The Price of Connectivity for Cycle Transversals*

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Abstract. For a family of graphs $\mathcal{F}$, an $\mathcal{F}$-transversal of a graph G is a subset $S \subseteq V(G)$ that intersects every subset of $V(G)$ that induces a subgraph isomorphic to a graph in $\mathcal{F}$. Let $t_{\mathcal{F}}(G)$ be the minimum size of an $\mathcal{F}$-transversal of G, and $ct_{\mathcal{F}}(G)$ be the minimum size of an $\mathcal{F}$-transversal of G that induces a connected graph. For a class of connected graphs $\mathcal{G}$, the price of connectivity for $\mathcal{F}$-transversals is the supremum of the ratios $ct_{\mathcal{F}}(G)/t_{\mathcal{F}}(G)$ over all $G \in \mathcal{G}$. We perform an in-depth study into the price of connectivity for various well-known graph families $\mathcal{F}$ that contain an infinite number of cycles and that, in addition, may contain one or more anticycles or short paths. For each of these families we study the price of connectivity for classes of graphs characterized by one forbidden induced subgraph H. We determine exactly those classes of H-free graphs for which this graph parameter is bounded by a multiplicative constant, bounded by an additive constant, or equal to 1. In particular, our tetrachotomies extend known results of Belmonte et al. (EuroComb 2012, MFCS 2013) for the case when $\mathcal{F}$ is the family of all cycles.

1 Introduction

Let $\mathcal{F}$ be a family of graphs. An $\mathcal{F}$-*transversal* of a graph $G = (V, E)$ is a subset $S \subseteq V$ that intersects every subset of V that induces a subgraph isomorphic to a graph in $\mathcal{F}$. Equivalently, S is an $\mathcal{F}$-transversal of G if $G - S$ is $\mathcal{F}$-*free*; that is, it does not contain an induced subgraph isomorphic to any graph in $\mathcal{F}$ (if $\mathcal{F} = \{H\}$ then we write H-free instead).

In certain cases, $\mathcal{F}$-transversals are well-studied. For example, a *vertex cover* is an $\mathcal{F}$-transversal for any family $\mathcal{F}$ that contains P_2 but not P_1 (here, P_k is the path on k vertices). Note that, for any $\{P_2\}$-transversal S of a graph G, the graph $G - S$ is an independent set. To give another example, a *feedback vertex set* is an $\mathcal{F}$-transversal for $\mathcal{F} = \{C_3, C_4, C_5, \ldots\}$. In this case, for any $\mathcal{F}$-transversal S

* This work was supported by a London Mathematical Society Scheme 4 Grant and by EPSRC Grant EP/K025090/1, and, in part, by the Slovenian Research Agency (I0-0035, research program P1-0285, research projects N1-0032, J1-5433, J1-6720, and J1-6743, and a Young Researchers Grant).

of a graph G, the graph $G - S$ is a forest. As the examples suggest, it is natural to study minimum size $\mathcal{F}$-transversals.

We can put an additional constraint on an $\mathcal{F}$-transversal S of a graph G by requiring that the subgraph of G induced by S is connected. Minimum size *connected* $\mathcal{F}$-transversals of a graph have also been considered. Minimum size connected vertex covers are well-studied, and minimum size connected feedback vertex sets have also received attention (see, for example, [7],[9]). We study the following question:

What is the effect of adding the connectivity constraint on the minimum size of a $\mathcal{F}$-transversal for a graph family $\mathcal{F}$?

To address this question we use the *price of connectivity* as our comparison measure. The price of connectivity was introduced by Cardinal and Levy [5] for vertex cover. We define it as follows. For a graph G, let $t_{\mathcal{F}}(G)$ denote the minimum size of an $\mathcal{F}$-transversal S of G, and $ct_{\mathcal{F}}(G)$ the minimum size of a connected $\mathcal{F}$-transversal S' of G. Then, for a class of connected graphs $\mathcal{G}$, the price of connectivity of $\mathcal{F}$-transversals is the supremum of the ratios $ct_{\mathcal{F}}(G)/t_{\mathcal{F}}(G)$ over all $G \in \mathcal{G}$.

We briefly survey existing work starting with a result of Cardinal and Levy [5], who proved that the price of connectivity for $\{P_2\}$-transversal (vertex cover) is at most $2/(1+\epsilon)$ for connected graphs with average degree ϵn. Camby et al. [3] proved that the price of connectivity for $\{P_2\}$-transversal is at most 2 for the class of all connected graphs and that this bound is asymptotically sharp for paths and cycles. They also gave forbidden induced subgraph characterizations of classes of graphs such that the price of connectivity for $\{P_2\}$-transversal for every connected induced subgraph is at most t, for each $t \in \{1, 4/3, 3/2\}$.

