Bounding the Clique-Width of H-free Chordal Graphs*

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Abstract. A graph is H-free if it has no induced subgraph isomorphic to H. Brandstädt, Engelfriet, Le and Lozin proved that the class of chordal graphs with independence number at most 3 has unbounded clique-width. Brandstädt, Le and Mosca erroneously claimed that the gem and the co-gem are the only two 1-vertex P_4-extensions H for which the class of H-free chordal graphs has bounded clique-width. In fact we prove that bull-free chordal and co-chair-free chordal graphs have clique-width at most 3 and 4, respectively. In particular, we prove that the clique-width is:

(i) bounded for four classes of H-free chordal graphs;
(ii) unbounded for three subclasses of split graphs.

Our main result, obtained by combining new and known results, provides a classification of all but two stubborn cases, that is, with two potential exceptions we determine *all* graphs H for which the class of H-free chordal graphs has bounded clique-width. We illustrate the usefulness of this classification for classifying other types of graph classes by proving that the class of $(2P_1 + P_3, K_4)$-free graphs has bounded clique-width via a reduction to K_4-free chordal graphs. Finally, we give a complete classification of the (un)boundedness of clique-width of H-free weakly chordal graphs.

1 Introduction

Clique-width is a well-studied graph parameter; see for example the surveys of Gurski [29] and Kamiński, Lozin and Milanič [30]. In particular, there are

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numerous graph classes, such as those that can be characterized by one or more forbidden induced subgraphs,[1] for which it has been determined whether or not the class is of *bounded clique-width* (i.e. whether there is a constant c such that the clique-width of every graph in the class is at most c). Clique-width is one of the most difficult graph parameters to deal with and our understanding of it is still very limited. We do know that computing clique-width is NP-hard [25] but we do not know if there exist polynomial-time algorithms for computing the clique-width of even very restricted graph classes, such as unit interval graphs. Also the problem of deciding whether a graph has clique-width at most c for some fixed constant c is only known to be polynomial-time solvable if $c \leq 3$ [13] and is a long-standing open problem for $c \geq 4$. Identifying more graph classes of bounded clique-width and determining what kinds of structural properties ensure that a graph class has bounded clique-width increases our understanding of this parameter. Another important reason for studying these types of questions is that certain classes of NP-complete problems become polynomial-time solvable on any graph class $\mathcal{G}$ of bounded clique-width.[2] Examples of such problems are those definable in Monadic Second Order Logic using quantifiers on vertices but not on edges.

Notation. The *disjoint union* $(V(G) \cup V(H), E(G) \cup E(H))$ of two vertex-disjoint graphs G and H is denoted by $G + H$ and the disjoint union of r copies of a graph G is denoted by rG. The *complement* of a graph G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and only if these vertices are not adjacent in G. If G is a graph, for $S \subseteq V(G)$, we let $G[S]$ denote the *induced* subgraph of G, which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. For two graphs G and H we write $H \subseteq_i G$ to indicate that H is an induced subgraph of G. The graphs $C_r, K_r, K_{1,r-1}$ and P_r denote the cycle, complete graph, star and path on r vertices, respectively. The graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, denotes the *subdivided claw*, that is the tree that has only one vertex x of degree 3 and exactly three leaves, which are of distance h, i and j from x, respectively. For a set of graphs $\{H_1, \ldots, H_p\}$, a graph G is $(H_1, \ldots, H_p)$*-free* if it has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$. A graph G is *chordal* if it is $(C_4, C_5, \ldots)$-free and *weakly chordal* if both G and $\overline{G}$ are $(C_5, C_6, \ldots)$-free. Every chordal graph is weakly chordal.

Research Goal and Motivation. The class of chordal graphs has unbounded clique-width, as it contains the classes of proper interval graphs and split graphs, both of which have unbounded clique-width as shown by Golumbic and Rotics [28] and Makowsky and Rotics [35], respectively. We want to determine all graphs H for which the class of H-free chordal graphs has *bounded* clique-width. Our motivation for this research is threefold.

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[1] For a record see also the Information System on Graph Classes and their Inclusions [22].

[2] This follows from results [15,24,31,38] that assume the existence of a so-called c-expression of the input graph $G \in \mathcal{G}$ combined with a result [37] that such a c-expression can be obtained in cubic time for some $c \leq 8^{\mathrm{cw}(G)} - 1$, where $\mathrm{cw}(G)$ is the clique-width of the graph G.

