A comparison of techniques for overcoming nonuniqueness of boundary integral equations for collocation Partition of Unity method in acoustics

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Abstract

In the last decade the Partition of Unity Method has become attractive as one approach for extending the allowable frequency range for wave simulations beyond that available using piecewise polynomial elements. It is well known that the conventional boundary integral equation obtained through Green's representation theorem suffers from a problem of nonuniqueness at certain frequencies. The standard methods of overcoming this problem are the so-called CHIEF method and that of Burton and Miller. The latter method introduces a hypersingular operator which may be treated in various ways. In this paper we present the collocation Partition of Unity BEM for Helmholtz problems and compare the performance of CHIEF against a Burton-Miller formulation regularised using the approach of Li and Huang.

Keywords: Nonuniqueness, CHIEF, Burton - Miller formulation, Partition of Unity BEM

1. Introduction

The theory of the boundary element method (BEM) for solving boundary integral equations (BIE) is well established [1]. It is well known that the Conventional BIE (CBIE) for an exterior acoustic problem results in a nonunique solution at irregular frequencies for the corresponding interior problem and that this is a purely mathematical phenomenon. A well known method to avoid this problem is the so called Combined Helmholtz Integral Equation Formulation (CHIEF) due to Schneck [2], where some additional Helmholtz integral equations evaluated at interior points are added in the original system matrix. Although this results in an over-determined system, CHIEF ensures a unique solution at an irregular frequency. Of course, one needs to set the interior points such that they do not lie on the nodal points of the eigenmodes of the interior Helmholtz problem. This however can introduce uncertainties for complicated geometries at high wavenumbers as the nodal points become densely packed in the interior which makes it difficult to find suitable locations for the placement of interior points. Another method to avoid the nonuniqueness problem is due to Burton and Miller [3]. They showed that the integral equation resulting from linear combination of the CBIE and its normal derivative at the collocation point always results in a unique solution. The potential problem with this method is the evaluation of the hypersingular integral which arises as a result of the differentiation of the CBIE at a collocation point. The Partition of Unity BEM (PUBEM) is based on the use of the wave nature of the solution and is shown to produce highly accurate results with improved efficiency in terms of degrees of freedom per wavelength. In the present study, we compare the CHIEF method with one regularized form of the Burton-Miller formulation for acoustic scattering from hard cylinders in two dimensions using PUBEM. The two methods are compared for their accuracy of the solution and efficiency.

2. Governing equation and the Boundary Integral Equations

The well known equation for time harmonic acoustic scattering and wave propagation is the Helmholtz equation

$$\nabla^2\phi(q) + k^2\phi(q) = 0 \qquad q \in \Omega \tag{1}$$

where k is the acoustic wavenumber, ϕ the spatially dependent ($e^{-i\omega t}$ time dependence) total acoustic potential that we seek in the computational domain Ω and ∇^2 is the Laplacian operator. For exterior acoustic problems, the total (or scattered) acoustic potential has to satisfy Sommerfeld's radiation condition given by

$$\lim_{r\to\infty} r^{\frac{n-1}{2}}\left(\frac{\partial}{\partial r} - ik\right)\phi = 0 \tag{2}$$

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where r is the distance of a point in Ω from the origin, $i = \sqrt{-1}$ and n the dimension of the space. The mathematical formulation for deriving the CBIE from the Helmholtz equation is well established [4]. The CBIE for an acoustic scattering (or radiation) problem governed by the Helmholtz differential equation is given by

$$c(p)\phi(p) + \int_\Gamma \frac{\partial G}{\partial n_q}\phi(q)d\Gamma(q) = \int_\Gamma G\frac{\partial \phi(q)}{\partial n_q}d\Gamma(q) + \phi^i(p) \tag{3}$$

where p is the collocation or source point, q the field point, G the free space Green's function for the Helmholtz problem, n_q and n_p the outward normals respectively at points q and p pointing away from acoustic domain Ω, $\phi(q)$ the unknown acoustic potential and $\phi^i(p)$ the known incident acoustic wave. $c(p)$ is the free coefficient which depends on the local geometry of Γ at p. In this study we assume Γ is smooth and take $c(p) = \frac{1}{2}$. The normal derivative of (3) at the collocation point p is given by

