3D EXTENDED ISOGEOMETRIC BOUNDARY ELEMENT METHOD (XIBEM) FOR ACOUSTIC WAVE SCATTERING

*Michael J. Peake, Jon Trevelyan and Graham Coates

Durham University, School of Engineering and Computing Sciences, South Road, Durham, DH1 3LE

*m.j.peake@dur.ac.uk

ABSTRACT

An isogeometric boundary element method based on NURBS is used to find solutions to the Helmholtz equation. The method is extended in a partition-of-unity sense, multiplying the NURBS functions with families of plane waves; this new method is called the eXtended Isogeometric Boundary Element Method (XIBEM). When compared to non-enriched boundary element simulations, using XIBEM significantly reduces the number of degrees of freedom required to obtain a solution with a given error.

The extension used here (and in similar Trefftz methods) requires a set of wave directions to be specified. Ideally, these directions are uniformly spaced points over the unit-sphere. Simple schemes (e.g. latitude/longitude and discretised cube methods) have been proposed previously. However, while these schemes provide adequate spacing of wave directions, they do not allow an arbitrary number of wave directions to the be chosen. The authors use a novel algorithm, based on a physical analogy of charged particles held in static equilibrium on a spherical surface.

The XIBEM formulation will be described, including a focus on the novel method of choosing a uniformly spaced set of plane wave directions for the enrichment. Numerical results show the reduction in degrees of freedom required to obtain approximations of engineering accuracy.

Key Words: acoustics; Helmholtz; boundary elements; isogeometric; wave scattering

1. Introduction

The *boundary element method* (BEM) is an effective tool for analysing homogeneous acoustic scattering problems: it inherently includes the conditions of infinite domains and only requires the boundary of a scatterer to be meshed.

Recently, in general BEM research, various authors have explored the possibility of taking the basis functions used to describe a geometry in CAD and using them directly in a numerical analysis [1, 2, 6]. This concept is known as *isogeometric analysis* (IGA).

The papers above have shown that an IGA-BEM approach is viable and produces accurate results. However, simulations in acoustics are still limited by an old heuristic rule that dictates that 10 degrees of freedom per wavelength per coordinate direction are required to obtain approximations of engineering accuracy. A number of enriched (or Trefftz) methods have been developed to overcome this. In particular, the authors are interested in the partition-of-unity BEM (PU-BEM) [5]. In this, the approximation basis of the boundary elements is enriched by a family of plane waves. This greatly reduces the numbers of degrees of freedom that are required to solve a problem of a given wavelength.

When using the PU-BEM, it has been found important to use an analytical description of the geometry. This has only been available a few geometries, thus far. However, by using the basis functions used in CAD, this requirement is automatically fulfilled. This paper shows how the combination of IGA-BEM and partition-of-unity enrichment provides for an accurate and computationally efficient algorithm that can interface well with a CAD environment.

The authors have successfully used a partition-of-unity enrichment with 2D isogeometric BEM simulations—a combination termed the eXtended Isogeometric Boundary Element Method (XIBEM)

[3]. This conference paper considers the advance to 3D simulations. The authors focus on a novel approach to determine the directions of propagation of the enriching waves. Some initial results of 3D XIBEM simulations are given.

2. XIBEM

2.1. Analytical formulation

 $\Omega \subset \mathbb{R}^3$ is an unbounded, homogeneous domain containing a smooth scatterer of boundary $\Gamma := \partial \Omega$. Acoustic waves, considered in the frequency domain with $\exp(-\iota\omega t)$ time dependence, are governed by the Helmholtz equation:

$$\nabla^2 \phi(\mathbf{p}) + k^2 \phi(\mathbf{p}) = 0, \quad \phi \in \mathbb{C}, \mathbf{q} \in \Omega, \tag{1}$$

where $\nabla^2(\cdot)$ is the Laplacian operator, ϕ is a complex wave potential, and k the is wavenumber—directly related to the wavelength $\lambda = 2\pi/k$. ι is used to denote the imaginary number, to avoid confusion with subscripts *i* and *j* later. The scatterer is impinged by an incident plane wave,

$$\phi^{\text{inc}}(\mathbf{p}) = A^{\text{inc}} \exp\left(\iota k \mathbf{d}^{\text{inc}} \cdot \mathbf{p}\right), \quad \left|\mathbf{d}^{\text{inc}}\right| = 1,$$
 (2)

where A^{inc} is the incident wave amplitude and the unit vector, \mathbf{d}^{inc} , is the direction of propagation.

