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A direct SIF approach for anisotropic materials using the Boundary Element Method

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ABSTRACT

Recently developments in numerical methods such as the Extended Finite Element Method (X-FEM) and the Extended Boundary Element Method (X-BEM) have significantly improved the accuracy in linear elastic fracture mechanics (LEFM) problems. Nevertheless, the postprocessing involved in the calculation of the Stress Intensity Factors still presents a computational burden. The most usual SIF evaluation techniques are the J-integral or the more general Interaction Integral (M-integral), which may require considerable computational resources of the order of those required to obtain the solution of the LEFM problem. A direct approach, where the SIFs are revealed in the solution vector, has been developed by [1] for isotropic materials and is further extended to anisotropic materials in this work.

Key Words: *Boundary element method; Anisotropic materials; Enrichment functions; Fracture mechanics*

1. Introduction

The boundary element method (BEM) is a well established discretisation method when dealing with fracture mechanics problems. It offers high accuracy and stability in providing results for the discontinuous, singular stress fields at crack tips than domain discretisation methods such as the Finite Element Method (FEM). However, in the last 15 years, the use of the partition of unity has improved the performance of the FEM, being further called Extended Finite Element Method (X-FEM) [1]. It was verified that the accuracy of the X-FEM could be similar to that found in BEM (see [2] for instance).

The partition of unity approach was first employed in [3] in a dual BEM formulation, in order to be more general than the quarter-point [4]. The quarter-point approach displaces the position of the central node to capture the proper $\sqrt{r}$ behaviour at the crack tip (r being the distance from the crack tip). However, enrichment functions as specified in [2, 3] cause the linear system to suffer from ill-conditioning arising from the addition of new degrees of freedom to the problem. In a recent work, the authors [5] have developed an implicit enrichment, where only a couple of degrees of freedom are added to the problem for each crack tip, hence adding the enrichment to more elements and not increasing the total number of degrees of freedom of the fracture problem. This methodology was called direct approach, and was obtained using the displacement field around the crack tip with the Williams' expansion, valid for isotropic materials only.

In this work, we propose a more general form of enrichment functions for anisotropic materials, obtained from the Stroh formalism following the work of [2].

2. Governing equations

Consider an anisotropic elastic domain Ω, the static equilibrium equations in the presence of body forces $\mathbf{b}$ are defined as

$$\sigma_{ij,j} + b_i = 0 \tag{1}$$

Symmetry holds for the stress and strain tensors, i.e.: $\sigma_{ij} = \sigma_{ji}$; $\varepsilon_{ij} = \varepsilon_{ji}$, where $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

The linear constitutive equations are given by the generalized Hooke's law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{2}$$

where C_{ijkl} define the material constants tensor, satisfying the following symmetry relations

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \tag{3}$$

3. Enrichment formulation

Adopting a polar coordinate system (r, θ) with origin at the crack tip, the asymptotic displacement field around a crack-tip in a plane anisotropic domain can be expressed by means of the Stroh formalism [6] as

$$u_i(r,\theta) = \sqrt{\frac{2}{\pi}}\Re\left(K_\alpha A_{im} B^{-1}_{m\alpha}\sqrt{r\left(\cos\theta + \mu_m \sin\theta\right)}\right) \tag{4}$$

where the summation convention over repeated indices holds; $i, m = 1, 2$; $\alpha = I,\ II$ is associated with the fracture modes; and $\Re(\cdot)$ is the real part of $(\cdot)$; $\mathbf{A}$, $\mathbf{B}$ and $\boldsymbol{\mu}$ are obtained from the following eigenvalue problem

$$\left(\begin{array}{c|c} -\mathbf{C_{22}}^{-1}\mathbf{C_{21}} & -\mathbf{C_{22}}^{-1} \\ \hline \mathbf{C_{11}} - \mathbf{C_{21}}^{T}\mathbf{C_{22}}^{-1}\mathbf{C_{21}} & -\mathbf{C_{21}}^{T}\mathbf{C_{22}}^{-1} \end{array}\right)\left(\begin{array}{c} \mathbf{A}_m \\ \hline \mathbf{B}_m \end{array}\right) = \mu_m \left(\begin{array}{c} \mathbf{A}_m \\ \hline \mathbf{B}_m \end{array}\right) \quad \text{(no sum on } m\text{)} \tag{5}$$

with

$$\mathbf{C_{11}} := \mathbf{C}_{1i1};\quad \mathbf{C_{21}} := \mathbf{C}_{2i1};\quad \mathbf{C_{22}} := \mathbf{C}_{2i2} \tag{6}$$