$$c(p)\frac{\partial \phi(p)}{\partial n_p} + \int_\Gamma \frac{\partial^2 G}{\partial n_p \partial n_q}\phi(q)d\Gamma(q) = \int_\Gamma \frac{\partial G}{\partial n_p}\frac{\partial \phi(q)}{\partial n_q}d\Gamma(q) + \frac{\partial \phi^i(p)}{\partial n_p} \tag{4}$$

and the Combined Hypersingular BIE (CHBIE) due to Burton and Miller [3] is

$$\begin{aligned} &c(p)\phi(p) + \alpha c(p)\frac{\partial \phi(p)}{\partial n_p} + \int_\Gamma \frac{\partial G}{\partial n_q}\phi(q)d\Gamma(q) + \alpha\int_\Gamma \frac{\partial^2 G}{\partial n_p \partial n_q}\phi(q)d\Gamma(q) = \\ &\int_\Gamma G\frac{\partial \phi(q)}{\partial n_q}d\Gamma(q) + \alpha\int_\Gamma \frac{\partial G}{\partial n_p}\frac{\partial \phi(q)}{\partial n_q}d\Gamma(q) + \phi^i(p) + \alpha\frac{\partial \phi^i(p)}{\partial n_p} \end{aligned} \tag{5}$$

where α is a coupling constant most commonly taken as i/k. In the present study, we analyse the acoustic scattering from sound hard cylinders. A sound hard surface is where the normal derivative of the total acoustic potential vanishes. Therefore, all the terms involving the normal derivative of acoustic potential vanish. Although (5) results in a unique solution, its main drawback remains the numerical treatment of the hypersingular integral, i.e. the last integral on the left hand side. Li and Huang [5] give the following weakly singular form of the hypersingular integral

$$\begin{aligned} \int_\Gamma \frac{\partial^2 G}{\partial n_p \partial n_q}\phi(q)d\Gamma(q) = &\int_\Gamma \left[\frac{\partial^2 G}{\partial n_p \partial n_q} - \frac{\partial^2 G_0}{\partial n_p \partial n_q}\right]\phi(q)\Gamma(q) + \int_\Gamma [\phi(q) - \phi(p) - \nabla\phi(p)\cdot(q-p)]\frac{\partial^2 G_0}{\partial n_p \partial n_q}d\Gamma(q) \\ &+ \int_\Gamma \nabla\phi(p)\cdot n_q\frac{\partial G_0}{\partial n_p}d\Gamma(q) - \frac{1}{2}\nabla\phi(p)\cdot n_p \end{aligned} \tag{6}$$

where G_0 is the free space Green's function for the Laplace equation. Again, for the present case of a hard boundary, the last term in the right hand side of (6) vanishes. Consequently, the final equation for this case of a hard boundary can be expanded as

$$\begin{aligned} &c(p)\phi(p) + \int_\Gamma \frac{\partial G}{\partial n_q}\phi(q)d\Gamma(q) + \alpha\int_\Gamma \left[\frac{\partial^2 G}{\partial n_p \partial n_q} - \frac{\partial^2 G_0}{\partial n_p \partial n_q}\right]\phi(q)\Gamma(q) + \alpha\int_\Gamma [\phi(q) - \phi(p) - \nabla\phi(p)\cdot(q-p)]\frac{\partial^2 G_0}{\partial n_p \partial n_q}d\Gamma(q) + \\ &\alpha\int_\Gamma \nabla\phi(p)\cdot n_q\frac{\partial G_0}{\partial n_p}d\Gamma(q) = \phi^i(p) + \alpha\frac{\partial \phi^i(p)}{\partial n_p} \end{aligned} \tag{7}$$

3. Plane wave basis and and discretization of CHBIE

The injection of enrichment based on the wave nature of the solution into the basis functions for wave problems is very well established. The dramatic improvement in the accuracy of the solution and an overall efficiency achieved in the solution process has been widely reported, see [6],[7],[8]. The acoustic potential at a point $\mathbf{x}$ in the domain Ω using plane wave basis can be approximated as

$$\phi(\mathbf{x}) = \sum_{j=1}^{3} N_j \sum_{m}^{M_j} A_{jm} e^{ik\mathbf{d}_{jm}\cdot\mathbf{x}} \qquad \mathbf{x} \in \Omega \tag{8}$$

where N_j is the j^{th} shape function, A_{jm} the unknown which can be thought of as the amplitude of the m^{th} plane wave with wave number k associated with node j. The direction of the m^{th} plane wave at node j is given by unit vector $\mathbf{d}_{jm}$ and $\mathbf{x}$ is the location of the point where the potential ϕ is sought. The element considered here is three noded and M_j is the number of plane waves located around the j^{th} node. In the context of the BEM, the plane wave basis defined in (8) can be used to express the unknown acoustic potential on the boundary Γ. The only change from conventional polynomial collocation