Firstly, as discussed, such a classification might generate more graph classes for which a number of NP-complete problems can be solved in polynomial time. Although many of these problems, such as the Colouring problem [27], are polynomial-time solvable on chordal graphs, many others stay NP-complete for graphs in this class. Of course, in order to find new "islands of tractability", one may want to consider superclasses of H-free chordal graphs instead. However, already when one considers H-free weakly chordal graphs, one does not obtain any new tractable graph classes. Indeed, the clique-width of the class of H-free graphs is bounded if and only if H is an induced subgraph of P_4 [21], and as we prove later, the induced subgraphs of P_4 are also the only graphs H for which the class of H-free weakly chordal graphs has bounded clique-width. The same classification therefore also follows for superclasses, such as $(H, C_5, C_6, \ldots)$-free graphs (or H-free perfect graphs, to give another example). Since forests, or equivalently, $(C_3, C_4, \ldots)$-free graphs have bounded clique-width it follows that the class of $(H, C_3, C_4, \ldots)$-free graphs has bounded clique-width for every graph H. It is therefore a natural question to ask for which graphs H the class of $(H, C_4, C_5, \ldots)$-free (i.e. H-free chordal) graphs has bounded clique-width.

Secondly, we have started to extend known results [2,5,6,7,8,9,11,17,19,35] on the clique-width of classes of (H_1, H_2)-free graphs in order to try to determine the boundedness or unboundedness of the clique-width of every such graph class [18,21]. This led to a classification of all but 13 open cases (under some equivalence relation, see [21]). An important technique that we used for showing the boundedness of the clique-width of three new graph classes of (H_1, H_2)-free graphs [18] was to reduce these classes to some known subclass of perfect graphs of bounded clique-width (recall that perfect graphs form a superclass of chordal graphs). An example of such a subclass, which we used for one of the three cases, is the class of diamond-free chordal graphs (the diamond is the graph $\overline{2P_1 + P_2}$), which has bounded clique-width [28]. We believe that a full classification of the boundedness of clique-width for H-free chordal graphs would be useful to attack some of the remaining open cases, just as the full classification for H-free bipartite graphs [20] has already proven to be [18,21]. Examples of open cases included the class of $(2P_1 + P_3, K_4)$-free graphs and its superclass of $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graphs [21], the first of which turns out to have bounded clique-width, as we shall prove in this paper via a reduction to K_4-free chordal graphs. The second case is still open.

Thirdly, a classification of those graphs H for which the clique-width of H-free chordal graphs is bounded would complete a line of research in the literature, which we feel is an interesting goal on its own. As a start, using a result of Corneil and Rotics [14] on the relationship between treewidth and clique-width it follows that the clique-width of the class of K_r-free chordal graphs is bounded for all $r \geq 1$. Brandstädt, Engelfriet, Le and Lozin [5] proved that the class of $4P_1$-free chordal graphs has unbounded clique-width. Brandstädt, Le and Mosca [9] considered forbidding the graphs $\overline{P_1 + P_4}$ (gem) and $P_1 + P_4$ (co-gem) as induced subgraphs (see also Fig. 1). They showed that $(P_1 + P_4)$-free chordal graphs have clique-width at most 8 and also observed that $\overline{P_1 + P_4}$-free chordal graphs belong to the class of distance-hereditary graphs, which have clique-width at most 3 (as shown by Golumbic and Rotics [28]). Moreover, the same authors [9] erroneously claimed that the gem and co-gem are the only two 1-vertex P_4-extensions H for which the class of H-free chordal graphs has bounded clique-width. We prove that bull-free chordal graphs have clique-width at most 3, improving a known bound of 8, which was shown by Le [33]. We also prove that $\overline{S_{1,1,2}}$-free chordal graphs have clique-width at most 4, which Le posed as an open problem. Results [28,32,35] for split graphs and proper interval graphs lead to other classes of H-free chordal graphs of unbounded clique-width, as we shall discuss in Section 2. However, in order to obtain our almost-full dichotomy for H-free chordal graphs new results also need to be proved.

