New Geometric Representations and Domination Problems on Tolerance and Multitolerance Graphs*

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Abstract

Tolerance graphs model interval relations in such a way that intervals can tolerate a certain amount of overlap without being in conflict. In one of the most natural generalizations of tolerance graphs with direct applications in the comparison of DNA sequences from different organisms, namely *multitolerance* graphs, two tolerances are allowed for each interval – one from the left and one from the right side. Several efficient algorithms for optimization problems that are NP-hard in general graphs have been designed for tolerance and multitolerance graphs. In spite of this progress, the complexity status of some fundamental algorithmic problems on tolerance and multitolerance graphs, such as the *dominating set* problem, remained unresolved until now, three decades after the introduction of tolerance graphs. In this article we introduce two new geometric representations for tolerance and multitolerance graphs, given by points and line segments in the plane. Apart from being important on their own, these new representations prove to be a powerful tool for deriving both hardness results and polynomial time algorithms. Using them, we surprisingly prove that the dominating set problem can be solved in polynomial time on tolerance graphs and that it is APX-hard on multitolerance graphs, solving thus a longstanding open problem. This problem is the first one that has been discovered with a different complexity status in these two graph classes. Furthermore we present an algorithm that solves the independent dominating set problem on multitolerance graphs in polynomial time, thus demonstrating the potential of this new representation for further exploitation via sweep line algorithms.

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1 Introduction

A graph $G = (V, E)$ on n vertices is a *tolerance* graph if there exists a collection $I = \{I_v \mid v \in V\}$ of intervals on the real line and a set $t = \{t_v \mid v \in V\}$ of positive numbers (the tolerances), such that for any two vertices $u, v \in V$, $uv \in E$ if and only if $|I_u \cap I_v| \geq \min\{t_u, t_v\}$, where $|I|$ denotes the length of the interval I. The pair $\langle I, t\rangle$ is called a *tolerance representation* of G. If G has a tolerance representation $\langle I, t\rangle$, such that $t_v \leq |I_v|$ for every $v \in V$, then G is called a *bounded tolerance* graph.

If we replace in the above definition "min" by "max", we obtain the class of *max-tolerance* graphs. Both tolerance and max-tolerance graphs have attracted many research

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efforts [2,4,7,9,10,12,14–17] as they find numerous applications, especially in bioinformatics, among others [10,12,14]; for a more detailed account see the book on tolerance graphs [11]. One of their major applications is in the comparison of DNA sequences from different organisms or individuals by making use of a software tool like BLAST [1]. However, at some parts of the above genomic sequences in BLAST, we may want to be more tolerant than at other parts, since for example some of them may be biologically less significant or we have less confidence in the exact sequence due to sequencing errors in more error prone genomic regions. This concept leads naturally to the notion of *multitolerance* graphs which generalize tolerance graphs [11,15,19]. The main idea is to allow two different tolerances for each interval, one to each of its sides. Then, every interval tolerates in its interior part the intersection with other intervals by an amount that is a convex combination of these two border-tolerances.

Formally, let $I = [l, r]$ be an interval on the real line and $l_t, r_t \in I$ be two numbers between l and r, called *tolerant points*. For every $\lambda \in [0, 1]$, we define the interval $I_{l_t,r_t}(\lambda) = [l + (r_t - l)\lambda,\, l_t + (r - l_t)\lambda]$, which is the convex combination of $[l, l_t]$ and $[r_t, r]$. Furthermore, we define the set $\mathcal{I}(I, l_t, r_t) = \{I_{l_t,r_t}(\lambda) \mid \lambda \in [0, 1]\}$ of intervals. That is, $\mathcal{I}(I, l_t, r_t)$ is the set of all intervals that we obtain when we linearly transform $[l, l_t]$ into $[r_t, r]$. For an interval I, the *set of tolerance-intervals* τ of I is defined either as $\tau = \mathcal{I}(I, l_t, r_t)$ for some values $l_t, r_t \in I$ (the case of a *bounded* vertex), or as $\tau = \{\mathbb{R}\}$ (the case of an *unbounded* vertex). A graph $G = (V, E)$ is a *multitolerance* graph if there exists a collection $I = \{I_v \mid v \in V\}$ of intervals and a family $t = \{\tau_v \mid v \in V\}$ of sets of tolerance-intervals, such that: for any two vertices $u, v \in V$, $uv \in E$ if and only if $Q_u \subseteq I_v$ for some $Q_u \in \tau_u$, or $Q_v \subseteq I_u$ for some $Q_v \in \tau_v$. Then, the pair $\langle I, t\rangle$ is called a *multitolerance representation* of G. If G has a multitolerance representation with only bounded vertices, i.e., with $\tau_v \neq \{\mathbb{R}\}$ for every vertex v, then G is called a *bounded multitolerance* graph.

For several optimization problems that are NP-hard in general graphs, such as the coloring, clique, and independent set problems, efficient algorithms are known for tolerance and multitolerance graphs. However, only few of them have been derived using the (multi)tolerance representation (e.g. [10,19]), while most of these algorithms appeared as a consequence of the containment of tolerance and multitolerance graphs to weakly chordal (and thus also to perfect) graphs [20]. To design efficient algorithms for (multi)tolerance graphs, it seems to be essential to assume that a suitable representation of the graph is given along with the input, as it has been recently proved that the recognition of tolerance graphs is NP-complete [17]. Recently two new geometric intersection models in the 3-dimensional space have been introduced for both tolerance graphs (the *parallelepiped* representation [16]) and multitolerance graphs (the *trapezoepiped* representation [15]), which enabled the design of very efficient algorithms for such problems, in most cases with (optimal) $O(n \log n)$ running time [15,16]. In spite of this, the complexity status of some algorithmic problems on tolerance and multitolerance graphs still remains open, three decades after the introduction of tolerance graphs in [8]. Arguably the two most famous and intriguing examples of such problems are the *minimum dominating set* problem and the *Hamilton cycle* problem (see e.g. [20, page 314]). Both these problems are known to be NP-complete on the greater class of weakly chordal graphs [3,18] but solvable in polynomial time in the smaller classes of bounded tolerance and bounded multitolerance (i.e., trapezoid) graphs [6,13]. The reason that these problems resisted solution attempts over the years seems to be that the existing representations for (multi)tolerance graphs do not provide enough insight to deal with these problems.

In this article we introduce a new geometric representation for multitolerance graphs, which we call the *shadow representation*, given by a set of line segments and points in the plane. In the case of tolerance graphs, this representation takes a very special form, in which all line segments are horizontal, and therefore we call it the *horizontal shadow representation*. Note that both the shadow and the horizontal shadow representations are *not* intersection models for multitolerance graphs and for tolerance graphs, respectively, in the sense that two line segments may not intersect in the representation although the corresponding vertices are adjacent. However, the main advantage of these two new representations is that they provide substantially new insight for tolerance and multitolerance graphs and they can be used to interpret optimization problems (such as the dominating set problem and its variants) using computational geometry terms.