BEM is that now the unknowns are amplitudes of plane waves (A_{jm}) located around boundary element nodes. It is now convenient to write the following discretized form of (7) using (8)

$$C_1 + \sum_{s=1}^{s=4} \sum_{e=1}^{NEL} I_s^e = C_2 + C_3 \tag{9}$$

where

$$C_1 = c(p) \sum_{j=1}^{3} N_j^p \sum_{m=1}^{M_j} A_{jm}^p e^{ik\mathbf{d}_{jm}\cdot\mathbf{x}(p)} \tag{10}$$

$$I_1^e = \int_{\Gamma^e} \left(\frac{\partial G}{\partial n_q}\right) \sum_{j=1}^{3} N_j^q \sum_{m=1}^{M_j} A_{jm}^q e^{ik\mathbf{d}_{jm}\cdot\mathbf{x}(q)} d\Gamma^e(q) \tag{11}$$

$$I_2^e = \alpha \int_{\Gamma^e} \frac{\partial^2 G}{\partial n_q \partial n_p} \left(\sum_{j=1}^{3} N_j^q \sum_{m=1}^{M_j} A_{jm}^q e^{ik\mathbf{d}_{jm}\cdot\mathbf{x}(q)} - \sum_{j=1}^{3} N_j^p \sum_{m=1}^{M_j} A_{jm}^p e^{ik\mathbf{d}_{jm}\cdot\mathbf{x}(p)} \right) d\Gamma^e(q) \tag{12}$$

$$I_3^e = \alpha \int_{\Gamma^e} \frac{\partial^2 G_0}{\partial n_p \partial n_q} \left(\left(\sum_{j=1}^{3} N_j^q \sum_{m=1}^{M_j} A_{jm}^q e^{ik\mathbf{d}_{jm}\cdot\mathbf{x}(q)} - \sum_{j=1}^{3} N_j^p \sum_{m=1}^{M_j} A_{jm}^p e^{ik\mathbf{d}_{jm}\cdot\mathbf{x}(p)} \right) - \right. \tag{13}$$

$$\left. \left(\frac{\partial}{\partial x} \left[\sum_{j=1}^{3} N_j^p \sum_{m=1}^{M_j} A_{jm}^p e^{ik\mathbf{d}_{jm}\cdot\mathbf{x}(p)} \right] r_x + \frac{\partial}{\partial y} \left[\sum_{j=1}^{3} N_j^p \sum_{m=1}^{M_j} A_{jm}^p e^{ik\mathbf{d}_{jm}\cdot\mathbf{x}(p)} \right] r_y \right) d\Gamma^e(q)$$

$$I_4^e = \alpha \int_{\Gamma^e} \left(\frac{\partial}{\partial x} \left[\sum_{j=1}^{3} N_j^p \sum_{m=1}^{M_j} A_{jm}^p e^{ik\mathbf{d}_{jm}\cdot\mathbf{x}(p)} \right] n_{qx} + \frac{\partial}{\partial y} \left[\sum_{j=1}^{3} N_j^p \sum_{m=1}^{M_j} A_{jm}^p e^{ik\mathbf{d}_{jm}\cdot\mathbf{x}(p)} \right] n_{qy} \right) d\Gamma^e(q) \tag{14}$$

and

$$C_2 = \phi^i(p) \;\; ; \;\; C_3 = \alpha \frac{\partial \phi^i(p)}{\partial n_p} \tag{15}$$

where NEL is the total number of boundary elements dividing the boundary Γ and Γ^e is the division of Γ corresponding to the e^{th} boundary element and $r_x = x(q) - x(p),\ r_y = y(q) - y(p)$. Choosing appropriate locations on the boundary Γ as collocation point p yields the following set of linear equations

$$[\mathbf{H}]\{\mathbf{a}\} = \{\mathbf{b}\} \tag{16}$$

where the vector $\mathbf{a}$ contains the amplitudes of plane waves. Vector $\mathbf{b}$ is obtained as

$$\{\mathbf{b}\} = \{\mathbf{C_2} + \mathbf{C_3}\} \tag{17}$$

where $\{\mathbf{C_2}\}$ and $\{\mathbf{C_3}\}$ are the vectors formed using (15). The matrix $\mathbf{H}$ is obtained by evaluating the boundary integrals.