Our Results. In Section 2, in addition to some known results for H-free chordal graphs, we give our result that bull-free chordal graphs have clique-width at most 3. In Section 3 we present four new classes of H-free chordal graphs of bounded clique-width,[3] namely when $H \in \{\overline{K_{1,3} + 2P_1}, P_1 + \overline{P_1 + P_3}, P_1 + \overline{2P_1 + P_2}, \overline{S_{1,1,2}}\}$ (see also Fig. 1). We include most of the proof for the $\overline{S_{1,1,2}}$ case, but do not include any other proofs due to space restrictions. In the same section we present three new subclasses of split graphs that have unbounded clique-width, namely $\overline{\mathbb{H}}$-free, $(\overline{3P_1 + P_2})$-free and $(K_3 + 2P_1, K_4 + P_1, P_1 + \overline{P_1 + P_4})$-free split graphs. By combining all these results with a number of previously known results [5,9,28,32,33,35], we obtain an almost-complete classification for H-free chordal graphs, leaving only two open cases (see also Figs. 1 and 2). We omit the proof, which is based on case analysis.

Theorem 1. *Let H be a graph with $H \notin \{F_1, F_2\}$. The class of H-free chordal graphs has bounded clique-width if and only if*
- $H = K_r$ *for some* $r \geq 1$*;*
- $H \subseteq_i$ bull*;*
- $H \subseteq_i P_1 + P_4$*;*
- $H \subseteq_i \overline{P_1 + P_4}$*;*
- $H \subseteq_i \overline{K_{1,3} + 2P_1}$*;*
- $H \subseteq_i P_1 + \overline{P_1 + P_3}$*;*
- $H \subseteq_i P_1 + \overline{2P_1 + P_2}$ *or*
- $H \subseteq_i \overline{S_{1,1,2}}$.

We also present our full classification for H-free weakly chordal graphs. We omit the proof.

Theorem 2. *Let H be a graph. The class of H-free weakly chordal graphs has bounded clique-width if and only if H is an induced subgraph of P_4.*

[3] In Theorems 8, 9 and 10, we do not specify our upper bounds as this would complicate our proofs for negligible gain. In our proofs we repeatedly apply graph operations that exponentially increase the upper bound on the clique-width, which means that the bounds that could be obtained from our proofs would be very large and far from being tight. We use different techniques to prove Lemma 5 and Theorem 11, and these allow us to give good bounds for these cases.

Finally, we illustrate the usefulness of having a classification for H-free chordal graphs by proving that the class of $(2P_1+P_3, K_4)$-free graphs has bounded clique-width via a reduction to K_4-free chordal graphs, and mention future research directions.

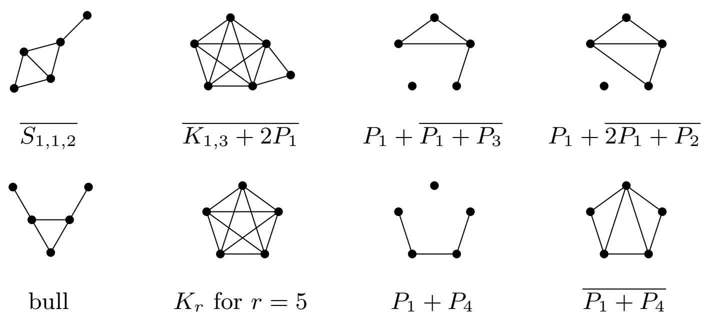

Fig. 1. The graphs H for which the class of H-free chordal graphs has bounded clique-width; the four graphs at the top are new cases proved in this paper.

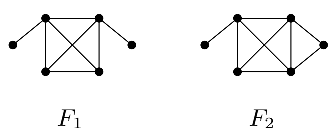

Fig. 2. The graphs H for which boundedness of clique-width of the class of H-free chordal graphs is open.

2 Preliminaries

All graphs considered in this paper are finite, undirected and have neither multiple edges nor self-loops. Let $G=(V,E)$ be a graph. Let $S,T \subseteq V$ with $S \cap T = \emptyset$. We say that S is *complete* to T if every vertex in S is adjacent to every vertex in T, and we say that S is *anti-complete* to T if every vertex in S is non-adjacent to every vertex in T. Similarly, a vertex $v \in V \setminus T$ is *complete* or *anti-complete* to T if it is adjacent or non-adjacent, respectively, to every vertex of T. A set of vertices M is a *module* if every vertex not in M is either complete or anti-complete to M. We say that a vertex v *distinguishes* two vertices x and y if v is adjacent to precisely one of x and y. Note that if a set $M \subseteq V$ is not a module then there must be vertices $x,y \in M$ and a vertex $v \in V \setminus M$ such that v distinguishes x and y.