Apart from being important on their own, these new representations enable us to establish the complexity of the *minimum dominating set* problem on both tolerance and multitolerance graphs, thus solving a longstanding open problem. Given a horizontal shadow representation of a tolerance graph G, we present an algorithm that computes a minimum dominating set in polynomial time. On the other hand, using the shadow representation, we prove that the minimum dominating set problem is APX-hard on multitolerance graphs by providing a reduction from a special case of the set cover problem. That is, there exists no Polynomial Time Approximation Scheme (PTAS) for this problem unless P=NP. This is the first problem that has been discovered with a different complexity status in these two graph classes. Therefore, given the (seemingly) small difference between the definition of tolerance and multitolerance graphs, this dichotomy result appears to be surprising. Furthermore we present an easy algorithm that solves (using the shadow representation) the independent dominating set problem on multitolerance graphs in polynomial time. This algorithm demonstrates the potential of this new representation for further exploitation via sweep line algorithms. Due to lack of space, full proofs are given in a clearly marked appendix.

Throughout the article we consider simple undirected graphs with no loops or multiple edges. In an undirected graph G the edge between two vertices u and v is denoted by uv, and in this case u and v are said to be *adjacent* in G. We denote by $N(u) = \{v \in V : uv \in E\}$ the set of neighbors of a vertex u in G, and $N[u] = N(u) \cup \{u\}$. Given a graph $G = (V, E)$ and a subset $S \subseteq V$, $G[S]$ denotes the induced subgraph of G on the vertices in S. A subset $S \subseteq V$ is a *dominating set* of G if every vertex $v \in V \setminus S$ has at least one neighbor in S. A subset $S \subseteq V$ is an *independent set* of G if $G[S]$ has no edges. Furthermore $S \subseteq V$ is an *independent dominating set* of G if S is both an independent set and a dominating set of G. Note that any inclusion maximal independent set is also an independent dominating set. The *(independent) dominating set problem* is the problem of computing an (independent) dominating set of minimum size in a given graph G. Finally, given a set $X \subseteq \mathbb{R}^2$ of points in the plane, we denote by $H_{\text{convex}}(X)$ the *convex hull* defined by the points of X, and by $\overline{X} = \mathbb{R}^2 \setminus X$ the complement of X in $\mathbb{R}^2$. For simplicity of the presentation we make the following notational convention throughout the paper: whenever we need to compute a set S with the smallest cardinality among a family $\mathcal{S}$ of sets, we write $S = \min\{\mathcal{S}\}$.

2 Tolerance and Multitolerance Graphs

In this section we briefly revise the 3-dimensional intersection models for tolerance graphs [16] and multitolerance graphs [15] and some useful properties of these models that are needed for the remainder of the paper. Consider a multitolerance graph $G = (V, E)$ that is given along with a multitolerance representation R. Let V_B and V_U denote the set of bounded and unbounded vertices of G in this representation, respectively. Consider now two parallel lines L_1 and L_2 in the plane. For every vertex $v \in V = V_B \cup V_U$, we appropriately construct a

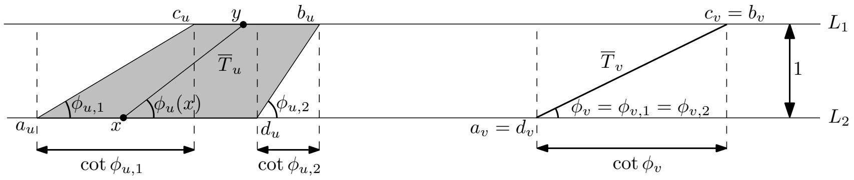

Figure 1 The trapezoid $\overline{T}_u$ corresponds to the bounded vertex $u \in V_B$, while the line segment $\overline{T}_v$ corresponds to the unbounded vertex $v \in V_U$.

trapezoid $\overline{T}_v$ with its parallel lines on L_1 and L_2, respectively (for details of this construction of the trapezoids we refer to [15]). According to this construction, for every unbounded vertex $v \in V_U$ the trapezoid $\overline{T}_v$ is trivial, i.e., a line [15]. For every vertex $v \in V = V_B \cup V_U$ we denote by a_v, b_v, c_v, d_v the lower left, upper right, upper left, and lower right endpoints of the trapezoid $\overline{T}_v$, respectively. Note that for every unbounded vertex $v \in V_U$ we have $a_v = d_v$ and $c_v = b_v$, since $\overline{T}_v$ is just a line segment. An example is depicted in Figure 1, where $\overline{T}_u$ corresponds to a bounded vertex u and $\overline{T}_v$ corresponds to an unbounded vertex v.

We now define the left and right angles of these trapezoids. For every angle ϕ, the values $\tan \phi$ and $\cot \phi = \frac{1}{\tan \phi}$ denote the tangent and the cotangent of ϕ, respectively. Furthermore, $\phi = arc \cot x$ is the angle ϕ, for which $\cot \phi = x$.

▶ **Definition 1** ([15]). For every vertex $v \in V = V_B \cup V_U$, the values $\phi_{v,1} = arc \cot (c_v - a_v)$ and $\phi_{v,2} = arc \cot (b_v - d_v)$ are the *left angle* and the *right angle* of $\overline{T}_v$, respectively. Moreover, for every unbounded vertex $v \in V_U$, $\phi_v = \phi_{v,1} = \phi_{v,2}$ is the *angle* of $\overline{T}_v$.

Note that without loss of generality we can assume that all endpoints and angles of the trapezoids are distinct, i.e., $\{a_u, b_u, c_u, d_u\} \cap \{a_v, b_v, c_v, d_v\} = \emptyset$ and $\{\phi_{u,1}, \phi_{u,2}\} \cap \{\phi_{v,1}, \phi_{v,2}\} = \emptyset$ for every $u, v \in V$ with $u \neq v$, as well as that $0 < \phi_{v,1}, \phi_{v,2} < \frac{\pi}{2}$ for all angles $\phi_{v,1}, \phi_{v,2}$ [15]. It is important to note here that this set of trapezoids $\{\overline{T}_v : v \in V = V_B \cup V_U\}$ is *not* an intersection model for the graph G, as two trapezoids $\overline{T}_v, \overline{T}_w$ may have a non-empty intersection although $vw \notin E$. However the subset of trapezoids $\{\overline{T}_v : v \in V_B\}$ that corresponds to the *bounded* vertices is an intersection model of the induced subgraph $G[V_B]$, i.e., $uv \in E$ if and only if $\overline{T}_u \cap \overline{T}_v \neq \emptyset$ where $u, v \in V_B$.

In order to construct an intersection model for the whole graph G (i.e., including also the set V_U of the unbounded vertices), we exploit the third dimension as follows. Let $\Delta = \max\{b_v : v \in V\} - \min\{a_u : u \in V\}$ (where we consider the endpoints b_v and a_u as real numbers on the lines L_1 and L_2, respectively). First, for every unbounded vertex $v \in V_U$ we construct the line segment $T_v = \{(x, y, z) : (x, y) \in \overline{T}_v, z = \Delta - \cot \phi_v\}$. For every bounded vertex $v \in V_B$, denote by $\overline{T}_{v,1}$ and $\overline{T}_{v,2}$ the left and the right line segment of the trapezoid $\overline{T}_v$, respectively. We construct two line segments $T_{v,1} = \{(x, y, z) : (x, y) \in \overline{T}_{v,1}, z = \Delta - \cot \phi_{v,1}\}$ and $T_{v,2} = \{(x, y, z) : (x, y) \in \overline{T}_{v,2}, z = \Delta - \cot \phi_{v,2}\}$. Then, for every $v \in V_B$, we construct the 3-dimensional object T_v as the convex hull $H_{\text{convex}}(\overline{T}_v, T_{v,1}, T_{v,2})$; this 3-dimensional object T_v is called the *trapezoepiped* of vertex $v \in V_B$. The resulting set $\{T_v : v \in V = V_B \cup V_U\}$ of objects in the 3-dimensional space is called the *trapezoepiped representation* of the multitolerance graph G [15]. This is an *intersection model* of G, i.e., two vertices v, w are adjacent if and only if $T_v \cap T_w \neq \emptyset$. For a proof of this fact and for more details about the trapezoepiped representation of multitolerance graphs we refer to [15].

