On the formulation of thermodynamically-consistent viscoplastic-damage constitutive models

T.J. Birchall[a,*], A.S. Osman[a]

[a]School of Engineering and Computing Sciences, Durham University, South Road, Durham, DH1 3LE

Abstract

This paper illustrates the formulation of viscoplastic-damage constitutive models using the framework of hyperplasticity. The entire constitutive behaviour is derived from only two scalar potentials; a free energy potential and a dissipation potential. This ensures that the model obeys the laws of thermodynamics.

Keywords: viscoplasticity, damage, creep, constitutive model, hyperplasticity.

1. Introduction

Time-dependent constitutive models are essential for modelling the deformation of underground cavities excavated in creeping rock and are very useful in producing a successful design. Creep behaviour is prominent in soft rocks such as salt rock and potash, and has even been observed in hard rocks, highlighting the importance of designing for such behaviour.

In this paper we present a new class of time-dependent constitutive models which are able to take into account both the secondary and tertiary stages of creep deformation. Three stages of creep are typically observed in laboratory creep tests: a primary stage considered to be viscoelastic, a secondary stage considered to be viscoplastic and a tertiary stage considered to be caused by material damage, through the formation of cracks. Thus here we develop a viscoplastic-damage constitutive model. These models are novel as they are derived using the hyperplastic framework of Houlsby and Puzrin [1] and therefore automatically obeys the laws of thermodynamics. In this approach the entire constitutive behaviour can be derived from two scalar potentials; a free energy potential which provides the elasticity law, and a dissipation potential which provides the yield function, the direction of plastic flow and the evolution of a damage variable. No additional assumptions are required. We assume that the viscoplasticity and damage are coupled, therefore they occur simultaneously. The classical sign convention in geomechanics is adopted; where compressive stresses are taken to be positive. The constutive models are formulated in term of effective stresses.

2. Formulation of a Von-Mises viscoplastic-damage constitutive model

This section presents the derivation of the constitutive equations using the hyperplastic framework. The free energy potential can be written for a damaged material as follows

$$\psi = \frac{1}{2}\{\varepsilon^e\}(1-\alpha_d)[D^e]\{\varepsilon^e\}, \tag{1}$$

where $\{\varepsilon^e\}$ is the elastic strain vector, $[D^e]$ is the elastic stiffness matrix and α_d is a scalar damage parameter.

Differentiating the free energy potential with respect to the elastic strain vector produces the elasticity law

$$\{\sigma\} = \frac{\partial\psi}{\partial\{\varepsilon^e\}} = (1-\alpha_d)[D^e]\{\varepsilon^e\}. \tag{2}$$

The free energy potential (equation (1)) can be expressed as follows in terms of deviatoric and mean stresses

$$\psi = \frac{q^2}{6G(1-\alpha_d)} + \frac{p^2}{2K(1-\alpha_d)}, \tag{3}$$

*T.J. Birchall
Email address: t.j.birchall@durham.ac.uk (T.J. Birchall)

where q is the von Mises equivalent stress defined as $q = \sqrt{3J_2}$, where $J_2 = \frac{1}{2}\mathrm{tr}([s][s])$, where the deviatoric stress is $[s] = [\sigma] - p[1]$ and [1] is the three by three identity matrix, p is the mean stress, G is the shear modulus and K is the bulk modulus.

The stresses associated with the damage can then be derived as follows

$$\chi_d = \frac{\partial\psi}{\partial\alpha_d} = \frac{q^2}{6G(1-\alpha_d)^2} + \frac{p^2}{2K(1-\alpha_d)^2}. \tag{4}$$