Belmonte et al. [1,2] studied the price of connectivity for feedback vertex set, that is, for $\mathcal{F}$-transversals where $\mathcal{F} = \{C_3, C_4, C_5 \ldots\}$. They characterized exactly those finite families $\mathcal{H}$ for which the price of connectivity for feedback vertex set is bounded by a constant [2]. If $|\mathcal{H}| = 1$ they also considered additive bounds: they determined exactly those graphs classes $\mathcal{G}$ of H-free graphs for which, for all $G \in \mathcal{G}$, $ct_{\mathcal{F}}(G) - t_{\mathcal{F}}(G)$ is bounded by a constant (and they found exactly when that constant is zero) [1].

The price of connectivity can also be defined for other graph measures that are defined as the size of a smallest subset of vertices that satisfies a prescribed constraint. We give two further examples. A result of Duchet and Meyniel [6] implies that the price of connectivity for dominating set is at most 3 for all connected graphs. A result of Zverovich [10] implies that the price of connectivity for dominating set is exactly 1 for connected (P_5, C_5)-free graphs. Camby and Schaudt [4] gave an additive bound of 1 for every connected (P_6, C_6)-free graph G. The same authors proved that the price of connectivity for dominating set is at most 2 for connected (P_8, C_8)-free graphs and at most 3 for connected (P_9, C_9)-free graphs; both bounds were shown to be sharp. Camby and Schaudt [4] proved that the problem of deciding whether the price of connectivity for dominating set is at most r is $P^{\text{NP}[\log]}$-complete for fixed r, $1 < r < 3$. Grigoriev and Sitters [7] proved that the price of connectivity for face hitting set is at most 11 for connected planar graphs of minimum degree at least 3. Schweitzer and Schweitzer [9] reduced this bound to 5 and proved tightness.

We consider a number of families $\mathcal{F}$ that contain cycles, paths and complements of cycles. We study the price of connectivity of $\mathcal{F}$-transversals for graph classes characterized by one forbidden induced subgraph. Before we can present our results we need to introduce the following terminology and notation.

Definition 1. *Let H be a graph and let $\mathcal{G}$ be the class of connected H-free graphs. Let $\mathcal{F}$ be a family of graphs. We say that $\mathcal{G}$ is:*

(a) $\mathcal{F}$-unbounded *if for every function $f : \mathbb{N} \to \mathbb{N}$ there exists a graph $G \in \mathcal{G}$ such that $ct_{\mathcal{F}}(G) > f(t_{\mathcal{F}}(G))$;*
(b) $\mathcal{F}$-multiplicative *if $ct_{\mathcal{F}}(G) \leq c_H t_{\mathcal{F}}(G)$ for some constant c_H and for every $G \in \mathcal{G}$;*
(c) $\mathcal{F}$-additive *if $ct_{\mathcal{F}}(G) \leq t_{\mathcal{F}}(G) + d_H$ for some constant d_H and for every $G \in \mathcal{G}$; and*
(d) $\mathcal{F}$-identical *if $ct_{\mathcal{F}}(G) = t_{\mathcal{F}}(G)$ for every $G \in \mathcal{G}$.*

For graphs F and G, we write $F \subseteq_i G$ to denote that F is an induced subgraph of G. We let C_n, K_n and P_n denote the cycle, complete graph, and path on n vertices, respectively. The *disjoint union* of two vertex-disjoint graphs G and H is the graph $G + H$ that has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ where $V(G) \cap V(H) = \emptyset$. We denote the disjoint union of r copies of G by rG. A graph is a *linear forest* if it is the disjoint union of a set of paths.

The *complement* $\overline{G}$ of a graph G has the same vertex set as G and an edge between two distinct vertices if and only if these vertices are not adjacent in G. A *hole* is a cycle of length at least 4. An *antihole* is the complement of a hole. A cycle, hole or antihole is *even* if it contains an even number of vertices; otherwise it is *odd*. A hole is *long* if it is of length at least 5, and a *long antihole* is the complement of a long hole. A graph is *odd-hole-free* or *odd-antihole-free* if it contains no induced odd holes or no induced odd antiholes, respectively. An *even-hole-free* graph is defined similarly. A graph is *chordal* if it has no induced hole, that is, if it has no induced cycles of length at least 4. A graph is *weakly chordal* if it has no induced long hole and no induced long antihole. A graph is *perfect* if the chromatic number of every induced subgraph equals the size of a largest clique in that subgraph. By the Strong Perfect Graph Theorem, a graph is perfect if and only if it is odd-hole-free and odd-antihole-free. A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set. Split graphs coincide with the $(2K_2, C_4, C_5)$-free graphs. A graph is *threshold* if it is $(2P_2, P_4)$-free, *trivially perfect* if it is (C_4, P_4)-free, *cotrivially perfect* if it is $(C_4, 2P_2)$-free and a *cograph* if it is P_4-free.