An example of this construction is given in Figure 2. A multitolerance graph G with six vertices $\{v_1, v_2, \ldots, v_6\}$ is depicted in Figure 2a, while the trapezoepiped representation of G is illustrated in Figure 2b. The set of bounded and unbounded vertices in this representation are $V_B = \{v_3, v_4, v_6\}$ and $V_U = \{v_1, v_2, v_5\}$, respectively.

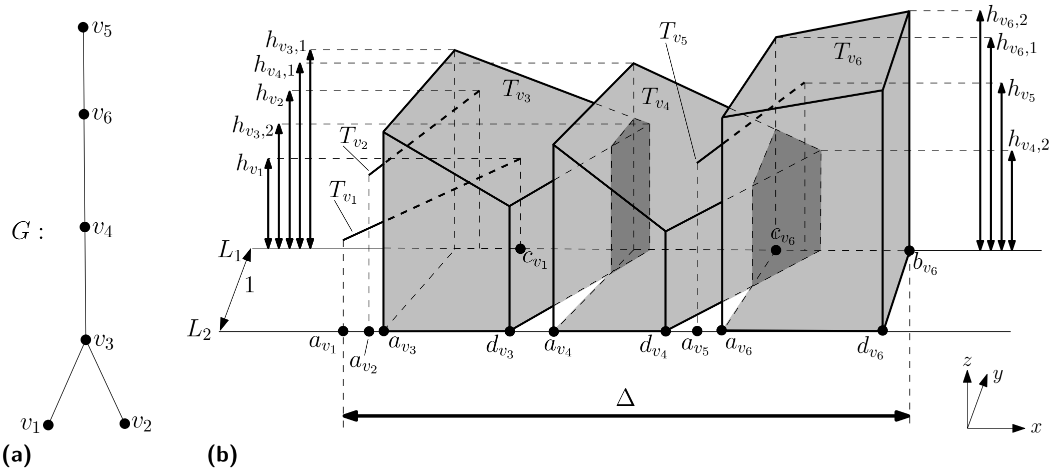

Figure 2 (a) A multitolerance graph G and (b) a trapezoepiped representation R of G. Here, $h_{v_i,j} = \Delta - \cot\phi_{v_i,j}$ for every bounded vertex $v_i \in V_B$ and $j \in \{1,2\}$, while $h_{v_i} = \Delta - \cot\phi_{v_i}$ for every unbounded vertex $v_i \in V_U$.

▶ **Definition 2** ([15]). An unbounded vertex $v \in V_U$ is *inevitable* if replacing T_v by $H_{\text{convex}}(T_v, \overline{T}_v)$ creates a new edge uv in G; then u is a *hovering vertex* of v and the set $H(v)$ of all hovering vertices of v is the *hovering set* of v. A trapezoepiped representation of a multitolerance graph G is called *canonical* if every unbounded vertex is inevitable.

In the example of Figure 2, v_2 and v_5 are inevitable unbounded vertices, v_1 and v_4 are hovering vertices of v_2 and v_5, respectively, while v_1 is not an inevitable unbounded vertex. Therefore, this representation is not canonical for the graph G. However, if we replace T_{v_1} by $H_{\text{convex}}(T_{v_1}, a_{v_1}, c_{v_1})$, we get a canonical representation for G in which vertex v_1 is bounded.

Let G be a multitolerance graph and R be a trapezoepiped representation of G, where $\phi_{u,1} = \phi_{u,2}$ for every bounded vertex $u \in V_B$. Then, for every $u \in V_B$, T_u becomes a *parallelepiped* and it can be proved that G is a *tolerance graph* [15]. This particular 3-dimensional intersection model for tolerance graphs is known as a *parallelepiped representation* [16].

3 The New Geometric Representations

In this section we introduce the *shadow representation* of multitolerance graphs, which is given by a set of line segments and points in the plane. Given a trapezoepiped representation of a multitolerance graph G with n vertices, we can compute a shadow representation of G in $O(n)$ time. Whenever G admits a parallelepiped representation (i.e., G is a tolerance graph) all line segments in the shadow representation of G become horizontal, and in this case we call it a *horizontal shadow representation.*

▶ **Definition 3** (shadow representation). Let $G = (V, E)$ be a multitolerance graph, R be a trapezoepiped representation of G, and V_B, V_U be the sets of bounded and unbounded vertices of G in R, respectively. We associate the vertices of G with the following points and line segments in the plane:

- for every $v \in V_B$, the points $p_{v,1} = (a_v, \Delta - \cot\phi_{v,1})$ and $p_{v,2} = (d_v, \Delta - \cot\phi_{v,2})$ and the line segment $L_v = (p_{v,1}, p_{v,2})$,
- for every $v \in V_U$, the point $p_v = (a_v, \Delta - \cot\phi_v)$.

The tuple $(\mathcal{P}, \mathcal{L})$, where $\mathcal{L} = \{L_v : v \in V_B\}$ and $\mathcal{P} = \{p_v : v \in V_U\}$, is the *shadow representation* of G. If $\phi_{v,1} = \phi_{v,2}$ for every $v \in V_B$, then $(\mathcal{P}, \mathcal{L})$ is the *horizontal shadow representation* of the tolerance graph G. Furthermore, the representation $(\mathcal{P}, \mathcal{L})$ is *canonical* if the initial trapezoepiped representation R is also canonical.

As an example we illustrate in Figure 3 the shadow representation $(\mathcal{P}, \mathcal{L})$ of the multitolerance graph G of Figure 2.

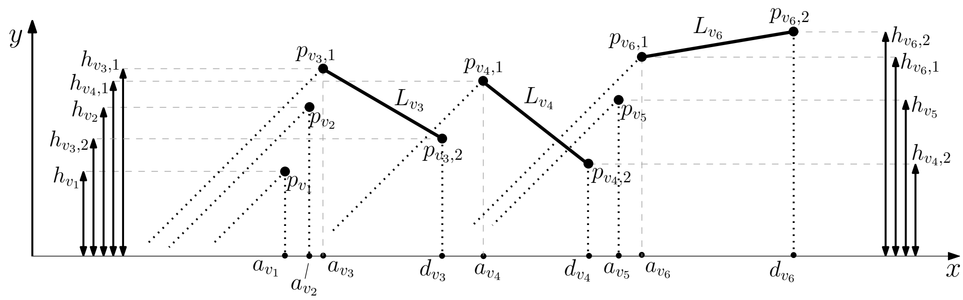

Figure 3 The shadow representation $(\mathcal{P}, \mathcal{L})$ of the multitolerance graph G of Figure 2. The unbounded vertices $V_U = \{v_1, v_2, v_5\}$ are associated with the points $\mathcal{P} = \{p_{v_1}, p_{v_2}, p_{v_5}\}$, while the bounded vertices $V_B = \{v_3, v_4, v_6\}$ are associated with the line segments $\mathcal{L} = \{L_{v_1}, L_{v_2}, L_{v_5}\}$, respectively.