For a coupled viscoplastic-damage material the dissipation potential $\dot{D}$ can be defined as

$$\dot{D} = \sigma_y \sqrt{(r_p\dot{\alpha}_{s1} + r_p\eta^{n-1}\dot{\alpha}_{s2}^n)^2 + (r_d\rho\dot{\alpha}_d)^2}, \tag{5}$$

where $\dot{\alpha}_{s1}$ and $\dot{\alpha}_{s2}$are internal variables representing the equivalent deviatoric plastic strain rate $\dot{\alpha}_s(\alpha_{s1} = \alpha_{s2} = \alpha_s)$, η is the viscous coefficient which controls the extent of plastic strain, n is a material constant, whilst r_p and r_d are constants governing the ratio of viscoplasticity and damage. $\sigma_y = \sigma_{y0}\Pi(\alpha_d)$, where σ_{y0} is the uniaxial yield stress. ρ is defined as

$$\rho = \frac{q}{6G(1-\alpha_d)^2} + \frac{p^2}{2K(1-\alpha_d)^2 q}, \tag{6}$$

It should be noted that for a rate-independent von Mises material, the above dissipation potential reduces to the form given by Einav et al. [2].

Differentiation of the dissipation potential with respect to the plastic strains gives the dissipative stresses.

$$\chi_q = \frac{\partial\dot{D}^*}{\partial\dot{\alpha}_{s1}} + \frac{1}{n}\frac{\partial\dot{D}^*}{\partial\dot{\alpha}_{s2}}. \tag{7}$$

Noting that $\alpha_{s1} = \alpha_{s2} = \alpha_s$, we can show that

$$\chi_q = \frac{\sigma_y(r_p\dot{\alpha}_s + r_p\eta^{n-1}\dot{\alpha}_s^n)r_p(1+(\eta\dot{\alpha}_s)^n)}{\sqrt{(r_p\dot{\alpha}_s + r_p\eta^{n-1}\dot{\alpha}_s^n)^2 + (r_d\rho\dot{\alpha}_d)^2}}. \tag{8}$$

Differentiating with respect to $\dot{\alpha}_d$ gives the dissipative stress due to damage

$$\chi_d = \frac{\partial\dot{D}^*}{\partial\dot{\alpha}_d} = \frac{\sigma_y\dot{\alpha}_d(r_d\rho)^2}{\sqrt{(r_p\dot{\alpha}_s + r_p\eta^{n-1}\dot{\alpha}_s^n)^2 + (r_d\rho\dot{\alpha}_d)^2}}. \tag{9}$$

Since there is no kinematic hardening, the true stresses and the dissipative stresses are identical. From equations (8)and (9):

$$\left(\frac{q}{r_p\sigma_y(1+(\eta\dot{\alpha}_s)^{n-1})}\right)^2 + \left(\frac{q}{r_d\sigma_y}\right)^2 = 1. \tag{10}$$

Rearranging equation (10) we obtain

$$\dot{\alpha}_s = \frac{1}{\eta}\left(\frac{1}{\sqrt{\left(\frac{\sigma_y r_p}{q}\right)^2 - \left(\frac{r_p}{r_d}\right)^2}} - 1\right)^{\frac{1}{n-1}} \tag{11}$$

The plastic strain rate $\dot{\alpha}_s$ is a non negative quantity, therefore the expression inside the bracket in equation (11) must be greater than or equal to zero. Therefore the yield criterion is given by

$$f = \left(\frac{q}{\sigma_y}\right)^2\left(\frac{1}{r_p^2} + \frac{1}{r_d^2}\right) - 1 \tag{12}$$

From this equation we can find that the constants r_p and r_d are related by the following expression

$$\frac{1}{r_p^2} + \frac{1}{r_d^2} = 1, \tag{13}$$

from which it follows that $r_p \geq 1$ and $r_d \geq 1$. In the limiting case when $r_p = 1$, $r_d \to \infty$ and the constitutive model simplifies to the well-known Perzyna viscoplastic model [4].

An expression for the damage internal variable can be found from equation (9), bearing in mind that $\rho = \frac{\chi_d}{q}$

$$\dot{\alpha}_d = \frac{\dot{\alpha}_s \left(1 + (\dot{\alpha}_s \eta)^{n-1}\right)^2}{\rho} \left(\frac{r_p}{r_d}\right)^2. \tag{14}$$