Our Results. Table 1 summarizes our results together with related past work. Results can be seen both according to the family $\mathcal{F}$ and the corresponding property of the graph $G - S$, where S is an $\mathcal{F}$-transversal of G. We note that when $\mathcal{F}$ is the family of even cycles or of holes there is an open case. In all other cases, the stated conditions in Table 1 are both necessary and sufficient for $\mathcal{F}$-multiplicativity ($\mathcal{F}$-boundedness), $\mathcal{F}$-additivity, and $\mathcal{F}$-identity, respectively, in the class of connected H-free graphs.

Table 1. The price of connectivity of $\mathcal{F}$-transversals for various families of graphs $\mathcal{F}$ on graph classes defined by one forbidden induced subgraph H. The results on cycles in the first row are due to Belmonte et al. [1] and the multiplicativity result on cycles and P_2 in the ninth row is due to Camby et al. [3]. All other results are new and presented in this paper. †For even cycles and holes the condition is not complete as in these cases we do not know if H-free graphs are $\mathcal{F}$-additive for $H \subseteq_i P_3 + P_2 + sP_1$.

$\mathcal{F}$	Property of $G - S$	Condition for $\mathcal{F}$-multiplicativity (for $\mathcal{F}$-boundedness)	Condition for $\mathcal{F}$-additivity	Condition for $\mathcal{F}$-identity
cycles	forest	H is a linear forest [1]	$H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$ [1]	$H \subseteq_i P_3$ [1]
odd cycles	bipartite	H is a linear forest	$H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$	$H \subseteq_i P_3$
even cycles† (equiv.: even holes)	even-hole-free	H is a linear forest	$H \subseteq_i P_4 + sP_1$ †	$H \subseteq_i P_3$
holes†	chordal	H is a linear forest	$H \subseteq_i P_4 + sP_1$ †	$H \subseteq_i P_3$
odd holes	odd-hole-free	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
odd holes and odd antiholes	perfect	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
long holes	long-hole-free	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
long holes and long antiholes	weakly chordal	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
cycles and P_2 (equiv.: $\{P_2\}$)	edgeless	no restriction [3]	$H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$	$H \subseteq_i P_3$
holes and $2P_2$ (equiv.: $\{C_4, C_5, 2P_2\}$)	split	no restriction	$H \subseteq_i P_4 + sP_1$ or $H \subseteq_i P_3 + sP_2$	$H \subseteq_i P_3$
holes and $2P_2, P_4$ (equiv.: $\{C_4, 2P_2, P_4\}$)	threshold	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
holes and P_4 (equiv.: $\{C_4, P_4\}$)	trivially perfect	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
long holes and $2P_2$ (equiv.: $\{C_5, 2P_2\}$)	$(C_5, 2P_2)$-free	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$ $H \subseteq_i P_2 + P_1$
long holes and $2P_2, P_4$ (equiv.: $\{2P_2, P_4\}$)	cotrivially perfect	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$ or $H \subseteq_i P_2 + P_1$
long holes and P_4 (equiv.: $\{P_4\}$)	cograph	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$

From Table 1 we can draw a number of conclusions. If a transversal that intersects (small) paths is wanted, we obtain multiplicative bounds for any class of H-free graphs. In all other cases, H may not contain a cycle or a claw (so is a linear forest). We also see that when we add a requirement that all triangles are intersected, there is always a jump from $H = P_4 + sP_1$ to $H = P_5 + sP_1$ for the additive bound. In general, it can be noticed that adding small graphs to $\mathcal{F}$

has differing effects. We say that a family of graphs $\mathcal{F}$ or a graph F *positively (negatively) influences* a family of graphs $\mathcal{F}'$ if the row in the table for their union contains more (fewer) bounded cases than the row for $\mathcal{F}'$. So, for example, $2P_2$ does not influence $\{C_4, C_5, C_6, \ldots\} \cup \{P_4\}$, and P_4 does not influence the family of long holes. Moreover, odd holes do not influence even holes, whereas even holes influence odd holes positively.

In the remainder of our paper, after presenting some known and new basic results in Section 2, we present a number of general theorems, from which the results in Table 1 directly follow. We emphasize that all proofs of these theorems are algorithmic in nature, that is, they can be translated directly into polynomial-time algorithms that modify an $\mathcal{F}$-transversal into a connected $\mathcal{F}$-transversal of appropriate cardinality.