▶ **Definition 4** (shadow). For an arbitrary point $t = (t_x, t_y) \in \mathbb{R}^2$ the *shadow* of t is the region $S_t = \{(x, y) \in \mathbb{R}^2 : x \leq t_x,\ y - x \leq t_y - t_x\}$. Furthermore, for every line segment L_u, where $u \in V_B$, the *shadow* of L_u is the region $S_u = \bigcup_{t \in L_u} S_t$.

▶ **Definition 5** (reverse shadow). For an arbitrary point $t = (t_x, t_y) \in \mathbb{R}^2$ the *reverse shadow* of t is the region $F_t = \{(x, y) \in \mathbb{R}^2 : x \geq t_x,\ y - x \geq t_y - t_x\}$. Furthermore, for every line segment L_i, where $u \in V_B$, the *reverse shadow* of L_i is the region $F_i = \bigcup_{t \in L_i} F_t$.

▶ **Lemma 6.** *Let G be a multitolerance graph and $(\mathcal{P}, \mathcal{L})$ be a shadow representation of G. Let $u \in V_B$ be a bounded vertex of G such that the corresponding line segment L_u is not trivial, i.e., L_u is not a single point. Then the angle of the line segment L_u with a horizontal line (i.e., parallel to the x-axis) is at most $\frac{\pi}{4}$ and at least $-\frac{\pi}{2}$.*

Recall now that two unbounded vertices $u, v \in V_U$ are never adjacent. The connection between a multitolerance graph G and a shadow representation of it is given in Lemmas 7 and 8. Furthermore Lemma 9 describes how the hovering vertices of an unbounded vertex $v \in V_U$ (cf. Definition 2) can be seen in a shadow representation $(\mathcal{P}, \mathcal{L})$.

▶ **Lemma 7.** *Let $(\mathcal{P}, \mathcal{L})$ be a shadow representation of a multitolerance graph G. Let $u, v \in V_B$ be two bounded vertices of G. Then $uv \in E$ if and only if $L_v \cap S_u \neq \emptyset$ or $L_u \cap S_v \neq \emptyset$.*

▶ **Lemma 8.** *Let $(\mathcal{P}, \mathcal{L})$ be a shadow representation of a multitolerance graph G. Let $v \in V_U$ and $u \in V_B$ be two vertices of G. Then $uv \in E$ if and only if $p_v \in S_u$.*

▶ **Lemma 9.** *Let $(\mathcal{P}, \mathcal{L})$ be a shadow representation of a multitolerance graph G. Let $v \in V_U$ be an unbounded vertex of G and $u \in V \setminus \{v\}$ be another arbitrary vertex. If $u \in V_B$ (resp. $u \in V_U$), then u is a hovering vertex of v if and only if $L_u \cap S_v \neq \emptyset$ (resp. $p_u \in S_v$).*

In the example of Figure 3 the shadows of the points in $\mathcal{P}$ and of the line segments in $\mathcal{L}$ are shown with dotted lines. For instance, $p_{v_2} \in S_{v_3}$ and $p_{v_2} \notin S_{v_4}$, and thus the unbounded vertex v_2 is adjacent to the bounded vertex v_3 but not to the bounded vertex v_4. Furthermore $L_{v_3} \cap S_{v_4} \neq \emptyset$, and thus v_3 and v_4 are adjacent. On the other hand, $L_{v_3} \cap S_{v_6} = L_{v_6} \cap S_{v_3} = \emptyset$, and thus v_3 and v_4 are not adjacent. Finally $p_{v_1} \in S_{v_2}$ and $L_{v_4} \cap S_{v_5} \neq \emptyset$, and thus v_1 is a hovering vertex of v_2 and v_4 is a hovering vertex of v_5. These facts can be also checked in the trapezoepiped representation of the same multitolerance graph G in Figure 2b.

4 Dominating Set is APX-hard on Multitolerance Graphs

In this section we prove that the dominating set problem on multitolerance graphs is APX-hard via an approximation-preserving reduction [21] from a special case of the set cover problem, namely Special 3-Set Cover [5].

▶ **Theorem 10.** Dominating Set *is APX-hard on multitolerance graphs.*

5 Bounded Dominating Set on Tolerance Graphs

In this section we use the *horizontal shadow representation* of tolerance graphs (cf. Section 3) to provide a polynomial time algorithm for a variation of the minimum dominating set problem on tolerance graphs, namely Bounded Dominating Set, formally defined below. This problem variation may be interesting on its own, but we use our algorithm for Bounded Dominating Set as a subroutine in our algorithm for the minimum dominating set problem on tolerance graphs, cf. Sections 6 and 7. Note that, given a horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ of a tolerance graph $G = (V, E)$, the representation $(\mathcal{P}, \mathcal{L})$ defines a partition of the vertex set V into the set V_B of bounded vertices and the set V_U of unbounded vertices. We denote $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$, where $|\mathcal{P}| + |\mathcal{L}| = |V_U| + |V_B| = |V|$.

With a slight abuse of notation, for any two elements $x_1, x_2 \in \mathcal{P} \cup \mathcal{L}$, we may say in the following that x_1 is adjacent with x_2 (or x_1 is a neighbor of x_2) if the vertices that correspond to x_1 and x_2 are adjacent in the graph G. Moreover, whenever $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}$ and $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}$, we may say that the set $\mathcal{P}_1 \cup \mathcal{L}_1$ *dominates* $\mathcal{P}_2 \cup \mathcal{L}_2$ if the vertices corresponding to $\mathcal{P}_1 \cup \mathcal{L}_1$ dominate the subgraph of G induced by the vertices corresponding to $\mathcal{P}_2 \cup \mathcal{L}_2$.

Bounded Dominating Set on Tolerance Graphs
Input: A horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ of a tolerance graph G.
Output: A set $Z \subseteq \mathcal{L}$ of minimum size that dominates $(\mathcal{P}, \mathcal{L})$, or the announcement that $\mathcal{L}$ does not dominate $(\mathcal{P}, \mathcal{L})$.

5.1 Notation and Terminology

For an arbitrary point $t = (t_x, t_y) \in \mathbb{R}^2$ we define two (infinite) lines passing through t:
- the vertical line $\Gamma_t^{\text{vert}} = \{(t_x, s) \in \mathbb{R}^2 : s \in \mathbb{R}\}$, i.e., the line that is parallel to the y-axis,
- the diagonal line $\Gamma_t^{\text{diag}} = \{(s, s + (t_y - t_x)) \in \mathbb{R}^2 : s \in \mathbb{R}\}$, i.e., the line that is parallel to the main diagonal $\{(s, s) \in \mathbb{R}^2 : s \in \mathbb{R}\}$.