4. Numerical results for scattering problems

We present a PUBEM solution for two acoustic scattering problems for which either an analytical solution or an approximate series solution is available. For both the examples presented we use two 3-noded continuous elements to model the cylinder boundary. The shape functions are trigonometric and the collocation points are always equally spaced. It has been shown recently that the use of trigonometric shape functions improves the accuracy of the PUBEM solution over the conventional polynomial shape functions [9]. The integration points are always placed on analytical geometry.

4.1. Sound hard single cylinder

The first example is that of scattering of a plane incident wave due to a sound hard cylinder of radius a and of infinite length placed in a homogeneous, unbounded acoustic medium (air). We assume the incident wave is of unit amplitude and with direction vector $(-1, 0)$ i.e. travelling in the negative x direction. This problem has an analytical solution for scattered potential, ϕ^s, given by an infinite series, see [10].

$$\phi^s(\mathbf{x}) = -\frac{\mathrm{J}_0'(ka)}{\mathrm{H}_0'(ka)} \mathrm{H}_0(kr) - 2 \sum_{\nu=1}^{\infty} i^\nu \frac{\mathrm{J}_\nu'(ka)}{\mathrm{H}_\nu'(ka)} \mathrm{H}_\nu(kr) \cos(\nu\theta), \tag{18}$$

where $\mathbf{x} = r(\cos(\theta), \sin(\theta))$, $\mathrm{H}_\nu(kr)$ is the Hankel function of the first kind and order ν, $\mathrm{J}_\nu(kr)$ is the Bessel function of the first kind and order ν. The prime sign denotes derivative with respect to kr. Fig. 1 shows the plot of relative L^2 error in ϕ^s against the wavenumber k for CHIEF and Burton - Miller schemes. The relative L^2 error is defined as

$$E^2 = \sqrt{\frac{\sum_{j=1}^{NOP} \left(\tilde{\phi}_j - \phi_j\right)^2}{\sum_{j=1}^{NOP} \tilde{\phi}_j^{\,2}}} \tag{19}$$

where ϕ_j is the numerically computed solution and $\tilde{\phi}_j$ is the analytical solution at the j^{th} point computed using the series in (18) on the boundary Γ. NOP is the total number of points on the boundary Γ used for computing the relative L^2 error. All results corresponding to CHIEF for this example are obtained with 50 randomly placed collocation points in the interior of the cylinder. The coefficient matrix **H** generated using the plane wave basis is always highly ill-conditioned. For this reason we use Singular Value Decomposition (SVD) in order to ensure solvability of the system of equations. All the singular values below a threshold of 10^{-10} are set to zero when computing the inverse. Fig. 2 shows the comparison for 2-norm condition numbers for the coefficient matrix **H** for CHIEF and Burton-Miller. The 2-norm condition number, $\kappa(A)$, for a general matrix A is calculated using

$$\kappa(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)} \tag{20}$$

where $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ are the maximum and minimum singular values of the matrix A. As seen from Fig.1, CHIEF

