. This means that $v_i$ and $v_k$ must have different labels in $G(x)$. We conclude that $v_1, v_{(m+1)+1}, v_{2(m+1)+1}, v_{3(m+1)+1}, \ldots, v_{(\lfloor \frac{n-1}{m+1} \rfloor)(m+1)+1}$ must all have different labels in $G(x)$. Hence, the $k$-expression of $G$ uses at least $\lfloor \frac{n-1}{m+1} \rfloor + 1$ labels. □

We now use Theorem 3 to determine two new graph classes that have unbounded clique-width. We omit the proofs.

**Theorem 4.** *The class of* $(P_6, \overline{2P_1 + P_2})$*-free graphs has unbounded clique-width.*

**Theorem 5.** *The class of* $(3P_2, P_2 + P_4, P_6, \overline{P_1 + P_4})$*-free graphs has unbounded clique-width.*

## 5 Classifying Classes of $(H_1, H_2)$-Free Graphs

In this section we study the boundedness of clique-width of classes of graphs defined by two forbidden induced subgraphs. Recall that this study is partially motivated by the fact that it is easy to obtain a full classification for the boundedness of clique-width of graph classes defined by one forbidden induced subgraph, as shown in the next theorem (we omit the proof). This classification does not seem to have previously been explicitly stated in the literature.

**Theorem 6.** *Let $H$ be a graph. The class of $H$-free graphs has bounded clique-width if and only if $H$ is an induced subgraph of $P_4$.*

We are now ready to study classes of graphs defined by two forbidden induced subgraphs. Given four graphs $H_1, H_2, H_3, H_4$, we say that the class of $(H_1, H_2)$-free graphs and the class of $(H_3, H_4)$-free graphs are *equivalent* if the unordered pair $H_3, H_4$ can be obtained from the unordered pair $H_1, H_2$ by some combination of the following operations:

1. complementing both graphs in the pair;
2. if one of the graphs in the pair is $K_3$, replacing it with $\overline{P_1 + P_3}$ or vice versa.

By Fact 2 and Lemma 5, if two classes are equivalent then one has bounded clique-width if and only if the other one does. Given this definition, we can now classify all classes defined by two forbidden induced subgraphs for which it is known whether or not the clique-width is bounded. This includes both the already-known results and our new results. We will later show that (up to equivalence) this leaves only 13 open cases.

**Theorem 7.** *Let $\mathcal{G}$ be a class of graphs defined by two forbidden induced subgraphs. Then:*

*(i) $\mathcal{G}$ has bounded clique-width if it is equivalent to a class of $(H_1, H_2)$-free graphs such that one of the following holds:*
  1. *$H_1$ or $H_2 \subseteq_i P_4$;*
  2. *$H_1 = sP_1$ and $H_2 = K_t$ for some $s, t$;*
  3. *$H_1 \subseteq_i P_1 + P_3$ and $\overline{H_2} \subseteq_i K_{1,3} + 3P_1,\ K_{1,3} + P_2,\ P_1 + S_{1,1,2},\ P_6$ or $S_{1,1,3}$;*
  4. *$H_1 \subseteq_i 2P_1 + P_2$ and $\overline{H_2} \subseteq_i 2P_1 + P_3,\ 3P_1 + P_2$ or $P_2 + P_3$;*
  5. *$H_1 \subseteq_i P_1 + P_4$ and $\overline{H_2} \subseteq_i P_1 + P_4$ or $P_5$;*
  6. *$H_1 \subseteq_i 4P_1$ and $\overline{H_2} \subseteq_i 2P_1 + P_3$;*
  7. *$H_1, \overline{H_2} \subseteq_i K_{1,3}$.*

*(ii) $\mathcal{G}$ has unbounded clique-width if it is equivalent to a class of $(H_1, H_2)$-free graphs such that one of the following holds:*
  1. *$H_1 \notin \mathcal{S}$ and $H_2 \notin \mathcal{S}$;*
  2. *$\overline{H_1} \notin \mathcal{S}$ and $\overline{H_2} \notin \mathcal{S}$;*
  3. *$H_1 \supseteq_i K_{1,3}$ or $2P_2$ and $\overline{H_2} \supseteq_i 4P_1$ or $2P_2$;*
  4. *$H_1 \supseteq_i P_1 + P_4$ and $\overline{H_2} \supseteq_i P_2 + P_4$;*
  5. *$H_1 \supseteq_i 2P_1 + P_2$ and $\overline{H_2} \supseteq_i K_{1,3},\ 5P_1,\ P_2 + P_4$ or $P_6$;*
  6. *$H_1 \supseteq_i 3P_1$ and $\overline{H_2} \supseteq_i 2P_1 + 2P_2,\ 2P_1 + P_4,\ 4P_1 + P_2,\ 3P_2$ or $2P_3$;*
  7. *$H_1 \supseteq_i 4P_1$ and $\overline{H_2} \supseteq_i P_1 + P_4$ or $3P_1 + P_2$.*