Let $G = (V, E)$ be a graph. The graph G is a *split graph* if it has a *split partition*, i.e. a partition of V into two (possibly empty) sets K and I, where K is a clique and I is an independent set; if K and I are complete to each other, then G is a *complete* split graph. Every split graph is chordal. It is well known [26] that a graph is split if and only if it is $(C_4, C_5, 2P_2)$-free.

Clique-width. The *clique-width* of a graph G, denoted by $\mathrm{cw}(G)$, is the minimum number of labels needed to construct G by using the following four operations:

1. creating a new graph consisting of a single vertex v with label i (denoted by $i(v)$);
2. taking the disjoint union of two labelled graphs G_1 and G_2 (denoted by $G_1 \oplus G_2$);
3. joining each vertex with label i to each vertex with label j ($i \neq j$, denoted by $\eta_{i,j}$);
4. renaming label i to j (denoted by $\rho_{i\rightarrow j}$).

An algebraic term that represents such a construction of G and uses at most k labels is said to be a k*-expression* of G (i.e. the clique-width of G is the minimum k for which G has a k-expression). For instance, an induced path on four consecutive vertices a, b, c, d has clique-width equal to 3, and the following 3-expression can be used to construct it:

$$\eta_{3,2}(3(d) \oplus \rho_{3\rightarrow 2}(\rho_{2\rightarrow 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))).$$

The following lemma tells us that if $\mathcal{G}$ is a hereditary graph class then in order to determine whether $\mathcal{G}$ has bounded clique-width we may restrict ourselves to the graphs in $\mathcal{G}$ that are prime.

Lemma 3 ([16]). *Let G be a graph and let $\mathcal{P}$ be the set of all induced subgraphs of G that are prime. Then* $\mathrm{cw}(G) = \max_{H\in\mathcal{P}} \mathrm{cw}(H)$.

Known Results on H-free Chordal Graphs. To prove our results, we need to use a number of known results. We present these results as lemmas below; a number of relevant graphs are displayed in Figs. 1 and 3. For a graph G, let $\mathrm{tw}(G)$ denote the treewidth of G. Corneil and Rotics [14] showed that $\mathrm{cw}(G) \leq 3 \times 2^{\mathrm{tw}(G)-1}$ for every graph G. Because the treewidth of a chordal graph is equal to the size of a maximum clique minus 1 (see e.g. [1]), this result leads to the following well-known lemma.

Lemma 4. *The class of K_r-free chordal graphs has bounded clique-width for all* $r \geq 1$.

The *bull* is the graph obtained from the cycle $abca$ after adding two new vertices d and e with edges ad, be (see also Fig. 1). In [9], Brandstädt, Le and Mosca erroneously mentioned that the clique-width of $\overline{S_{1,1,2}}$-free chordal graphs and of bull-free chordal graphs is unbounded. Using a general result of De Simone [23], Le [33] proved that every bull-free chordal graph has clique-width at most 8. Using a result of Olariu [36] we can show the following (we omit the proof).

Lemma 5. *Every bull-free chordal graph has clique-width at most* 3.

Lemma 6 ([9]). *Every* $P_1 + P_4$*-free chordal graph has clique-width at most* 8 *and every* $\overline{P_1 + P_4}$*-free chordal graph has clique-width at most* 3.

Lemma 7 ([5,28,32,35]). *The class of* H*-free chordal graphs has unbounded clique-width if* $H \in \{4P_1, K_{1,3}, 2P_2, C_4, C_5, \text{net}, \overline{\text{net}}\}$.

3 New Classes of Bounded and Unbounded Clique-width

We first present four new classes of H-free chordal graphs that have bounded clique-width. We omit the proofs for the first three of these.