We provide a brief guide to Table 1. Theorem 2 implies the second row. Theorem 3 implies the third and fourth row, and Theorem 4 implies the next four rows. Proofs for the remaining rows have been omitted due to space restrictions.

2 Preliminaries and Some Basic Results

We consider finite undirected graphs with no multiple edges and no self-loops. Let $G = (V, E)$ be a connected graph. The *distance* between two vertices u and v is the length of a shortest path between them. The maximum distance in G is called the *diameter* of G. A set $D \subseteq V$ *dominates* G if every vertex $u \in V \setminus D$ is adjacent to at least one vertex in D. We also say that $G[D]$ dominates G. If $D = \{u, v\}$ for two adjacent vertices u, v, then uv is called a *dominating edge* of G. A set $D \subseteq V$ *dominates* a set $S \subseteq V \setminus D$ if every vertex in S is adjacent to at least one vertex in D. We say that two vertex-disjoint subgraphs of a graph G are *adjacent* if there is at least one edge between a vertex in one of them and a vertex in the other one. Similarly, we say that a vertex of G is adjacent to a subgraph of $G - v$ if G has an edge joining v with a vertex of the subgraph. The *join* of two vertex-disjoint graphs G and H is the graph obtained from the disjoint union $G + H$ by adding to it all possible edges of the form xy with $x \in V(G)$ and $y \in V(H)$.

For $r \geq 1$, $s \geq 1$, the *complete bipartite graph* $K_{r,s}$ is a bipartite graph whose vertex set can be partitioned into two sets of sizes r and s such that there is an edge joining each pair of vertices from distinct sets. The graph $K_{1,3}$ is also called a *claw*.

We now give a number of new results that we use as lemmas in our other results. Some proofs are omitted for reasons of space.

Lemma 1. *For every family $\mathcal{F}$ of graphs, the class of connected P_4-free graphs is $\mathcal{F}$-additive.*

We also need to generalize a result that was proved by Belmonte et al. [1] for the graph $H = P_5$.

Lemma 2. *For a family of graphs $\mathcal{F}$ and a graph H, if the class of connected H-free graphs is $\mathcal{F}$-additive then so is the class of connected $(H + sP_1)$-free graphs for all $s \geq 1$.*

Lemma 3. *Let G be a connected graph with diameter d. Let A be a subgraph of G consisting of r components. Then G has a connected subgraph A' that contains A and that has less than $|V(A)| + (r-1)d$ vertices.*

The following theorem is used in all our tetrachotomies. The third part was shown by Belmonte et al. [1] for the case when $\mathcal{F}$ is the family of all cycles, and our proof is a modification of theirs.

Theorem 1. *Let $\mathcal{F}$ be a family of graphs and let H be a graph. Then, the following statements hold:*

(i) If $\mathcal{F}$ contains a linear forest, then the class of all graphs is $\mathcal{F}$-multiplicative.
(ii) If H is a linear forest, then the class of H-free graphs is $\mathcal{F}$-multiplicative.
(iii) If $\mathcal{F}$ contains an infinite number of cycles and no linear forests and H is not a linear forest, then the class of H-free graphs is $\mathcal{F}$-unbounded.

3 Transversals of Families of Odd Cycles

In this section we assume we are given a family $\mathcal{F}$ that contains all odd cycles, although we will show more general results whenever possible. To prove our results we need a number of lemmas, the first of which has been proven by Belmonte et al. [1] for the special case when the family $\mathcal{F}$ consists of all cycles.

Lemma 4. *For any family of graphs $\mathcal{F}$ with $K_r \in \mathcal{F}$ for some integer $r \geq 1$, the class of connected P_5-free graphs is $\mathcal{F}$-additive.*

We now give a technical lemma (which we also apply in some other proofs).

Lemma 5. *Let $s \geq 1$ be an integer and let G be an sP_3-free connected graph with a subset $S \subseteq V(G)$ and an independent set $U \subseteq V(G) \setminus S$. Suppose that some component of $G[S]$, say Z, contains an induced copy of $(s-1)P_3$. Then there exists a set S' with $S \subseteq S'$ of size at most $|S| + 8s^2 + 2s$ such that*

- *$G[S']$ has a component Z' that contains all vertices of $V(Z) \cup (S' \setminus S)$;*
- *every vertex of $U' = U \setminus S'$ is adjacent to at most one component of $G[S']$ not equal to Z';*
- *every component of $G[S']$ not equal to Z' is adjacent to at most one vertex of U'.*

The following lemma generalizes the corresponding result of Belmonte et al. [1] when $\mathcal{F}$ is the family of all cycles. We use a similar approach as used in their proof but our arguments (which are based on bipartiteness instead of cycle-freeness) are different and this proof demonstrates some techniques used several times in obtaining our results.