*Proof.* We first consider the bounded cases. Statement (i).1 follows from Theorem 6. To prove Statement (i).2 note that if $H_1 = sP_1$ and $H_2 = K_t$ for some $s, t$ then by Ramsey's Theorem, all graphs in the class of $(H_1, H_2)$-free graphs have a bounded number of vertices and therefore the clique-width of graphs in this class is bounded. By the definition of equivalence, when proving Statement (i).3, we may assume that $H_1 = K_3$. Then Statement (i).3 follows from Fact 2 combined with the fact that $(K_3, H)$-free graphs have bounded clique-width if $H$ is $K_{1,3} + 3P_1$ (Theorem 2), $K_{1,3} + P_2$ [16], $P_1 + S_{1,1,2}$ (Theorem 1), $P_6$ [5] or $S_{1,1,3}$ [16]. Statement (i).4 follows from Fact 2 and the fact that $(\overline{2P_1 + P_2}, 2P_1 + P_3)$-free, $(\overline{2P_1 + P_2}, 3P_1 + P_2)$-free and $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs have bounded clique-width [15]. Statement (i).5 follows from Fact 2 and the fact that both $(P_1 + P_4, \overline{P_1 + P_4})$-free graphs [7] and $(P_5, \overline{P_1 + P_4})$-free graphs [8] have bounded clique-width. Statement (i).6 follows from Fact 2 and the fact that $(2P_1 + P_3, K_4)$-free graphs have bounded clique-width [3]. Statement (i).7 follows from the fact that $(K_{1,3}, \overline{K_{1,3}})$-free graphs have bounded clique-width [1,9].

We now consider the unbounded cases. Statements (ii).1 and (ii).2 follow from Lemma 1 and Fact 2. Statement (ii).3 follows from the fact that the classes of $(C_4, K_{1,3}, K_4, \overline{2P_1 + P_2})$-free [4], $(K_4, 2P_2)$-free [4] and $(C_4, C_5, 2P_2)$-free graphs (or equivalently, split graphs) [34] have unbounded clique-width. Statement (ii).4 follows from Fact 2 and the fact that the class of $(P_2 + P_4, 3P_2, P_6, \overline{P_1 + P_4})$-free (Theorem 5) graphs have unbounded clique-width. Statement (ii).5 follows from Fact 2 and the fact that $(C_4, K_{1,3}, K_4, \overline{2P_1 + P_2})$-free [4], $(5P_1, \overline{2P_1 + P_2})$-free [14], $(\overline{2P_1 + P_2}, P_2 + P_4)$-free (see arXiv version of [15]) and $(P_6, \overline{2P_1 + P_2})$-free (Theorem 4) graphs have unbounded clique-width. To prove Statement (ii).6, suppose $H_1 \supseteq_i 3P_1$ and $\overline{H_2} \supseteq_i 2P_1+2P_2, 2P_1+P_4, 4P_1+P_2, 3P_2$ or $2P_3$. Then $\overline{H_1} \notin \mathcal{S}$, so $\overline{H_2} \in \mathcal{S}$, otherwise we are done by Statement (ii).2. By Lemma 3, $\overline{H_2}$ is not an induced subgraph of any graph in $\{K_{1,3}+3P_1, K_{1,3}+P_2, P_1+S_{1,1,3}, S_{1,2,3}\}$. The class of $(H_1, H_2)$-free graphs contains the class of complements of $\overline{H_2}$-free bipartite graphs. By Fact 2 and Lemma 2, this latter class has unbounded clique-width. Statement (ii).7 follows from the Fact 2 and the fact that the classes of $(K_4, P_1+P_4)$-free graphs (Lemma 6) and $(4P_1, \overline{3P_1+P_2})$-free graphs [14] have unbounded clique-width. □

As we will prove in Theorem 8, the above classification leaves exactly 13 open cases (up to equivalence).

**Open Problem 1** *Does the class of $(H_1, H_2)$-free graphs have bounded clique-width when:*

1. $H_1 = 3P_1, \overline{H_2} \in \{P_1+P_2+P_3, P_1+2P_2, P_1+P_5, P_1+S_{1,1,3}, P_2+P_4, S_{1,2,2}, S_{1,2,3}\}$*;*
2. $H_1 = 2P_1+P_2, \overline{H_2} \in \{P_1+P_2+P_3, P_1+2P_2, P_1+P_5\}$*;*
3. $H_1 = P_1+P_4, \overline{H_2} \in \{P_1+2P_2, P_2+P_3\}$ *or*
4. $H_1 = \overline{H_2} = 2P_1+P_3$*.*

Note that the two pairs $(3P_1, \overline{S_{1,1,2}})$ and $(3P_1, \overline{S_{1,2,3}})$, or equivalently, the two pairs $(K_3, S_{1,2,2})$ and $(K_3, S_{1,2,3})$ are the only pairs that correspond to open cases in which both $H_1$ and $H_2$ are connected. We also observe the following. Let $H_2 \in \{P_1+P_2+P_3, P_1+2P_2, P_1+P_5, P_1+S_{1,1,3}, P_2+P_4, S_{1,2,2}, S_{1,2,3}\}$. Lemma 2 shows that all bipartite $H_2$-free graphs have bounded clique-width. Moreover, the graph $P_1+2P_2$ is an induced subgraph of $H_2$. Hence, for investigating whether the boundedness of the clique-width of bipartite $H_2$-free graphs can be extended to $(K_3, H_2)$-free graphs, the $H_2 = P_1+2P_2$ case is the starting case.