Theorem 8. *The class of* $\overline{K_{1,3} + 2P_1}$*-free chordal graphs has bounded clique-width.*

Theorem 9. *The class of* $(P_1 + \overline{P_1 + P_3})$*-free chordal graphs has bounded clique-width.*

Theorem 10. *The class of* $(P_1 + \overline{2P_1 + P_2})$*-free chordal graphs has bounded clique-width.*

To prove Theorem 8, we make use of the celebrated Menger's Theorem and a tool developed by Lozin and Rautenbach, who proved that a graph G has bounded clique-width if and only if every block of G has bounded clique-width [34]. To the best of our knowledge, this technique has not been explored in previous research on clique-width. For Theorem 9, one may get the impression that the class of $(P_1 + \overline{P_1 + P_3})$-free chordal graphs is not much more complicated than the class of $\overline{P_1 + P_3}$-free chordal graphs and therefore expect it to have bounded clique-width (and similarly for the class of $(P_1 + \overline{2P_1 + P_2})$-free chordal graphs). We point out, however, that clique-width has a subtle transition from bounded to unbounded even if the class of graphs under consideration has a "slight" enlargement. For instance, the class of $(2P_1 + \overline{3P_1})$-free chordal (or even split) graphs (see Theorem 17) turns out to have unbounded clique-width. In fact, our proofs for Theorems 9 and 10 are rather involved. We now present a (detailed) proof sketch of our last new result for boundedness.

Theorem 11. *Every* $\overline{S_{1,1,2}}$*-free chordal graph has clique-width at most* 4.

We first provide a structural description of prime $\overline{S_{1,1,2}}$-free chordal graphs, and then Theorem 11 follows easily from our structural result. To this end, we appeal to the well-developed technique of *prime extension*. Results on prime extension effectively say that a prime graph that contains a particular pattern H as an induced subgraph must contain some extension of H (in the sense of being a supergraph of H) from a prescribed list of graphs (see e.g. [23,33]). The following two structural lemmas, both of which play fundamental roles in the proof of Theorem 11, are of this flavour. The first is due to Brandstädt, Le and de Ridder and the second is due to Brandstädt.

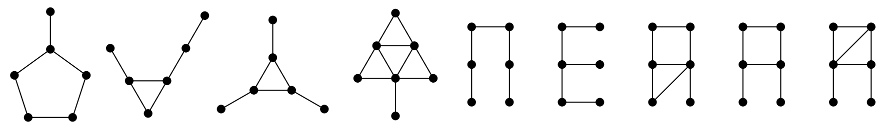

Fig. 3. The minimal prime extensions of $P_1 + P_4$.

Lemma 12 ([10]). *If a prime graph G contains an induced subgraph isomorphic to $P_1 + P_4$ then it contains one of the graphs in Fig. 3 as an induced subgraph.*

Lemma 13 ([3]). *If a prime graph G contains an induced $\overline{2P_1 + P_2}$ then it contains an induced $\overline{P_1 + P_4}$, d-$\mathbb{A}$ or d-domino. (See also Figs. 1 and 3. The d-domino is the graph with vertex set $\{x_1, \dots, x_6\}$ and edge set $\{x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1, x_1x_3, x_1x_4\}$.)*

A graph G is a *thin spider* if its vertex set can be partitioned into a clique K, an independent set I and a set R such that $|K| = |I| \geq 2$, the set R is complete to K and anti-complete to I and the edges between K and I form an induced matching (that is, every vertex of K has a unique neighbour in I and vice versa). Note that if a thin spider is prime then $|R| \leq 1$. A *thick spider* is the complement of a thin spider. A graph is a *spider* if it is either a thin or a thick spider. Spiders play an important role in our result for $\overline{S_{1,1,2}}$-free chordal graphs and we will need the following lemma (due to Brandstädt and Mosca).

Lemma 14 ([12]). *If G is a prime $S_{1,1,2}$-free split graph then it is a spider.*

We now show that the clique-width of $\overline{S_{1,1,2}}$-free chordal graphs is bounded. Switching to the complement, we study $S_{1,1,2}$-free co-chordal graphs which are a subclass of $(2P_2, C_5, S_{1,1,2})$-free graphs. The main step consists of the following structural result.