Lemma 6. *For any family of graphs $\mathcal{F}$ containing either all odd cycles or P_2 and for any fixed $s \geq 1$, the class of connected sP_3-free graphs is $\mathcal{F}$-additive.*

Proof. The proof is by induction on s. Let $s = 1$. Then every connected sP_3-free graph G is complete. Hence, every minimum $\mathcal{F}$-transversal of G is connected.

Now let $s \geq 2$. Let G be a connected sP_3-free graph. We may assume by induction that G contains an induced copy Γ_0 of an $(s-1)P_3$. Let S be a minimum $\mathcal{F}$-transversal of G. Let Γ be a minimum connected induced subgraph of G that contains Γ_0. Because G is sP_3-free, G has diameter less than $4s$. Then, by Lemma 3, we find that Γ has size less than $3(s-1)+(s-2)4s = 4s^2-5s-3$. Let $S' = S \cup V(\Gamma)$. Then we have that $|S'| \leq |S| + 4s^2 - 5s - 3$.

If S' is connected then we take $d_{sP_3} = 4s^2-5s-3$ as our desired constant and we are done. Suppose S' is not connected. Below we describe how to refine S'. During this process, we always use Z to denote the component of S' containing Γ, and we will never remove a vertex of Z from S'; in fact, one can think of the proof as "growing" Z and connecting it to the other vertices of S' until $Z = S'$.

Observe that the sP_3-freeness of G implies that every component of S' other than Z is complete. Throughout the proof, we let A denote the union of clique components of S', so $V(A) = S' \setminus V(Z) = S \setminus V(Z)$. We also note that the graph $G - S'$ is bipartite, as even its supergraph $G - S$ contains no odd cycles by the definition of S. Hence we can partition $G - S'$ into two (possibly empty) sets U_1 and U_2 so that U_1 and U_2 are independent sets.

We start with the following two claims, both of which follow from Lemma 5, which we apply twice, namely once with respect to U_1 and once with respect to U_2. By Lemma 5 this leads to a total increase of S' by an additive factor of at most $2(8s^2 + 2s) = 16s^2 + 4s$.

Claim 1: Without loss of generality, we may assume that every vertex of $U_1 \cup U_2$ is adjacent to at most one component of A.

Claim 2: Without loss of generality, we may assume that every component of A is adjacent to at most one vertex of U_1 and to at most one vertex of U_2.

Using Claims 1 and 2 we prove the following crucial claim.

Claim 3: Without loss of generality, every vertex of every component of A has exactly one neighbour in U_1 and exactly one neighbour in U_2.

We prove Claim 3 as follows. Let A^* be the union of components for which the statement of Claim 3 does not hold. Let D be a component of A^*. By Claim 2, D is adjacent to at most one vertex of U_1 and to at most one vertex of U_2. First suppose that D is non-adjacent to U_1 or to U_2, say D is not adjacent to U_1. Because G is connected, this means that D is adjacent to (exactly one) vertex $z \in U_2$, say $v \in D$ is adjacent to z. As D belongs to A^*, we find that D contains a vertex v' not adjacent to z. Hence, $vv'z$ is an induced P_3. Now suppose that D is adjacent to U_1 and to U_2, say D has vertices u, v (possibly $u = v$) so that u is adjacent to $x \in U_1$ and v is adjacent to $z \in U_2$. Then, as D is in A^*, there exists a vertex v' that is non-adjacent to at least one of x, z, say to z. Again, $vv'z$ is an induced P_3. As G is sP_3-free and no vertex in $U_1 \cup U_2$ is adjacent to more than one component of A by Claim 1, we deduce that A^* contains at most $s-1$ components. Moreover, each vertex $z \in U_1 \cup U_2$ involved in an induced P_3 as described above must be adjacent to Z (due to sP_3-freeness of G and the fact that Z contains an induced $(s-1)P_3$). Hence, we can add these vertices to Z increasing the size of Z, and thus the size of S', by at most $s-1$. The remaining components of A have the desired property. Moreover, Claims 1 and 2 are still valid. This completes the proof of Claim 3.