For every point $t = (t_x, t_y) \in \mathbb{R}^2$, each of the lines $\Gamma_t^{\text{vert}}, \Gamma_t^{\text{diag}}$ separates $\mathbb{R}^2$ into two regions. With respect to the line Γ_t^{vert} we define the regions $\mathbb{R}^2_{\text{left}}(\Gamma_t^{\text{vert}}) = \{(x, y) \in \mathbb{R}^2 : x \leq t_x\}$ and $\mathbb{R}^2_{\text{right}}(\Gamma_t^{\text{vert}}) = \{(x, y) \in \mathbb{R}^2 : x \geq t_x\}$ of the points to the left and to the right of Γ_t^{vert}, respectively. With respect to the line Γ_t^{diag}, we define the regions $\mathbb{R}^2_{\text{left}}(\Gamma_t^{\text{diag}}) = \{(x, y) \in \mathbb{R}^2 : y - x \geq t_y - t_x\}$ and $\mathbb{R}^2_{\text{right}}(\Gamma_t^{\text{diag}}) = \{(x, y) \in \mathbb{R}^2 : y - x \leq t_y - t_x\}$ of the points to the left and to the right of Γ_t^{diag}, respectively.

Furthermore, for an arbitrary point $t = (t_x, t_y) \in \mathbb{R}^2$ we define the region A_t (resp. B_t) that contains all points that are both to the right (resp. to the left) of Γ_t^{vert} and to the right (resp. to the left) of Γ_t^{diag}. That is, $A_t = \mathbb{R}^2_{\text{right}}(\Gamma_t^{\text{vert}}) \cap \mathbb{R}^2_{\text{right}}(\Gamma_t^{\text{diag}})$ and $B_t = \mathbb{R}^2_{\text{left}}(\Gamma_t^{\text{vert}}) \cap \mathbb{R}^2_{\text{left}}(\Gamma_t^{\text{diag}})$.

Consider an arbitrary horizontal line segment $L_i \in \mathcal{L}$. We denote by l_i and r_i its left and its right endpoint, respectively; note that possibly $l_i = r_i$. Denote by $\mathcal{A} = \{l_i, r_i : 1 \leq i \leq |\mathcal{L}|\}$ the set of all endpoints of all line segments of $\mathcal{L}$. Furthermore denote by $\mathcal{B} = \{\Gamma_t^{\text{diag}} \cap \Gamma_{t'}^{\text{vert}} : t, t' \in \mathcal{A}\}$ the set of all intersection points of the vertical and the diagonal lines that pass from points of $\mathcal{A}$. Note that $\mathcal{A} \subseteq \mathcal{B}$.

Given a horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ we always assume that the points $p_1, p_2, \ldots, p_{|\mathcal{P}|}$ are ordered increasingly with respect to their x-coordinates. Similarly we assume that the horizontal line segments $L_1, L_2, \ldots, L_{|\mathcal{L}|}$ are ordered increasingly with respect to the x-coordinates of their endpoint r_i.

▶ **Definition 11.** Let $1 \leq i, i' \leq |\mathcal{L}|$. The pair (i, i') is a *right-crossing pair* if $r_{i'} \in S_{r_i}$. Furthermore the pair (i, i') is a *left-crossing pair* if $l_i \in S_{l_{i'}}$. For every right-crossing pair (i, i'), we define $\mathcal{L}^{\text{left}}_{i,i'} = \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq B_t$, where $t = \Gamma_{r_i}^{\text{vert}} \cap \Gamma_{r_{i'}}^{\text{diag}}\}$ and for every left-crossing pair (i, i') we define $\mathcal{L}^{\text{right}}_{i,i'} = \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq A_t$, where $t = \Gamma_{l_i}^{\text{vert}} \cap \Gamma_{l_{i'}}^{\text{diag}}\}$

▶ **Definition 12.** Let $S \subseteq \mathcal{P} \cup \mathcal{L}$ be an arbitrary set. Let (i, i') be a right-crossing pair and (j, j') be a left-crossing pair. If $L_i, L_{i'} \in S$ and $S \subseteq \mathcal{L}^{\text{left}}_{i,i'}$, then (i, i') is the *end-pair* of the set S. If $L_j, L_{j'} \in S$ and $S \subseteq \mathcal{L}^{\text{right}}_{j,j'}$, then (j, j') is the *start-pair* of the set S.

5.2 The Algorithm

In this section we present our algorithm for Bounded Dominating Set on tolerance graphs, cf. Algorithm 1. Given a horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ of a tolerance graph G, we first add two dummy line segments L_0 and $L_{|\mathcal{L}|+1}$ (with endpoints l_0, r_0 and $l_{|\mathcal{L}|+1}, r_{|\mathcal{L}|+1}$, respectively) such that all elements of $\mathcal{P} \cup \mathcal{L}$ are contained in A_{r_0} and in $B_{l_{|\mathcal{L}|+1}}$. Let $\mathcal{L}' = \mathcal{L} \cup \{L_0, L_{|\mathcal{L}|+1}\}$. Note that $(\mathcal{P}, \mathcal{L}')$ is a horizontal shadow representation of some tolerance graph G', where the bounded vertices V'_B of G' correspond to the line segments of $\mathcal{L}'$ and the unbounded vertices V'_U of G' correspond to the points of $\mathcal{P}$. Furthermore note that $V'_B = V_B \cup \{v_0, v_{|\mathcal{L}|+1}\}$ and $V'_U = V_U$, where v_0 and $v_{|\mathcal{L}|+1}$ are the (isolated) bounded vertices of G' that correspond to the line segments L_0 and $L_{|\mathcal{L}|+1}$, respectively.

For simplicity of the presentation, we refer in the following to the augmented set $\mathcal{L}'$ of horizontal line segments by $\mathcal{L}$. In the remainder of this section we will write $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$ with the understanding that the first and the last line segments L_1 and $L_{|\mathcal{L}|}$ of $\mathcal{L}$ are dummy. Furthermore, we will refer to the augmented tolerance graph G' by G.

For every pair of points $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $b \in \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})$, define $X(a, b)$ to be the set of all points of $\mathcal{P}$ and all line segments of $\mathcal{L}$ that are contained in the region $B_b \setminus \Gamma_b^{\text{vert}}$ and to the right of the line Γ_a^{diag}, i.e., $X(a, b) = \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq (B_b \setminus \Gamma_b^{\text{vert}}) \cap \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})\}$.

▶ **Definition 13.** Let $(a, b) \in \mathcal{A} \times \mathcal{B}$ be a pair of points such that $b \in \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})$. Furthermore let (i, i') be a right-crossing pair such that $b \in \mathbb{R}^2_{\text{left}}(\Gamma_{r_i}^{\text{vert}})$. Then $BD_{(\mathcal{P},\mathcal{L})}(a, b, i, i')$ is a dominating set $Z \subseteq \mathcal{L}$ of $X(a, b)$ with the smallest size, in which (i, i') is its end-pair. If such a dominating set $Z \subseteq \mathcal{L}$ of $X(a, b)$ does not exist, we define $BD_{(\mathcal{P},\mathcal{L})}(a, b, i, i') = \perp$.

Note that always $L_i \in BD_{(\mathcal{P},\mathcal{L})}(a, b, i, i')$. However, since $b \in \mathbb{R}^2_{\text{left}}(\Gamma_{r_i}^{\text{vert}})$ in Definition 13, it follows that $L_i \nsubseteq B_b \setminus \Gamma_b^{\text{vert}}$, and thus $L_i \notin X(a, b)$. For simplicity of the presentation we may refer to the set $BD_{(\mathcal{P},\mathcal{L})}(a, b, i, i')$ as $BD_G(a, b, i, i')$, where $(\mathcal{P}, \mathcal{L})$ is the horizontal shadow representation of the tolerance graph G, or just as $BD(a, b, i, i')$ whenever the horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ is clear from the context.