Lemma 15. *If a prime $(2P_2, C_5, S_{1,1,2})$-free graph G contains an induced subgraph isomorphic to the net (see Fig. 3) then G is a thin spider.*

Proof. Suppose that G is a prime $(2P_2, C_5, S_{1,1,2})$-free graph and suppose that G contains a net, say N with vertices $a_1, a_2, a_3, b_1, b_2, b_3$ such that a_1, a_2, a_3 is an independent set (the *end-vertices* of N), b_1, b_2, b_3 is a clique (the *mid-vertices* of N), and the only edges between a_1, a_2, a_3 and b_1, b_2, b_3 are $a_ib_i \in E(G)$ for $i \in \{1, 2, 3\}$.

Let $M = V(G) \setminus V(N)$. We partition M as follows: For $i \in \{1, \dots, 5\}$, let M_i be the set of vertices in M with exactly i neighbours in $V(N)$. Let U be the set of vertices in M adjacent to every vertex of $V(N)$. Let Z be the set of vertices in M with no neighbours in $V(N)$. Note that Z is an independent set in G, since G is $2P_2$-free. We now analyse the structure of G through a series of claims. The proofs of these claims have been omitted.

Claim 1. *$M_1 \cup M_2 \cup M_5 = \emptyset$.*
Next, we show that vertices in $M_3 \cup M_4$ have a restricted type of neighbourhood in $V(N)$:

Claim 2. *Every $x \in M_3$ is adjacent to either exactly one end-vertex a_i and its two opposite mid-vertices b_j and b_k ($j \neq i$, $k \neq i$) or to all three mid-vertices of N.*
The situation for M_4 is similar to that of M_3, as shown in the following claim.

Claim 3. *If $x \in M_4$ then it is adjacent to exactly one end-vertex and all mid-vertices.*
Let Mid_3 denote the set of vertices in M_3 that are adjacent to all three mid-vertices of N (and non-adjacent to any end-vertex of N).

Claim 4. *U is complete to $(M_3 \cup M_4)$.*
Let Z_1 denote the set of vertices in Z that have a neighbour in $M_3 \cup M_4$, and let $Z_0 = Z \setminus Z_1$. The next two claims show the adjacency between Z_1 and other subsets of $V(G)$.

Claim 5. *Z_1 is anti-complete to $((M_3 \cup M_4) \setminus Mid_3)$.*
Claim 6. *U is complete to Z_1.*

Let $X = V(N) \cup M_3 \cup M_4 \cup Z_1$. Then X is a module: every vertex in U is complete to X (due to the definition of U, together with Claims 4 and 6) and every vertex in Z_0 is anti-complete to X (due to the definitions of Z, Z_0 and Z_1, together with the fact that Z is an independent set). Since G is prime, X must be a trivial module. Since X contains more than one vertex, it follows that $V(G) = X = V(N) \cup M_3 \cup M_4 \cup Z_1$. Hence $U \cup Z_0 = \emptyset$. It remains to show that $G = G[V(N) \cup M_3 \cup M_4 \cup Z_1]$ is a thin spider. For $i \in \{1, 2, 3\}$ let $M'_i = (M_3 \cup M_4) \cap N(a_i)$. Note that $M_3 \cup M_4 = Mid_3 \cup M'_1 \cup M'_2 \cup M'_3$. The next two claims show how each M'_i is connected to other subsets of $V(G)$.

Claim 7. *For $i \neq j$, M'_i is complete to M'_j.*
Claim 8. *For every $i = 1, 2, 3$, M'_i is complete to Mid_3.*

By Claims 2, 3, 5, 7 and 8 we find that, for every $i \in \{1, 2, 3\}$, $M'_i \cup \{b_i\}$ is a module, so $M'_i = \emptyset$ (since G is prime). Consequently, $V(G) = V(N) \cup Mid_3 \cup Z_1$. Next, we show the following:

Claim 9. *Mid_3 is a clique.*
By Claim 9 and the definition of Mid_3, we find that $\{b_1, b_2, b_3\} \cup Mid_3$ is a clique. By the definition of Z and the fact that Z is independent, $\{a_1, a_2, a_3\} \cup Z_1$ is an independent set. Therefore G is a split graph. By Lemma 14, since G is prime and $S_{1,1,2}$-free, it must be a spider. Since G contains an induced net, it must be a thin spider. □

The following is our new structural theorem on prime chordal $\overline{S_{1,1,2}}$-free graphs.