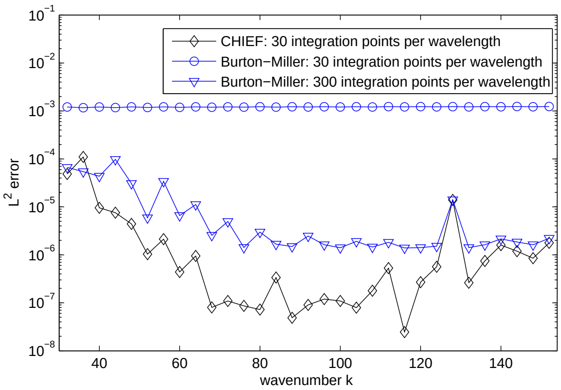

Figure 1: L^2 errors: acoustic scattering by unitradius hard cylinder

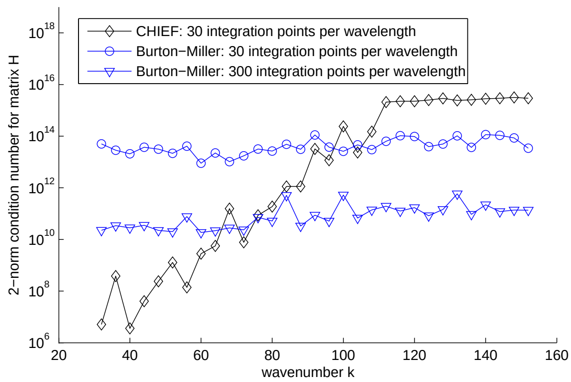

Figure 2: 2−norm condition numbers for single cylinder problem

provides better accuracy compared to Burton-Miller at 30 integration points per wavelength. Burton-Miller needs at least 300 integration points per wavelength to achieve the same accuracy as that of CHIEF at 30 integration points per wavelength. It should be mentioned here that although the regularised form of the Burton-Miller formulation used here is only weakly singular, the first integral on the right hand side of (6) converges extremely slowly. Due to the slow convergence of the said integral in (6), Burton-Miller needs a very high number of integration points to be used in order to achieve the accuracy comparable to that from the CHIEF method. The condition numbers for CHIEF for $k < 100$ are better compared with Burton-Miller. However, the conditioning for Burton-Miller improves for higher wavenumbers ($k > 100$) (Fig.2). Interestingly an accurately computed Burton-Miller solution provides a better conditioning of the system matrix.

4.2. Array of sound hard cylinders

This is an example problem where the use of PUBEM is the most justified because of the multiple reflections from the individual cylinders. Consider a setting where four sound hard cylinders of infinite extent and unit radius are placed in an unbounded homogeneous acoustic medium (air). The cylinder centres are given as (-2,-2), (2,-2), (2,2) and (-2,2). Consider a plane wave of unit amplitude with wavenumber 2.4048 incident upon the cylinder array at an angle $\theta^I = 45°$ with the horizontal where the θ^I is measured anticlockwise. For this example the results are compared with an approximate series given in [11]. In figures 3-6 we compare the PUBEM solution (absolute value of total potential on the surface of each cylinder) from CHIEF and Burton - Miller with the series solution from [11] for $k = 2.4048$. It may be noted that $k = 2.4048$ is the first zero of the Bessel function $\mathrm{J}_0(ka)$ for $a = 1$ and thus corresponds to the first eigenmode of the interior Dirichlet problem. Only one interior collocation point is used for CHIEF. Table 1 gives a comparison of the