Algorithm 1 Bounded Dominating Set on Tolerance Graphs

Input: A horizontal shadow representation $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$

Output: A set $Z \subseteq \mathcal{L}$ of minimum size that dominates $(\mathcal{P}, \mathcal{L})$, or the announcement that $\mathcal{L}$ does not dominate $(\mathcal{P}, \mathcal{L})$

Add two dummy line segments L_0 and $L_{|\mathcal{L}|+1}$ completely to the left and to the right of $\mathcal{P} \cup \mathcal{L}$, respectively
$\mathcal{L} \leftarrow \mathcal{L} \cup \{L_0, L_{|\mathcal{L}|+1}\}$; denote $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$, where now $L_1, L_{|\mathcal{L}|}$ are dummy
$\mathcal{A} \leftarrow \{l_i, r_i : 1 \leq i \leq |\mathcal{L}|\}$; $\mathcal{B} \leftarrow \{\Gamma_t^{\text{diag}} \cap \Gamma_{t'}^{\text{vert}} : t, t' \in \mathcal{A}\}$
for every pair of points $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $b \in \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})$ **do**
  $X(a, b) \leftarrow \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq (B_b \setminus \Gamma_b^{\text{vert}}) \cap \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})\}$
  for every $i, i' \in \{1, 2, \ldots, |\mathcal{L}|\}$ **do**
    if $r_{i'} \in S_{r_i}$ **then** $\{(i, i')$ is a right-crossing pair$\}$
      if $\{L_i\} \cup \{L_{i'}\}$ dominates all elements of $X(a, b)$ **then** $BD_{(\mathcal{P},\mathcal{L})}(a, b, i, i') \leftarrow \{L_i\} \cup \{L_{i'}\}$ {initialization}
      $\mathcal{L}_{i,i'}^{\text{left}} \leftarrow \{L_k \subseteq B_t : t = \Gamma_{r_i}^{\text{vert}} \cap \Gamma_{r_{i'}}^{\text{diag}}\}$
      if $\mathcal{L} \cap \mathcal{L}_{i,i'}^{\text{left}}$ does not dominate all elements of $X(a, b)$ **then** $BD_{(\mathcal{P},\mathcal{L})}(a, b, i, i') \leftarrow \perp$ {initialization}
        else $BD_{(\mathcal{P},\mathcal{L})}(a, b, i, i') \leftarrow \mathcal{L} \cap \mathcal{L}_{i,i'}^{\text{left}}$ {initialization}
for every pair of points $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $b \in \mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})$ **do**
  for every $i, i' \in \{1, 2, \ldots, |\mathcal{L}|\}$ **do**
    if $r_{i'} \in S_{r_i}$ **then** $\{(i, i')$ is a right-crossing pair$\}$
      Compute $Z_1 = \{L_i\} \cup \min_{c,j,j'}\{BD_{(\mathcal{P},\mathcal{L})}(a, c, j, j')\}$ by Lemma 14
      if $|Z_1| < |BD_{(\mathcal{P},\mathcal{L})}(a, b, i, i')|$ **then** $BD_{(\mathcal{P},\mathcal{L})}(a, b, i, i') \leftarrow Z_1$
      Compute $Z_2 = \min_c\{BD_{(\mathcal{P},\mathcal{L})}(a, c, i, i') \cup BD_{(\mathcal{P},\mathcal{L})}(c, b, i, i')\}$ by Lemma 15
      if $|Z_2| < |BD_{(\mathcal{P},\mathcal{L})}(a, b, i, i')|$ **then** $BD_{(\mathcal{P},\mathcal{L})}(a, b, i, i') \leftarrow Z_2$
if $BD_{(\mathcal{P},\mathcal{L})}(l_1, r_{\mathcal{L}}, |\mathcal{L}|, |\mathcal{L}|) = \perp$ **then return** $\mathcal{L}$ does not dominate $(\mathcal{P}, \mathcal{L})$
  else return $BD_{(\mathcal{P},\mathcal{L})}(l_1, r_{\mathcal{L}}, |\mathcal{L}|, |\mathcal{L}|) \setminus \{L_1, L_{|\mathcal{L}|}\}$

▶ **Lemma 14.** *Let $(a, b) \in \mathcal{A} \times \mathcal{B}$ and let (i, i') be a right-crossing pair such that $BD(a, b, i, i') \neq \perp$. If $BD(a, b, i, i') \setminus L_i$ dominates all elements of $\{x \in X(a, b) : x \cap (S_i \cup F_i) \neq \emptyset\}$ then $BD(a, b, i, i') = \{L_i\} \cup \min_{c,j,j'}\{BD(a, c, j, j')\}$, where the minimum is taken over all c, j, j' such that:*

- *$c = \Gamma_{r_j}^{vert} \cap \Gamma_b^{diag}$ if $r_j \in \mathbb{R}^2_{left}(\Gamma_b^{vert})$, and $c = b$ otherwise,*
- *(j, j') is a right-crossing pair of $\mathcal{L}_{i,i'}^{left} \setminus \{L_i\}$, where $j' = i'$ whenever $i \neq i'$, and*
- *$\{L_j\} \cup \{L_{j'}\}$ dominates all elements of the set $X(a, b) \setminus X(a, c)$.*

▶ **Lemma 15.** *Let $(a, b) \in \mathcal{A} \times \mathcal{B}$ and let (i, i') be a right-crossing pair such that $BD(a, b, i, i') \neq \perp$. If $BD(a, b, i, i') \setminus L_i$ does not dominate all elements of $\{x \in X(a, b) : x \cap (S_i \cup F_i) \neq \emptyset\}$ then $BD(a, b, i, i') = \min_c\{BD(a, c, i, i') \cup BD(c, b, i, i')\}$ where the minimum is taken over all c such that:*

- *$c \in \mathcal{B} \cap \mathbb{R}^2_{right}(\Gamma_{l_i}^{vert}) \cap \mathbb{R}^2_{right}(\Gamma_a^{diag}) \cap (B_b \setminus \Gamma_b^{vert})$ and*
- *$\mathcal{P} \cap X(a, b) \cap F_c \cap F_i = \emptyset$.*

▶ **Theorem 16.** *Given a horizontal shadow representation $(\mathcal{P}, \mathcal{L})$ of a tolerance graph G, Algorithm 1 computes* Bounded Dominating Set *in polynomial time.*

Algorithm 2 RESTRICTED BOUNDED DOMINATING SET on Tolerance Graphs

Input: A 6-tuple $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$, where $(\mathcal{P}, \mathcal{L})$ is a horizontal shadow representation of a tolerance graph G, (j, j') is a left-crossing pair of $(\mathcal{P}, \mathcal{L})$, and (i, i') is a right-crossing pair of $(\mathcal{P}, \mathcal{L})$.

Output: A set $Z \subseteq \mathcal{L}$ of minimum size that dominates $(\mathcal{P}, \mathcal{L})$, where (j, j') is the start-pair and (i, i') is the end-pair of Z, or the announcement that $\mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ does not dominate $(\mathcal{P}, \mathcal{L})$.