Due to Claim 3 we may assume without loss of generality that each vertex v in each component D of A has exactly two neighbours in $G - S'$, namely one neighbour in U_1 and one neighbour in U_2. By Claim 2, these neighbours are the same for all vertices in D. Hence, we may denote these two neighbours by s_D and t_D, respectively,

Consider a component D of A. If one of its neighbours in $U_1 \cup U_2$, say s_D, is adjacent to Z, then replacing S' with $(S' \cup \{s_D\}) \setminus \{v\}$ and Z with the connected component of S' containing $Z \cup \{s_D\}$ does not result in an odd cycle in $G - S'$. Moreover, such a swap does not increase the size of S' either. It does, however, reduce the number of vertices of S' that are not in Z (which is our goal). Consequently, we perform these swaps until, in the end, both the neighbours s_D and t_D of each component of A are not adjacent to Z. In particular this implies that s_D and t_D are adjacent, so $V_D \cup \{s_D, t_D\}$ is a clique. Then, due to Claims 1–3, the components in A together with their neighbours in $U_1 \cup U_2$ induce a union of complete graphs. This union is a disjoint union, as otherwise G would contain an induced P_3 not adjacent to Z and, as Z has an induced $(s-1)P_3$, we would obtain an induced sP_3 in G. Note that the swaps did not change the size of S'.

Let U_1' and U_2' denote the subsets of U_1 and U_2, respectively, that consist of vertices adjacent to no components of A. Let W_1 consist of all vertices s_D adjacent to U_2' and let W_2 consist of all vertices t_D adjacent to U_1'. Note that $W_1 \subseteq U_1 \setminus U_1'$ and that $W_2 \subseteq U_2 \setminus U_2'$. Because G is connected and no s_D or t_D is adjacent to Z or to some other component of A not equal to D, we find that $W_1 \cup W_2$ contain at least one of s_D, t_D for each component D of A.

We choose smallest sets U_1'' and U_2'' in U_1' and U_2', respectively, that dominate W_2 and W_1, respectively. By minimality, each vertex $u \in U_1''$ must have a "private" neighbour t_D in W_2, and hence together with t_D and s_D, corresponds to a "private" P_3. Consequently, as G is sP_3-free and $U_1'' \subseteq U_1$ is an independent set, U_1'' has size at most $s-1$. Similarly, U_2'' has size at most $s-1$. Moreover, each vertex in $U_1'' \cup U_2''$ is adjacent to Z (again due to the sP_3-freeness of G).

Figure 1 shows an example in which the components of A consist on three cliques (the first two of size two and the last one of size one) to illustrate the situation.

We now do as follows. First, for each component D of A we pick one of its vertices v and swap v with s_D if $s_D \in W_1$ and otherwise we swap v with t_D (note that $t_D \in W_2$ in that case). We also add all vertices of $U_1'' \cup U_2''$ to Z and thus to S'. The results of these swaps are as follows. First, $G[S']$ has become connected. Second, S' has increased in size at most by $2(s-1)$, which is allowed. Third, $G - S'$ is still bipartite (as swapping a vertex of a component D of A with s_D

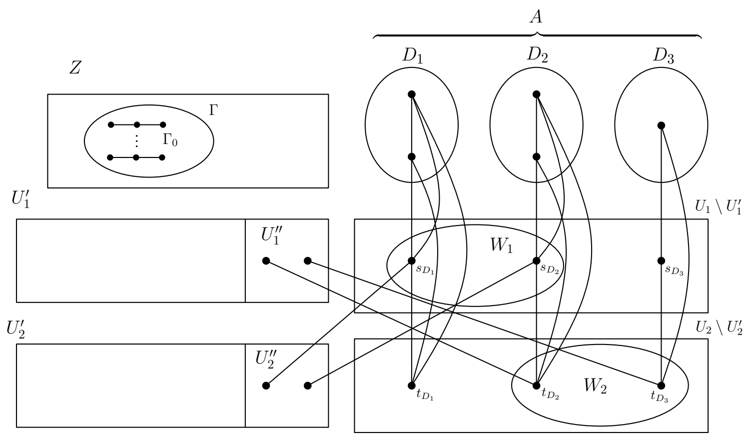

Fig. 1. The situation in the proof of Lemma 6.

or t_D does not create any odd cycles). Consequently, we have found a connected $\mathcal{F}$-transversal of size at most $|S|+4s^2-5s-3+16s^2+4s+(s-1)+2(s-1) = |S|+20s^2+2s-6$, so we can take $d_{sP_3} = 20s^2+2s-6$. $\square$

Belmonte et al. [1] proved that the class of connected (P_2+P_4, P_6)-free graphs is not $\mathcal{F}$-additive if $\mathcal{F}$ is the class of all cycles. We have the following more general result.

Lemma 7. *For any family of cycles $\mathcal{F}$ with $C_3 \in \mathcal{F}$, the class of connected (P_2+P_4, P_6)-free graphs is not $\mathcal{F}$-additive.*

We now state the following result. To prove it we use the previous lemmas for the first two claims. For the third claim we observe that connected P_3-free graphs are complete (proving the case $H \subseteq_i P_3$) and that $K_{2,2,2}$ is a counterexample for the case $H \not\subseteq_i P_3$.