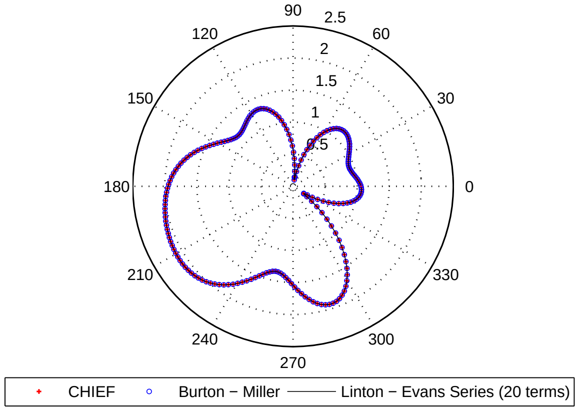

Figure 3: $|\phi|$ for Cylinder 4, $k = 2.4048$

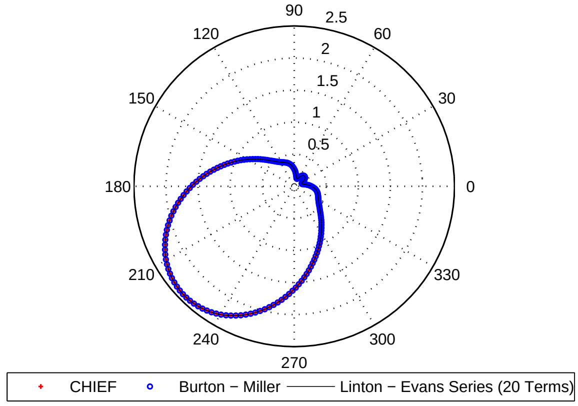

Figure 4: $|\phi|$ for Cylinder 3, $k = 2.4048$

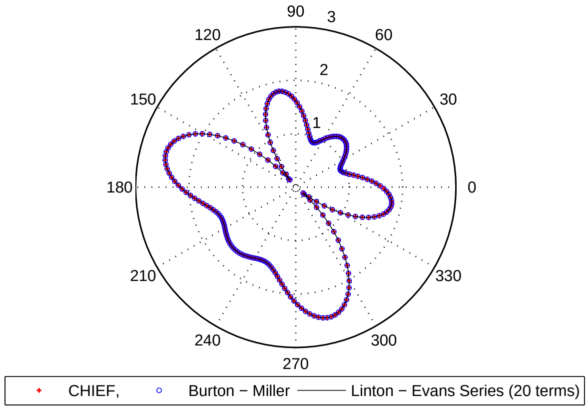

Figure 5: $|\phi|$ for Cylinder 1, $k = 2.4048$

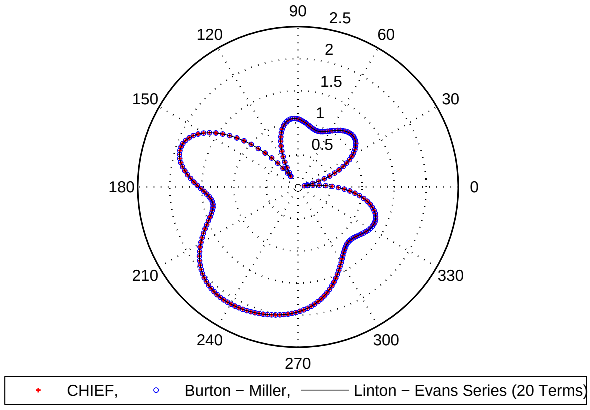

Figure 6: $|\phi|$ for Cylinder 2, $k = 2.4048$

relative L^2 errors for various integration orders relative to the wavelength. The relative L^2 errors for this example are computed using (19) except now $\tilde{\phi}_j$ is the approximate series solution obtained using the series from reference [11].

	Cylinder 1		Cylinder 2		Cylinder 3		Cylinder 4	
NGP /λ	CHIEF	BM	CHIEF	BM	CHIEF	BM	CHIEF	BM
30	6.27e-007	1.52e-004	7.8e-007	1.29e-004	4.13e-007	1.23e-004	7.64e-007	1.29e-004
60	9.28e-007	2.0e-005	6.92e-007	1.70e-005	8.59e-007	1.62e-005	9.27e-007	1.70e-005
100	9.48e-007	4.38e-006	8.41e-007	4.55e-006	9.3e-007	3.59e-006	1.04e-006	4.07e-006
300	1.12e-006	6.22e-007	1.02e-006	7.618e-007	8.358e-007	3.27e-007	1.2e-006	8.04e-007

Table 1: PUBEM results - Relative L^2 errors for scattering from four cylinder array for $k = 2.4048$ and $\theta^I = 45°$ (NGP/λ: no. of Gauss points per wavelength)

As seen from Table1, CHIEF outperforms Burton-Miller for the case of 30 integration points per wavelength which is practical. Again for a multiple cylinder case, Burton-Miller needs about 300 integration points to be in the same accuracy range as that of CHIEF. The polar plots for total potential are for the case of 30 integration points per wavelength. Despite the difference seen in the table1 it is not possible to distinguish CHIEF and Burton-Miller results as they virtually lie on top of each other.

5. Conclusions

A comparison of CHIEF and Burton-Miller schemes for the use in the PUBEM has been presented for the first time. From the results presented it is clear that CHIEF outperforms Burton-Miller for both single cylinder and multiple cylinder problems. Accuracy of Burton-Miller improves only when a large number of integration points are used. CHIEF thus may be preferred over Burton-Miller at least for simple geometries and moderate frequencies where choosing interior points is easy. Burton-Miller offers only a moderate advantage in improving the conditioning of the coefficient matrix $\mathbf{H}$ at larger k. It needs to be investigated if Burton - Miller offers any advantage for complex geometries where it might be difficult to choose the interior points at high frequencies.

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