1: **if** $(\mathcal{P}, \mathcal{L})$ contains a bad point $p \in \mathcal{P}$ or a bad line segment $L_k \in \mathcal{L}$ (cf. Definition 17) **then**
2: **return** $\mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ does not dominate $(\mathcal{P}, \mathcal{L})$
3: Compute the set $Z_1 \subseteq \mathcal{L}$ of all irrelevant line segments (cf. Definition 17)
4: $\mathcal{L} \leftarrow \mathcal{L} \setminus Z_1$; $r \leftarrow \Gamma^{\text{vert}}_{r_i} \cap \Gamma^{\text{diag}}_{r_{i'}}$
5: Compute the representation $(\widehat{\mathcal{P}}, \widehat{\mathcal{L}})$ by adding the elements $\{x_{k,1}, x_{k,2} : k \in \{j, j'\}\}$ to $(\mathcal{P}, \mathcal{L})$ (cf. Lemma 18)
6: **return** $BD_{(\widehat{\mathcal{P}}, \widehat{\mathcal{L}})}(l_{x_{j,1}}, r, i, i')$ {by calling Algorithm 1}

6 Restricted Bounded Dominating Set on Tolerance Graphs

In this section we use Algorithm 1 of Section 5 to provide a polynomial time algorithm (cf. Algorithm 2) for a slightly modified version of BOUNDED DOMINATING SET on tolerance graphs, which we call RESTRICTED BOUNDED DOMINATING SET, formally defined below.

> RESTRICTED BOUNDED DOMINATING SET on Tolerance Graphs
>
> **Input:** A 6-tuple $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$, where $(\mathcal{P}, \mathcal{L})$ is a horizontal shadow representation of a tolerance graph G, (j, j') is a left-crossing pair of G, and (i, i') is a right-crossing pair of G.
>
> **Output:** A set $Z \subseteq \mathcal{L}$ of minimum size that dominates $(\mathcal{P}, \mathcal{L})$, where (j, j') is the start-pair and (i, i') is the end-pair of Z, or the announcement that $\mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ does not dominate $(\mathcal{P}, \mathcal{L})$.

▶ **Definition 17.** Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ be an instance of RESTRICTED BOUNDED DOMINATING SET on tolerance graphs. Let $l = \Gamma^{\text{vert}}_{l_j} \cap \Gamma^{\text{diag}}_{l_{j'}}$ and $r = \Gamma^{\text{vert}}_{r_i} \cap \Gamma^{\text{diag}}_{r_{i'}}$. A point $p \in \mathcal{P}$ is a *bad point* if $p \in B_l$ or $p \in \mathbb{R}^2_{\text{right}}(\Gamma^{\text{vert}}_r)$. A line segment $L_t \in \mathcal{L}$ is a *bad line segment* if $L_t \subseteq B_l$ or $L_t \subseteq A_r$. Furthermore a line segment $L_t \in \mathcal{L}$ is an *irrelevant line segment* if either $L_t \subseteq \overline{B_l} \cap \overline{A_r}$ and $L_t \notin \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$, or L_t has an endpoint in $B_l \cup A_r$ and another point in $\overline{B_l} \cap \overline{A_r}$.

The next lemma will enable us to reduce RESTRICTED BOUNDED DOMINATING SET to BOUNDED DOMINATING SET on tolerance graphs.

▶ **Lemma 18.** *Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$ be an instance of* RESTRICTED BOUNDED DOMINATING SET *on tolerance graphs, which has no bad points $p \in \mathcal{P}$ and no bad or irrelevant line segments $L \in \mathcal{L}$. Then for every $k \in \{j, j'\}$ we can add two new elements $x_{k,1}, x_{k,2}$ to the set $\mathcal{P} \cup \mathcal{L}$ such that L_k is the only neighbor of $x_{k,1}$ and $x_{k,2}$, $k \in \{j, j'\}$.*

▶ **Definition 19.** Let (j, j') be a left-crossing pair and (i, i') be a right-crossing pair in the horizontal shadow representation $(\mathcal{P}, \mathcal{L})$. Then $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i')$ is a dominating set

$Z \subseteq \mathcal{L} \cap \mathcal{L}^{\text{right}}_{j,j'} \cap \mathcal{L}^{\text{left}}_{i,i'}$ of $(\mathcal{P}, \mathcal{L})$ with the smallest size, in which (j, j') and (i, i') are the start-pair and the end-pair, respectively. If such a dominating set Z does not exist, we define $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i') = \bot$.

For simplicity of the presentation we may refer to the set $RD_{(\mathcal{P},\mathcal{L})}(j, j', i, i')$ as $RD_G(j, j', i, i')$, where $(\mathcal{P}, \mathcal{L})$ is the horizontal shadow representation of the tolerance graph G.

▶ **Theorem 20.** *Given a 6-tuple $\mathcal{I} = (\mathcal{P}, \mathcal{L}, j, j', i, i')$, where $(\mathcal{P}, \mathcal{L})$ is a horizontal shadow representation of a tolerance graph G, (j, j') is a left-crossing pair and (i, i') is a right-crossing pair of $(\mathcal{P}, \mathcal{L})$, Algorithm 2 computes* Restricted Bounded Dominating Set *in polynomial time.*

7 Dominating Set on Tolerance Graphs

In this section we present our main algorithm of the paper (cf. Algorithm 3) which computes in polynomial time a minimum dominating set of a tolerance graph G, given by a horizontal shadow representation $(\mathcal{P}, \mathcal{L})$. Algorithm 3 uses Algorithms 1 and 2 as subroutines (cf. Sections 5 and 6). Throughout this section we assume without loss of generality that the given tolerance graph G is connected and that G is given with a *canonical* horizontal shadow representation $(\mathcal{P}, \mathcal{L})$.

For every $p \in \mathcal{P}$ we denote by $N(p) = \{L_k \in \mathcal{L} : p \in S_k\}$ and $H(p) = \{x \in \mathcal{P} \cup \mathcal{L} : x \cap S_p \neq \emptyset\}$. Note that, due to Lemmas 8 and 9, $N(p)$ is the set of neighbors of p and $H(p)$ is the set of hovering vertices of p. Furthermore, for every $L_k \in \mathcal{L}$ we denote by $N(L_k) = \{p \in \mathcal{P} : p \in S_k\} \cup \{L_t \in \mathcal{L} : L_t \cap S_k \neq \emptyset \text{ or } L_k \cap S_t \neq \emptyset\}$. Note that, due to Lemmas 7 and 8, $N(L_k)$ is the set of neighbors of L_k.

Define now the subset $\mathcal{P}^* = \{p \in \mathcal{P} : \text{there exists no point } p' \in \mathcal{P} \text{ such that } p \in H(p')\}$. Note by the definition of the set $\mathcal{P}^*$ that for every $p_1, p_2 \in \mathcal{P}^*$ we have $p_1 \notin S_{p_2} \cup F_{p_2}$.