To solve this problem using boundary elements, a boundary integral equation is required. The derivation of the conventional BIE is well-known and yields:

$$\frac{1}{2}\phi(\mathbf{p}) = \int_{\Gamma} \left[\frac{\partial\phi(\mathbf{q})}{\partial n(\mathbf{q})} G(\mathbf{p}, \mathbf{q}) - \phi(\mathbf{q}) \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n(\mathbf{q})} \right] d\Gamma(\mathbf{q}) + \phi^{\text{inc}}(\mathbf{p}), \quad \mathbf{p}, \mathbf{q} \in \Gamma,$$
(3)

where **p** is an evaluation point, **q** is an integration point, and *n* is an outward-pointing, unit normal. Further, $G(\mathbf{p}, \mathbf{q})$ is the fundamental solution or *Green's function*, representing the field effect experienced at **q** due to a source radiating at **p**. The potential at any point **p** can be evaluated with (3) if $\phi(\mathbf{q})$ and $\partial \phi(\mathbf{q})/\partial n(\mathbf{q})$ over the entire the boundary is found.

2.2. Numerical implementation

Non-uniform rational B-splines (NURBS) are ubiquitous in CAD software and so it is NURBS basis functions that the authors use to discretise the scatter boundary, Γ . Another current development in IGA is T-splines, a superset of NURBS. Regardless of using NURBS or T-splines, both can be decomposed into their constituent Bézier patches. It is this decomposed mesh that the current authors use in the simulations of this work.

The boundary, Γ , is discretised into E + 1 boundary elements which provide the analytical geometry of the scatterer. On each element, Γ_e , the variation in ϕ is expressed in terms of the rational Bézier basis functions, R_{ii}^e ,

$$\phi^{e}(\mathbf{q}(\xi_{1},\xi_{2})) = \sum_{i=0}^{p} \sum_{j=0}^{q} R^{e}_{ij}(\xi_{1},\xi_{2})\phi^{e}_{ij},$$
(4)

where ϕ_{ij}^e is the potential associated with each NURBS basis function. A partition-of-unity enrichment is introduced, multiplying each basis function by a linear expansion of plane waves; (4) is rewritten

$$\phi^{e}(\mathbf{q}(\xi_{1},\xi_{2})) = \sum_{i=0}^{p} \sum_{j=0}^{q} R^{e}_{ij}(\xi_{1},\xi_{2}) \sum_{m=0}^{M} A^{e}_{ijm} \exp\left(\iota k \mathbf{d}^{e}_{ijm} \cdot \mathbf{q}\right), \quad \left|\mathbf{d}^{e}_{ijm}\right| = 1,$$
(5)

where there are M + 1 plane waves expanded on each basis function; each wave has a prescribed direction of propagation, $\mathbf{d}_{iim}^e \in \mathbb{R}^3$, and unknown amplitude, $A_{iim}^e \in \mathbb{C}$.

Substituting (5) into (3) and, for compact presentation, applying the *sound hard* boundary condition, $\partial \phi(\mathbf{q}) / \partial n(\mathbf{q}) = 0$, gives

$$\frac{1}{2}\phi(\mathbf{p}) + \sum_{e=0}^{E} \sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{m=0}^{M} \int_{-1}^{1} \frac{\partial G(\mathbf{p}, \mathbf{q}(\xi_{1}, \xi_{2}))}{\partial n(\mathbf{q}(\xi_{1}, \xi_{2}))} R_{ij}^{e}(\xi_{1}, \xi_{2}) \exp\left(\iota k \, \mathbf{d}_{ijm}^{e} \cdot \mathbf{q}(\xi_{1}, \xi_{2})\right) |J| \, \mathrm{d}\xi \, A_{ijm}^{e} = \phi^{\mathrm{inc}}(\mathbf{p}) \quad (6)$$

where |J| is the Jacobian of transformation of the mapping from global coordinates to local (ξ_1, ξ_2) coordinates.

(6) is collocated at a sufficient number of points to yield a linear system of equations can can be solved to find the unknown values of A_{iim}^e .

3. Equal spacing of d_{ijm}^e about the unit sphere

Works in plane-wave enriched FEM and BEM approaches most commonly involve a uniformly spaced set of directions \mathbf{d}_{ijm}^{e} . In 2D, it is a simple procedure to take equally spaced points about the unit circle. In 3D, the same process for the unit sphere is more complex.

The authors have developed an algorithm based upon the physical analogy of an arbitrary number of charged particles held in static equilibrium on a spherical surface [4]. M + 1 particles of unit mass and electrical charge lie on the surface S of a unit radius sphere at locations described by vectors \mathbf{u}_i . At time t, the Coulomb force vector acting on each particle is given by



Figure 1: Converged solution of Coulomb force algorithm [4]; M = 151.