**Theorem 8.** *Let $\mathcal{G}$ be a class of graphs defined by two forbidden induced subgraphs. Then $\mathcal{G}$ is not equivalent to any of the classes listed in Theorem 7 if and only if it is equivalent to one of the 13 cases listed in Open Problem 1.*

A graph $G$ is $(H_1, \ldots, H_p)$*-subgraph-free* if $G$ has no subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$. Let $G$ and $H$ be graphs. Then $G$ contains $H$ as a *minor* or *topological minor* if $G$ can be modified into $H$ by a sequence that consists of edge contractions, edge deletions and vertex deletions, or by a sequence that consists of vertex dissolutions, edge deletions and vertex deletions, respectively. If $G$ does not contain any of the graphs $H_1, \ldots, H_p$ as a (topological) minor, we say that $G$ is $(H_1, \ldots, H_p)$*-(topological-)minor-free*. We omit the proof of the following result, which completely characterizes which of these graph classes have bounded clique-width.

**Theorem 9.** *Let $\{H_1, \ldots, H_p\}$ be a finite set of graphs. Then the following statements hold:*

*(i) The class of $(H_1, \ldots, H_p)$-subgraph-free graphs has bounded clique-width if and only if $H_i \in \mathcal{S}$ for some $1 \le i \le p$.*

*(ii)* *The class of* $(H_1,\ldots,H_p)$*-minor-free graphs has bounded clique-width if and only if* $H_i$ *is planar for some* $1 \leq i \leq p$.

*(iii)* *The class of* $(H_1,\ldots,H_p)$*-topological-minor-free graphs has bounded clique-width if and only if* $H_i$ *is planar and has maximum degree at most* 3 *for some* $1 \leq i \leq p$.

## 6 Consequences for Colouring

One of the motivations of our research was to further the study of the computational complexity of the Colouring problem for $(H_1,H_2)$-free graphs. Recall that Colouring is polynomial-time solvable on any graph class of bounded clique-width by combining results of Kobler and Rotics [28] and Oum [38]. By combining a number of known results [11,12,16,22,29,35,41,42,44] with new results, Dabrowski, Golovach and Paulusma [14] presented a summary of known results for Colouring restricted to $(H_1,H_2)$-free graphs. Combining Theorem 7 with the results of Kobler and Rotics [28] and Oum [38] and incorporating a number of recent results [25,26,36] leads to an updated summary. This updated summary (and a proof of it) can be found in the recent survey paper of Golovach, Johnson, Paulusma and Song [21].

From this summary we note that not only the case when $H_1 = P_4$ or $H_2 = P_4$ but thirteen other maximal classes of $(H_1,H_2)$-free graphs for which Colouring is known to be polynomial-time solvable can be obtained by combining Theorem 7 with the results of Kobler and Rotics [28] and Oum [38] (see also [21]). One of these thirteen classes is one that we obtained in this paper (Theorem 2), namely the class of $(K_{1,3}+3P_1,\overline{P_1+P_3})$-free graphs, for which Colouring was not previously known to be polynomial-time solvable. Note that Dabrowski, Lozin, Raman and Ries [16] already showed that Colouring is polynomial-time solvable for $(\overline{P_1+P_3},P_1+S_{1,1,2})$-free graphs, but in Theorem 1 we strengthened their result by showing that the clique-width of this class is also bounded.

Theorem 8 shows that there are 13 classes of $(H_1,H_2)$-free graphs (up to equivalence) for which we do not know whether their clique-width is bounded. These classes correspond to 28+6+4+1=39 distinct classes of $(H_1,H_2)$-free graphs. The complexity of Colouring is unknown for only 15 of these classes. We list these cases below:

1. $\overline{H_1} \in \{3P_1,P_1+P_3\}$ and $H_2 \in \{P_1+S_{1,1,3},S_{1,2,3}\}$;
2. $H_1 = 2P_1+P_2$ and $\overline{H_2} \in \{P_1+P_2+P_3,P_1+2P_2,P_1+P_5\}$;
3. $H_1 = \overline{2P_1+P_2}$ and $H_2 \in \{P_1+P_2+P_3,P_1+2P_2,P_1+P_5\}$;
4. $H_1 = P_1+P_4$ and $\overline{H_2} \in \{P_1+2P_2,P_2+P_3\}$;
5. $\overline{H_1} = P_1+P_4$ and $H_2 \in \{P_1+2P_2,P_2+P_3\}$;
6. $H_1 = \overline{H_2} = 2P_1+P_3$.

Note that Case 1 above reduces to two subcases by Lemma 4. All classes of $(H_1,H_2)$-free graphs, for which the complexity of Colouring is still open and which are not listed above have unbounded clique-width. Hence, new techniques will need to be developed to deal with these classes.

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