Given a canonical horizontal shadow representation $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$, we add two dummy line segments L_0 and $L_{|\mathcal{L}|+1}$ (with endpoints l_0, r_0 and $l_{|\mathcal{L}|+1}, r_{|\mathcal{L}|+1}$, respectively) such that all elements of $\mathcal{P} \cup \mathcal{L}$ are contained in A_{r_0} and in $B_{l_{|\mathcal{L}|+1}}$. Denote $\mathcal{L}' = \mathcal{L} \cup \{L_0, L_{|\mathcal{L}|+1}\}$. Furthermore we add one dummy point $p_{|\mathcal{P}|+1}$ such that all elements of $\mathcal{P} \cup \mathcal{L}'$ are contained in $B_{p_{|\mathcal{P}|+1}}$. Denote $\mathcal{P}' = \mathcal{P} \cup \{p_{|\mathcal{P}|+1}\}$. For simplicity of the presentation, we refer in the following to the augmented sets $\mathcal{P}'$ and $\mathcal{L}'$ of points and horizontal line segments by $\mathcal{P}$ and $\mathcal{L}$, respectively. In the remainder of this section we will write $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$ with the understanding that the last point $p_{|\mathcal{P}|}$ of $\mathcal{P}$, as well as the first and the last line segments L_1 and $L_{|\mathcal{L}|}$ of $\mathcal{L}$, are dummy. Note that the last point $p_{|\mathcal{P}|}$ (i.e., the new dummy point) belongs to the set $\mathcal{P}^*$.

For every $p_i, p_j \in \mathcal{P}^*$ with $i < j$, we define $G_j = \{x \in \mathcal{P} \cup \mathcal{L} : x \subseteq B_{p_j} \setminus \Gamma^{\text{vert}}_{p_j}\}$ and $G(i, j) = \{x \in G_j : x \subseteq A_{p_i}\}$. Note that $p_j \notin G_j$ and $p_j \notin G(i, j)$.

▶ **Definition 21.** Let $p_j \in \mathcal{P}^*$ and (i, i') be a right-crossing pair in G_j. Then $D(j, i, i')$ is a *minimum dominating set* of G_j whose end-pair is (i, i'). If there exists no dominating set Z of G_j whose end-pair is (i, i'), we define $D(j, i, i') = \bot$.

▶ **Lemma 22.** *Let G be a tolerance graph, $(\mathcal{P}, \mathcal{L})$ be a canonical representation of G, $p_j \in \mathcal{P}^*$, and a right-crossing pair (i, i') of G_j such that $D(j, i, i') \neq \bot$. Then*

$$D(j, i, i') = \min_{q', z, z', w, w'} \begin{cases} D(q, z, z') \cup \{p_k \in \mathcal{P}^* : q \leq k \leq q'\} \cup RD_{G(q',j)}(w, w', i, i') \\ BD_{G_j}(l_1, b, i, i'), \ \textit{where } b = \Gamma^{vert}_{r_i} \cap \Gamma^{diag}_{r_{i'}} \end{cases}$$

where the minimum is taken over all q', z, z', w, w' such that:

Algorithm 3 DOMINATING SET on Tolerance Graphs

Input: A canonical horizontal shadow representation $(\mathcal{P},\mathcal{L})$, where $\mathcal{P}=\{p_1,p_2,\ldots,p_{|\mathcal{P}|}\}$ and $\mathcal{L}=\{L_1,L_2,\ldots,L_{|\mathcal{L}|}\}$.
Output: A set $D\subseteq\mathcal{L}\cup\mathcal{P}$ of minimum size that dominates $(\mathcal{P},\mathcal{L})$.

Add two dummy line segments L_0 and $L_{|\mathcal{L}|+1}$ completely to the left and to the right of $\mathcal{P}\cup\mathcal{L}$, respectively
Add a dummy point $p_{|\mathcal{P}|+1}$ completely to the right of $L_{|\mathcal{L}|+1}$
$\mathcal{P}\leftarrow\mathcal{P}\cup\{p_{|\mathcal{P}|+1}\}$; $\quad\mathcal{L}\leftarrow\mathcal{L}\cup\{L_0,L_{|\mathcal{L}|+1}\}$
Denote $\mathcal{P}=\{p_1,p_2,\ldots,p_{|\mathcal{P}|}\}$ and $\mathcal{L}=\{L_1,L_2,\ldots,L_{|\mathcal{L}|}\}$, where now $p_{|\mathcal{P}|}$, L_1, and $L_{|\mathcal{L}|}$ are dummy
$\mathcal{P}^*\leftarrow\{p\in\mathcal{P}:$ there exists no point $p'\in\mathcal{P}$ such that $p\in H(p')\}$
for every pair of points $(a,b)\in\mathcal{A}\times\mathcal{B}$ such that $b\in\mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})$ **do**
  $X(a,b)\leftarrow\{x\in\mathcal{P}\cup\mathcal{L}:x\subseteq(B_b\setminus\Gamma_b^{\text{vert}})\cap\mathbb{R}^2_{\text{right}}(\Gamma_a^{\text{diag}})\}$
for every $p_j\in\mathcal{P}^*$ **do**
  $G_j\leftarrow\{x\in\mathcal{P}\cup\mathcal{L}:x\subseteq B_{p_j}\setminus\Gamma_{p_j}^{\text{vert}}\}$
  for every $i,i'\in\{1,2,\ldots,|\mathcal{L}|\}$ **do**
    if $L_i,L_{i'}\in G_j$ and $r_{i'}\in S_{r_i}$ **then** $\{(i,i')$ is a right-crossing pair of $G_j\}$
      if $X(r_{i'},p_j)$ is not dominated by L_i and $L_{i'}$ **then** $D(j,i,i')\leftarrow\perp$
      if there exists a point $p\in\mathcal{P}\cap G_j$ such that $p\in\mathbb{R}^2_{\text{right}}(\Gamma_{r_i}^{\text{vert}})$ **then** $D(j,i,i')\leftarrow\perp$
      if $D(j,i,i')\neq\perp$ **then**
        Compute $D(j,i,i')$ by Lemma 22 {by calling Algorithms 1 and 2}
return $D(|\mathcal{P}|,|\mathcal{L}|,|\mathcal{L}|)\setminus\{L_1,L_{\mathcal{L}}\}$

- $1\le q'<j$,
- $i,i'\notin N(p_{q'})\cup H(p_{q'})$,
- (w,w') *is a left-crossing pair of* $G(q',j)$ *such that* $RD_{G(q',j)}(w,w',i,i')\neq\perp$,
- (z,z') *is a right-crossing pair of* $G_{q'}$,
- $q=\min\{1\le k\le q':p_k\in\mathcal{P}^*,\ p_k\in A_\omega,$ *where* $\omega=\Gamma_{r_z}^{vert}\cap\Gamma_{r_{z'}}^{diag}$,
- $D(q,z,z')\neq\perp$,
- $(H(p_q)\cup H(p_{q'}))\setminus\left(\bigcup_{q\le k\le q'}N(p_k)\right)$ *are dominated by the line segments* $L_z,L_{z'},L_w,L_{w'}$,
- $G(q,q')$ *is dominated by* $\{p_k\in\mathcal{P}^*:q\le k\le q'\}$.

▶ **Theorem 23.** *Given a canonical horizontal shadow representation $(\mathcal{P},\mathcal{L})$ of a connected tolerance graph G, Algorithm 3 computes in polynomial time a minimum dominating set of G.*

8 Independent Dominating Set on Multitolerance Graphs

In this section we provide a polynomial time sweep-line algorithm which, given a shadow representation $(\mathcal{P},\mathcal{L})$ of a multitolerance graph G, computes in polynomial time a minimum independent dominating set of G.

▶ **Theorem 24.** *Given a shadow representation $(\mathcal{P},\mathcal{L})$ of a multitolerance graph G, we can compute a minimum independent dominating set in polynomial time.*

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