Expanding and rearranging the terms of Eq. (4) in the same way as in [2], the enrichment functions are calculated as

$$F_{lj}(r,\theta) = \sqrt{\frac{2r}{\pi}}\begin{pmatrix} A_{11}B^{-1}_{11}\beta_1 + A_{12}B^{-1}_{21}\beta_2 & A_{11}B^{-1}_{12}\beta_1 + A_{12}B^{-1}_{22}\beta_2 \\ A_{21}B^{-1}_{11}\beta_1 + A_{22}B^{-1}_{21}\beta_2 & A_{21}B^{-1}_{12}\beta_1 + A_{22}B^{-1}_{22}\beta_2 \end{pmatrix} \tag{7}$$

where $\beta_i = \sqrt{\cos\theta + \mu_i \sin\theta}$, r is the distance between the crack tip and an arbitrary position, θ is the orientation measured from a coordinate system centred at the crack tip. Note that these enrichment functions are the equivalent of Williams' expansion for isotropic materials [5].

The anisotropic enrichment functions can also be used for isotropic materials, since this is a degenerated case from anisotropic materials (both eigenvalues are equal). For more details please refer to [2].

The displacement field can be defined in a similar fashion as [7]

$$u_j = \sum_{a=1}^{M} N^a u_j^a + \tilde{K}_I F_{Ij} + \tilde{K}_{II} F_{IIj} \tag{8}$$

where N^a represents the shape function for node a, u_j^a is a general coefficient rather than the nodal displacement, M is the number of nodes, $\tilde{K}_I$ and $\tilde{K}_{II}$ stand for the mode I and mode II Stress Intensity Factors (SIF), respectively, and they are now part of the solution vector instead of being calculated after the displacement solution is obtained. For the numerical discretisation of the fracture mechanics problem, the BEM is used.

4. Boundary Element Method (BEM)

The proposed enrichment is used with a Dual BEM formulation. The superposition of two boundaries (crack faces) can cause the degeneration of the linear system of equations if only the displacement boundary integral equation (DBIE) is employed. A traction boundary integral equation (TBIE) can be obtained from the differentiation of DBIE and further substitution into the constitutive laws equation (see [8] for details). The DBIE and the TBIE for the direct approach are given as [5].

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi}) + \int_\Gamma p^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) + \int_{\Gamma_c} p^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})\tilde{K}_{lj}F_{lj}(\boldsymbol{\xi})\mathbf{a}^\alpha_k d\Gamma = \int_\Gamma u^*_{ij}(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{9}$$

$$c_{ij}(\boldsymbol{\xi})p_j(\boldsymbol{\xi}) + N_r\int_\Gamma s^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) + N_r\int_{\Gamma_c} s^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})\tilde{K}_{lj}F_{lj}(\boldsymbol{\xi})\mathbf{a}^\alpha_k d\Gamma = N_r\int_\Gamma d^*_{rij}(\boldsymbol{x},\boldsymbol{\xi})p_j(\boldsymbol{x})d\Gamma(\boldsymbol{x}) \tag{10}$$

where $\Gamma_c = \Gamma_+ \cup \Gamma_-$ stands for the crack surfaces Γ_+ and Γ_- and N_r is the normal at the observation point. Let us recall that strongly singular and hypersingular terms arise from the integration of the p^*_{ij}, d^*_{rij} and s^*_{rij} kernels and they are regularised using the methodology proposed in [9], while the weakly singular terms are dealt using Telles transformation [10].