Theorem 2. *For any graph H and for any family of cycles $\mathcal{F}$ containing all odd cycles, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative if and only if H is a linear forest;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$ for some $s \geq 0$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_3$.*

4 Cycle Families with 4-Cycles but no 3-Cycles

In this section we consider families of cycles $\mathcal{F}$ such that $C_3 \notin \mathcal{F}$ but $C_4 \in \mathcal{F}$. To prove our results we need a lemma.

Lemma 8. *For any family $\mathcal{F}$ of cycles with $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$:*

– *the class of connected P_5-free graphs is not $\mathcal{F}$-additive.*

– *the class of connected $P_2 + P_4$-free graphs is not $\mathcal{F}$-additive.*
– *the class of connected $2P_3$-free graphs is not $\mathcal{F}$-additive.*
– *the class of connected $3P_2$-free graphs is not $\mathcal{F}$-additive.*

We now state our result for infinite families of cycles $\mathcal{F}$ with $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$. It does not provide a complete characterization as we are unable to give necessary and sufficient conditions for the class of H-free graphs to be $\mathcal{F}$-additive. This would be possible if it could be shown that $(P_3 + P_2 + sP_1)$-free graphs are $\mathcal{F}$-additive for all $s \geq 0$. Due to Lemma 2, this is the case if and only if $(P_3 + P_2)$-free graphs are $\mathcal{F}$-additive, which we conjecture to be true.

Theorem 3. *For any graph H and for any infinite family of cycles $\mathcal{F}$ with $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative if and only if H is a linear forest;*
- *$\mathcal{F}$-additive if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, but not if $H \nsubseteq_i P_4 + sP_1$ nor $H \nsubseteq_i P_3 + P_2 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_3$.*

Proof. The first claim follows from Theorem 1. We now prove the second claim. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from Lemmas 1 and 2. Now suppose $H \nsubseteq_i P_4 + sP_1$ and $H \nsubseteq_i P_3 + P_2 + sP_1$ for any $s \geq 0$. By Theorem 1, we may assume that H is a linear forest. Then $P_5 \subseteq_i H$, $P_2 + P_4 \subseteq_i H$, $2P_3 \subseteq_i H$ or $3P_2 \subseteq_i H$ and we can use Lemma 8.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, so the result follows directly. If $H \nsubseteq_i P_3$ then, by Theorem 1, we may assume that H is a linear forest. Hence, $3P_1 \subseteq_i H$ or $P_1 + P_2 \subseteq_i H$.

If $P_1 + P_2 \subseteq_i H$, then we have that the complete bipartite graph $G = K_{3,3}$ is a connected H-free graph (since it is $P_1 + P_2$-free). And $t_{\mathcal{F}}(G) = 2 < 3 = ct_{\mathcal{F}}(G)$ so the class of connected H-free graphs is not $\mathcal{F}$-identical.

Finally, suppose that $3P_1 \subseteq_i H$, and let G be the complement of the graph shown in Figure 2. Since $\overline{G}$ is triangle-free and every two vertices of $\overline{G}$ have a common non-neighbour, G is a connected $3P_1$-free graph. As every $\mathcal{F}$-transversal of G must intersect every induced $2P_2$ in $\overline{G}$, the minimum $\mathcal{F}$-transversals of G are in bijective correspondence with the four edges of the 4-cycle in $\overline{G}$. So $t_{\mathcal{F}}(G) = 2 < 3 = ct_{\mathcal{F}}(G)$, and the class of connected H-free graphs is also not $\mathcal{F}$-identical in this case. $\square$

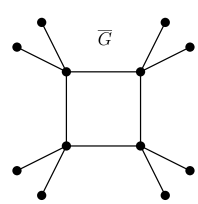

Fig. 2. The complement of a graph G with $t_{\mathcal{F}}(G) < ct_{\mathcal{F}}(G)$ whenever $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$.

5 Cycle Families with 5-Cycles but no 3- or 4-Cycles

In this section we consider families of cycles $\mathcal{F}$ such that $C_3, C_4 \notin \mathcal{F}$ but $C_5 \in \mathcal{F}$.

We first give the following lemma; note that C_3 and C_4 are both induced subgraphs of $\overline{2P_4}$.

Lemma 9. *Let $\mathcal{F}$ be a graph family with $C_5 \in \mathcal{F}$ that contains no induced subgraphs of $\overline{sP_4}$ for any $s \geq 1$. Then the class of connected $2P_2$-free graphs is not $\mathcal{F}$-additive.*

Proof. We describe a family of connected $2P_2$-free graphs that is not $\mathcal{F}$-additive, where $\mathcal{F}$ is any family of cycles as in the statement of the lemma. The graphs in the family consist of $k \geq 2$ copies of the join of two P_4s, say $H_1, \ldots, H_k$. For each of them there is a new vertex v_i adjacent to both endpoints of the two P_4s, and in addition there are all possible edges between vertices in different H_i's.