Figure 2: Scattering by a sphere at k = 20. Real-part of acoustic potential are shown.

$$\mathbf{F}_{i}^{t} = A \sum_{j=0}^{M} \frac{(1 - \delta_{ij}) \times \mathbf{r}}{|\mathbf{r}|^{3}},\tag{7}$$

where where A is a scalar multiplier, δ_{ij} is the Kronecker delta, and $\mathbf{r} = \mathbf{u}_i - \mathbf{u}_j$. \mathbf{F}_i^t will be oriented away from S, so the vector \mathbf{f}_i^t is defined as the projection of \mathbf{F}_i^t on S, given by

$$\mathbf{f} = (\mathbf{F}_i^t \times \mathbf{u}_i^t) \times \mathbf{u}_i^t.$$
(8)

The acceleration, $\ddot{\mathbf{u}}_i$, of each particle is

f

$$\ddot{\mathbf{u}}_i^t = \mathbf{f}_i^t - c \dot{\mathbf{u}}_i^t, \tag{9}$$

where *c* is a viscous damping coefficient and $\dot{\mathbf{u}}_i$ is the velocity of the particle. The velocity and position at the subsequent time, $t + \Delta t$, are given by

$$\dot{\mathbf{u}}_{i}^{t+\Delta t} = \dot{\mathbf{u}}_{i}^{t} + \ddot{\mathbf{u}}_{i}^{t}\Delta t, \qquad (10)$$

$$\mathbf{u}_{i}^{t+\Delta t} = \frac{\mathbf{u}_{i}^{t} + \dot{\mathbf{u}}_{i}^{t}\Delta t}{\left|\mathbf{u}_{i}^{t} + \dot{\mathbf{u}}_{i}^{t}\Delta t\right|},\tag{11}$$

where (11) normalises the position vectors to relocate the particles back onto *S*. Equations (7) to (11) are repeated in a time-stepping scheme to reach a converged solution, such as the one seen in Figure 1. Suitable values of *A*, *c* and Δt can be found in [4].

4. Numerical results

Here, numerical results from simulations of a plane wave scattering on the surface of a sound hard sphere (radius a = 1) are presented. The incident wave has unit amplitude and propagates in the direction $\mathbf{d}^{\text{inc}} = [1, 0, 0]$. An analytical solution exists such that the scattered acoustic potential ϕ^{scat} can be found at any point $x(r, \theta)$ by

$$\phi^{\text{scat}}(r,\theta) = \sum_{n=0}^{\infty} -\frac{i^n (2n+1) j'_n(ka)}{h'_n(ka)} P_n(\cos\theta) h_n(kr) \quad (12)$$

where j_n is the spherical Bessel function of the first kind, h_n is the spherical Hankel function of the first kind, and P_n is the Legendre function of the first kind. A visual representation of the real part of the potential over the surface of the sphere can be seen in Figure 2.

Figure 3 shows a comparison of the number of degrees of freedom required to obtain an L^2 error of engineering accuracy (1%) using a conventional BEM scheme and the proposed XIBEM scheme. It shows that far fewer degrees of freedom are required with XIBEM simulations. With a conventional BEM scheme, approximately 10 degrees of freedom per wavelength in each coordinate direction are required on each element. With the XIBEM, only 3 degrees of freedom per wavelength are required.



Figure 3: Comparison of N_{dof} required to obtain 1% error with conventional BEM and XIBEM.



Figure 4: L² errors of XIBEM of medium wavelength simulations of unit sphere problem.

Figure 4 shows the errors of XIBEM simulations using 3 degrees of freedom per wavelength in each coordinate direction for shorter wavelengths than Figure 3. All simulations exhibit reasonable levels of accuracy, with all but two achieving the 1% target accuracy. A small increase in the number of degrees of freedom used would guarantee that all simulations were below this threshold. The last simulation on the figure, for ka = 60, uses 10,322 degrees of freedom; a similar conventional BEM simulation would require $N_{dof} = 114,608$.

5. Conclusions

The PU-BEM reduces the number of degrees of freedom required to solve a Helmholtz problem such as the example in this paper. However, until now, only a few geometries have been considered due to the requirement of an analytically described geometry.

Using the functions used to describe geometry in CAD directly in a BEM simulation and enriching this basis in a partition-of-unity fashion—a combination termed XIBEM—has dramatically reduced the number of degrees of freedom required, per wavelength per coordinate direction, to solve a problem to engineering accuracy.

Using only 3 degrees of freedom per wavelength means that XIBEM needs approximately 9% of the number of equations that conventional BEM schemes need and the resulting matrix is < 1% of the size of the conventional BEM system matrix. This means, for a fixed computational resource, problems of much shorter wavelengths can now be solved.

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