3. Formulation of a Drucker-Prager visco-plastic constitutive model

For a frictional material, the dissipation function can be assumed to take the form:

$$\dot{D} = \sqrt{(r_p(c_2 p + c_1)\dot{\alpha}_{s1} + r_p \eta^{n-1} \dot{\alpha}_{s2}^n)^2 + (r_d \rho (c_2 p + c_1)\dot{\alpha}_d)^2}, \tag{15}$$

where $c_1 = d\Pi(\alpha_d)$ and $c_2 = \mu\Pi(\alpha_d)$, where d is a constant related to the cohesion, μ is a constant related to the frictional angle and $\Pi(\alpha_d)$ is the softening/hardening function. ρ is defined as

$$\rho = \frac{q^2}{6G(1-\alpha_d)^2(q - c_3 p)} + \frac{p^2}{2K(1-\alpha_d)^2(q - c_3 p)}, \tag{16}$$

where $c_3 = \beta\Pi(\alpha_d)$ and where β is a constant, related to the dilation angle. d, μ and β can be calculated from the cohesion, friction angle and dilation angle.
As a side condition we will impose a linear relationship between the volumetric and shear strain rates

$$\dot{\alpha}_v + c_3 \dot{\alpha}_s = 0, \tag{17}$$

where $\alpha_v = \mathrm{tr}([\varepsilon^p])$.

Starting from the dissipation function of equation (15) together with the constrain given by equation (17) and following the procedure illustrated in the previous section, it can be shown that

$$\dot{\alpha}_s = \frac{1}{\eta}\left((c_2 p + c_1)\left(\frac{1}{\sqrt{\left(\frac{r_p}{ck}\right)^2 - \left(\frac{r_p}{r_d}\right)^2}} - 1\right)\right)^{\frac{1}{n-1}} \tag{18}$$

where

$$ck = \frac{(q - c_3 p)}{(c_2 p + c_1)}. \tag{19}$$

And the damage internal variable is given by

$$\dot{\alpha}_d = \frac{\dot{\alpha}_s \left(1 + \frac{(\dot{\alpha}_s \eta)^{n-1}}{(c_2 p + c_1)}\right)^2}{\rho} \left(\frac{r_p}{r_d}\right)^2. \tag{20}$$

The yield criterion is given by

$$f = ck^2 \left(\frac{1}{r_p^2} + \frac{1}{r_d^2}\right) - 1 \tag{21}$$

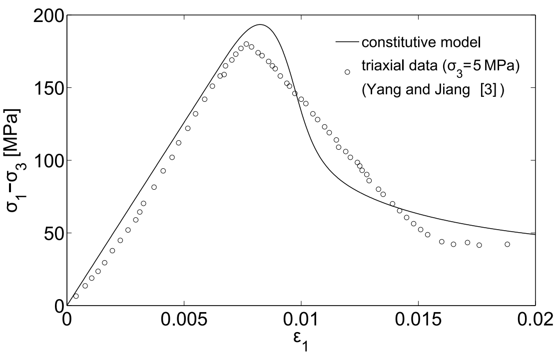
In order to use this constitutive model for practical application we must use a stress integration procedure. Here we use the iterative implicit backward Euler (bE) stress integration scheme.

4. Validation with triaxial data

The Drucker-Prager viscoplastic-damage constitutive model, which is applicable to geomaterials, has been validated using data from triaxial compression testing of sandstone carried out by Yang and Jiang [3]. The experimental data was obtained from short-term loading tests and long-term creep tests. Note that here we use a softening/hardening function $\Pi(\alpha_d)$ equal to $(1 - \alpha\alpha_d)^2$, where α is the softening hardening parameter. This function fits well with the experimental data.