We first show that every graph G in this family is $2P_2$-free. Every edge e of G has at least one endpoint in some H_i, say in H_1. Deleting the closed neighbourhood of e results in the subgraph induced by a subset of $\{v_1, \ldots, v_k\}$ (if $e \in E(H_1)$), or in the subgraph induced by $\{u, v_2, \ldots, v_k\}$ for some $u \in V(H_1)$ (otherwise). In either case, the resulting graph is edgeless. Therefore, G is $2P_2$-free. Let G be a graph in this family, and let k be the number of H_i's. We have $t_{\mathcal{F}}(G) \leq k$ since deleting the vertices $v_1, \ldots, v_k$ results in a graph that is isomorphic to $\overline{2kP_4}$ and thus $\mathcal{F}$-free. On the other hand, every connected $\mathcal{F}$-transversal S of G must contain at least two vertices from each subgraph induced by $V(H_i) \cup \{v_i\}$, for every i (otherwise it either misses an induced C_5 or contains only v_i, making it isolated in $G[S]$). Therefore, $ct_{\mathcal{F}}(G) \geq 2k$, which establishes the non-$\mathcal{F}$-additivity of the family. □

Lemma 10. *Let $\mathcal{F}$ be a family of graphs that contains C_5 but no induced subgraph of $\overline{4P_4}$. Then the class of connected $3P_1$-free graphs is not $\mathcal{F}$-identical.*

Theorem 4. *For any graph H and for any graph family $\mathcal{F}$ which only contains graphs with an induced P_4, including C_5 and an infinite number of other cycles but no linear forests and no induced subgraphs of $\overline{sP_4}$ for any $s \geq 1$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative if and only if H is a linear forest;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_4$.*

Proof. The first claim follows from Theorem 1. We now prove the second claim. First suppose that $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$. Then the class of connected H-free graphs is $\mathcal{F}$-additive due to Lemmas 1 and 2. Now suppose that $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$. By Theorem 1, we may assume that H is a linear forest. Hence, $2P_2 \subseteq_i H$ and we use Lemma 9. Finally, we show the third claim. Recall that if $H \subseteq_i P_4$ then any H-free graph is already $\mathcal{F}$-free. Suppose that $H \not\subseteq_i P_4$. If $2P_2 \subseteq_i H$ we use Lemma 9 again. Hence $3P_1 \subseteq_i H$. In that case we use Lemma 10. This completes the proof of Theorem 4. □

6 Conclusions

We extended the tetrachotomy result of Belmonte et al. [1] for the family $\mathcal{F}$ of all cycles by giving tetrachotomy results for a number of natural families $\mathcal{F}$ containing cycles and anticycles (see Table 1). Let us recall that a tetrachotomy for the price of connectivity of $\mathcal{F}$-transversals when $\mathcal{F}$ is the family of even cycles or of all holes is still an open case. To settle it, it would suffice to show that the set of connected $(P_3 + P_2)$-free graphs is $\mathcal{F}$-additive which we conjecture to be true. We also have no tetrachotomy for families $\mathcal{F}$ that contain C_3 but that miss some other odd cycle. The partial results below show that a more refined analysis is needed to obtain complete results in this direction.

We first summarize our current knowledge. By Theorem 1 we know that the class of H-free graphs is $\mathcal{F}$-multiplicative if and only if H is a linear forest. We also know, due to Lemma 7, that the class of connected (P_2+P_4, P_6)-free graphs is not $\mathcal{F}$-additive. Moreover, the class of connected H-free graphs is $\mathcal{F}$-identical if and only if $H \subseteq_i P_3$, as we can use the example of $G = K_{2,2,2}$ from Theorem 2. Hence, using Lemmas 1, 2, and 4, we see that what remains is to check, for every $s \geq 2$, whether the class of H-free graphs is $\mathcal{F}$-additive if $H = sP_3$. We can show that already for $s = 2$ this is true for some families $\mathcal{F}$ and false for others.

Proposition 1. *For any family of cycles $\mathcal{F}$ containing C_3 and C_5, the class of connected $2P_3$-free graphs is $\mathcal{F}$-additive.*

Proposition 2. *For any family $\mathcal{F}$ of cycles with $C_3 \in \mathcal{F}$ and $C_5 \notin \mathcal{F}$, the class of connected $2P_3$-free graphs is not $\mathcal{F}$-additive.*

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