The addition of $\tilde{K}_I$ and $\tilde{K}_{II}$ requires two more equations so the linear system of equations can be solved. The additional equations come from a restriction in the crack faces, in order to remove the displacement discontinuity observed at the crack tip. The displacement continuity can be enforced as

$$\sum_{a=1}^{L} N^a u_j^{a\ upper} = \sum_{a=1}^{L} N^a u_j^{a\ lower} \tag{11}$$

where L is the number of nodes used for the crack tip extrapolation. Eq. (11) is applied for both x and y directions, resulting in two different equations.

5. Numerical results

To validate the proposed approach, an isotropic material model is used. As specified previously, the isotropic case is a special case of an anisotropic material, where both the eigenvalues are equal, and special measures must be taken [11]. A small disturbance can be imposed into the eigenvalues, so they will be no longer be equal and the enrichment functions can be used. In this example, $E = 10000$ is the Young's modulus and $\nu = 0.3$ is the Poisson ratio.

Figure 1 illustrates a rectangular plate ($h/w = 0.5$) with a single edge crack of length a under a uniform loading σ. The size of the crack is defined by $a/w = 0.5$. Results for the X-FEM are obtained using topological and geometrical enrichment, with a fixed area of $r_e/a = 0.2$. For more information about both adopted enrichment types please refer to [2] for instance. Results are illustrated in Figure 2, where the proposed direct approach for calculating the SIFs presents similar accuracy to other numerical approaches, such as the J-integral and an X-BEM formulation in [3].

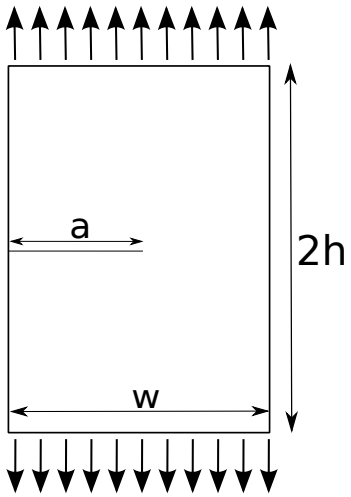

Figure 1: : Edge crack plate under a uniform load.

6. Conclusions

A direct approach for calculating the SIF in anisotropic materials has been presented in this work. The used enrichment functions have the advantage of depending only on the material properties, presented in a concise matrix form. Moreover, there are no further dependencies on the orientation of the material, which allow the enrichment functions to be as general as possible. The results are seen to match the reference solution, so we can conclude that the direct approach is an alternative method for obtaining the SIF, with similar precision to other evaluation methods for obtaining the SIF.

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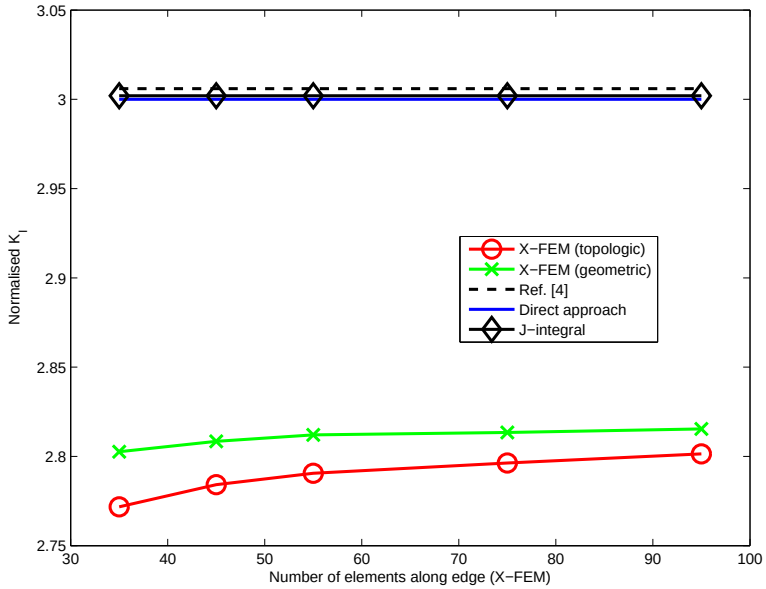

Figure 2: : Mode I SIFs for the edge crack problem.

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