The short-term loading tests were conducted by Yang and Jiang [3] under different confining pressures. We simulated these triaxial tests by using a single material point in MATLAB 7.9.0. We apply the initial confining pressure in the first time step and then apply an increment of strain over each subsequent time step, ensuring that the rate of applied strain corresponds to a rate of applied stress of 0.127 MPa/s, which is consistent with the rate adopted in the triaxial experiments. This rate of stress is applied over the full time period of the simulation (prior to and during material softening). Yang and Jiang [3] conducted a creep test under multi levels of stress. The applied rate of stress $(\sigma_1 - \sigma_3)$ was the same as that used in the short-term tests (0.127 MPa/s), with a confinement σ_3 of 5 MPa. When a required level of stress was obtained it was held constant and the sandstone was allowed to creep for 48 hours. The final and highest level of stress used was $\sigma_1 - \sigma_3 = 160$ MPa. Under this constant level of stress, the three typical stages of creep were observed. This stress was below the peak strength of the material, but corresponded to the point where yield was initiated. In order to simulate a creep test, we modified the program used for the short-term tests by keeping $\sigma_1 - \sigma_3$ constant when yield begins. Here we do not simulate a multi-stage creep test, but rather use a single level of stress equal to the final level of stress used by Yang and Jiang [3] $(\sigma_1 - \sigma_3 = 160$ MPa). This is justified from the results of triaxial tests carried out by Yang and Jiang [3]. Their findings showed that very little creep occurred below the yield stress of the sandstone.

Figure 1 compares the short-term stress-strain curves obtained by Yang and Jiang [3] and those obtained using the new frictional constitutive model. The model parameters used in the simulations are shown in Table 1. We observe that the constitutive model provides a good prediction of the experimental data for all three confining pressures, at both peak stress and in the region of material softening. The experimental data obtained from the long-term tests is shown in Figure 2, along with the response of the new constitutive model. Here we observe that the constitutive model underestimates the strain shown in the experimental data. However the overall shape of the curves is very similar, therefore the constitutive model provides a useful approximation.



(a)

Figure 1: Validating constitutive model using short-term stress-strain triaxial data

d[MPa]	μ	β	η[MPa h]	n	α	r_p
21.95	0	−2.38	1525	1.5	0.375	1.05

Table 1: Values of constitutive model parameters used in the triaxial simulation

It should be noted that more experimental data is required in order to determine the constitutive model parameters more accurately. Triaxial tests under different rates of loading are required, as the magnitude of the peaks of the short-term stress strain curves shown in Figure 1 is dependent on the rate of loading. A higher rate of loading could produce higher peaks, whilst a lower rate of loading could produce lower peaks. In order to obtain the strength parameters d, μ and β more accurately it is necessary to conduct triaxial tests under low rates of loading.

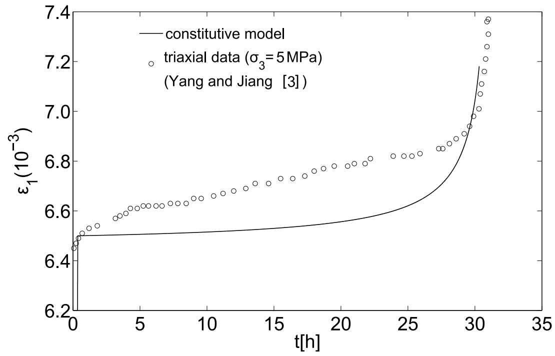


Figure 2: Creep response of model under yield stress

5. Conclusions

A novel viscoplastic-damage constitutive models have been derived within the framework of hyperplasticity. These models are able to describe both the secondary and tertiary stages of creep behaviour. The new frictional viscoplastic-damage model is validated with experimental data obtained by Yang and Jiang [3] for samples of sandstones. This model can be used for a variety of applications such as the creep analysis of underground tunnels and storage caverns as well as applications outside the field of geomechanics. The model can be generalised by incorporating an appropriate lode angle dependency function such as that suggested by Eekelen [5].

References

[1] G. T. Houlsby, A. M. Puzrin, Principles of hyperplasticity: an approach to plasticity theory based on thermodynamic principles, Springer-Verlag, 2006.

[2] I. Einav, G. T. Houlsby, G. D. Nguyen, Coupled damage and plasticity models derived from energy and dissipation potentials, International Journal of Solids and Structures 44 (2006) 2487–2508.

[3] S. Yang, Y. Jiang, Triaxial mechanical creep behaviour of sandstone, Mining Science and Technology 20 (2010) 339–349.

[4] P. Perzyna, Fundamental problems in viscoplasticity, Advances in Applied Mechanics, 9 (1966) 243–377.

[5] H. A. M. Eekelen, Isotropic yield surface in three dimensions for use in soil mechanics, International Journal for Numerical and Analytical Methods in Geomechanics 4 (1980) 89–101.