Theorem 16. *If G is a prime chordal $\overline{S_{1,1,2}}$-free graph then it is either a $\overline{2P_1 + P_2}$-free graph or a thick spider.*

Proof. Let G be a prime $\overline{S_{1,1,2}}$-free chordal graph. Note that since G is $\overline{S_{1,1,2}}$-free, it cannot contain d-$\mathbb{A}$ (see also Fig. 3) or d-domino as an induced subgraph. If G is $\overline{P_1+P_4}$-free then, by Lemma 13, it must therefore be $\overline{2P_1+P_2}$-free.

Now suppose that G contains an induced copy of $\overline{P_1+P_4}$. Since G is prime, $\overline{G}$ is also prime. Furthermore, $\overline{G}$ is $(2P_2, C_5, S_{1,1,2})$-free. By Lemma 12, $\overline{G}$ must contain one of the graphs in Fig. 3. The only graph in Fig. 3 which is $(2P_2, C_5, S_{1,1,2})$-free is the net, so $\overline{G}$ must contain a net. By Lemma 15, $\overline{G}$ is a thin spider, so G is a thick spider. ☐

We are now ready to prove Theorem 11.

Theorem 11 (restated) *Every $\overline{S_{1,1,2}}$-free chordal graph has clique-width at most* 4.

Proof. Let G be an $\overline{S_{1,1,2}}$-free chordal graph. By Lemma 3, we may assume that G is prime. If G is $\overline{2P_1+P_2}$-free then it has clique-width at most 3 by Lemma 6. By Theorem 16, we may therefore assume that G is a thick spider. Note that since a thick spider is the complement of a thin spider (see also the definition of a thin spider), K is an independent set, I is a clique and R is complete to I and anti-complete to K. Every vertex in K has exactly one non-neighbour in I and vice versa. Since G is prime and R is a module, R contains at most one vertex.

Let $i_1,\ldots,i_p$ be the vertices in I and let $k_1,\ldots,k_p$ be the vertices in K such that for each $j \in \{1,\ldots,p\}$, the vertex i_j is the unique non-neighbour of k_j in I. Let G_j be the labelled copy of $G[\{i_1,\ldots,i_j,k_1,\ldots,k_j\}]$ where every i_h is labelled 1 and every k_h is labelled 2. Now $G_1 = 1(i_1) \oplus 2(k_1)$ and for $j \in \{1,\ldots,p-1\}$ we can construct G_{j+1} from G_j as follows:

$$G_{j+1} = \rho_{3\to 1}(\rho_{4\to 2}(\eta_{1,3}(\eta_{1,4}(\eta_{2,3}(G_j \oplus 3(i_{j+1}) \oplus 4(k_{j+1})))))).$$

If $R = \emptyset$ then using the above recursively we get a 4-expression for G_p and therefore for G. If $R = \{x\}$ then we obtain a 4-expression for G using $\eta_{1,4}(G_p \oplus 4(x))$. Therefore G indeed has clique-width at most 4. This completes the proof. ☐

We now present three new subclasses of H-free split graphs that have unbounded clique-width. (The graph $\overline{\mathbb{H}}$ is the graph on six vertices whose complement looks like a capital letter "H".) We omit the proofs.

Theorem 17. *The following classes have unbounded clique-width:*
- $\overline{\mathbb{H}}$*-free split graphs*
- $\overline{3P_1+P_2}$*-free split graphs*
- $(K_3+2P_1, K_4+P_1, P_1+\overline{P_1+P_4})$*-free split graphs.*

4 Concluding Remarks

Using our new results and a significant amount of non-trivial case analysis, we are able to prove our classification theorems. As an application of this, we can prove the following theorem via a reduction to K_4-free chordal graphs (we omit the proof). This has reduced the number of open problems posed in [21] to 13.

Theorem 18. *The class of* $(K_4, 2P_1 + P_3)$*-free graphs has bounded clique-width.*

We still need to determine whether or not the classes of F_i-free chordal graphs have bounded clique-width when $i \in \{1, 2\}$. For this purpose, we recently managed to show that for $i \in \{1, 2\}$, the class of F_i-free split graphs has bounded clique-width [4] and we are currently exploring whether it is possible to generalize the proof of this result to the class of F_i-free chordal graphs.

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