Finding Secluded Places of Special Interest in Graphs[*][†]

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Abstract

Finding a vertex subset in a graph that satisfies a certain property is one of the most-studied topics in algorithmic graph theory. The focus herein is often on minimizing or maximizing the size of the solution, that is, the size of the desired vertex set. In several applications, however, we also want to limit the "exposure" of the solution to the rest of the graph. This is the case, for example, when the solution represents persons that ought to deal with sensitive information or a segregated community. In this work, we thus explore the (parameterized) complexity of finding such *secluded* vertex subsets for a wide variety of properties that they shall fulfill. More precisely, we study the constraint that the (open or closed) neighborhood of the solution shall be bounded by a parameter and the influence of this constraint on the complexity of minimizing separators, feedback vertex sets, $\mathcal{F}$-free vertex deletion sets, dominating sets, and the maximization of independent sets.

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1 Introduction

In many optimization problems on graphs, one searches for a minimum or maximum cardinality subset of vertices and edges satisfying certain properties, like a minimum *s-t* path, a maximum independent set, or a minimum dominating set. In several applications, however, it is important to also limit the *exposure* of the solution [5, 17]. For instance, we may want to find a way to send sensitive information that we want to protect from a vertex *s* to a vertex *t* in a network. If we assume that the information is exposed to all vertices on the way and all of their neighbors, limiting the exposure means to find an *s-t* path with a small closed neighborhood [5]. Another example is the search for segragated communities in social networks [17]. Herein, we search for dense subgraphs which are exposed to few neighbors in the rest of the graph. In addition to being a natural constraint in these applications, restricting the exposure of the solution may also yield more efficient algorithms [14, 16, 17, 18].

In accordance with previous work, we call a solution *secluded* if it has a small exposure [5]. Secluded paths and Steiner trees have been studied before [5, 11]. Our aim in this paper is to study the constraint of being secluded on the complexity of diverse vertex-subset optimization problems.

Inspired by Chechik et al. [5], we first measure the exposure of a solution S by the size of the closed neighborhood $N_G[S] = S \cup \bigcup_{v \in S} N_G(v)$ of S in the input graph G. Given a predicate $\Pi(G, S)$ that determines whether S is a solution for input graph G, we hence study the following problem.

SECLUDED Π
Input: A graph $G = (V, E)$ and an integer k.
Question: Is there a subset $S \subseteq V$ of vertices such that S satisfies $\Pi(G, S)$ and $|N_G[S]| \leq k$?

It makes sense to also control the size of the solution and its neighborhood in the graph directly. For example, when sending sensitive information from s to t as above, we may simultaneously aim to optimize latency, that is, minimize the number of vertices in the communication path and limit the exposure. Hence, our second measure of exposure of the solution is the size of the open neighborhood $N_G(S) = N_G[S] \setminus S$. This leads to the following problem formulation.

SMALL (LARGE) SECLUDED Π
Input: A graph $G = (V, E)$ and two integers k, ℓ.
Question: Is there a subset $S \subseteq V$ of vertices of G such that S satisfies $\Pi(G, S)$, $|S| \leq k$,
    and $|N_G(S)| \leq \ell$ (resp. $|S| \geq k$, and $|N_G(S)| \leq \ell$)?

We study both problems in the framework of parameterized complexity. As a parameter for SECLUDED Π we use the size k of the closed neighborhood and as parameters for SMALL SECLUDED Π we use the size k of the solution as well as the size ℓ of the open neighborhood.

The predicates $\Pi(G, S)$ that we study are *s-t* SEPARATOR, FEEDBACK VERTEX SET (FVS), $\mathcal{F}$-FREE VERTEX DELETION ($\mathcal{F}$-FVD) (for an arbitrary finite family $\mathcal{F}$ of graphs), encompassing CLUSTER VERTEX DELETION, for example, and INDEPENDENT SET (IS). Perhaps surprisingly, we find that SECLUDED *s-t*-SEPARATOR is polynomial-time solvable, whereas SMALL SECLUDED *s-t* SEPARATOR becomes NP-complete. The remaining problems are NP-complete. For them, roughly speaking, we prove that fixed-parameter tractability results for Π parameterized by the solution size carry over to SECLUDED Π parameterized by k. For SMALL SECLUDED Π parameterized by ℓ, however, we mostly obtain W[1]-hardness. On the positive side, for SMALL SECLUDED $\mathcal{F}$-FVD we prove fixed-parameter tractability when parameterized by $k + \ell$.

■ **Table 1** Overview of our results. PK stands for polynomial kernel. The results marked by an asterisk follow by a straightforward reduction from the non-secluded variant.

| | Complexity | | Parameterized Complexity / Kernelization | | | |
| | Secluded | Small Secl. | Secluded | Small Secluded | | |
Problem			k	k	ℓ	$k + \ell$
s-t-separator	P	NP-c.	P	W[1]-h.	W[1]-h.	?/noPK
$\mathcal{F}$-free VD	NP-c.	NP-c.[*]	FPT/PK	?	?	FPT/?
FVS	NP-c.	NP-c.[*]	FPT/PK	?	W[1]-h.	? / ?
q-DS, $2p \leq q$	NP-c.	NP-c.[*]	W[2]-h.	$\rightarrow$	$\rightarrow$	W[2]-h.
q-DS, $2p > q$	NP-c.	NP-c.[*]	FPT/noPK	?	?	FPT/noPK
Large Secluded IS	NP-c.[*]			$\rightarrow$	$\rightarrow$	W[1]-h.

We also study, for two integers $p < q$, the p-secluded version of q-DOMINATING SET (q-DS): a vertex set S is a q-*dominating set* if every vertex of $V \setminus S$ has distance at most q to some vertex in S. Herein, by p-secluded we mean that we upper bound the size of the distance-p-neighborhood of the solution S. Interestingly, this problem admits a complexity dichotomy: Whenever $2p > q$, (SMALL) p-SECLUDED q-DOMINATING SET is fixed-parameter tractable with respect to k (with respect to $k + \ell$), but it is W[2]-hard otherwise.

We also study polynomial-size problem kernels for our secluded problems. Here we observe that the polynomial-size problem kernels for FEEDBACK VERTEX SET and $\mathcal{F}$-FREE VERTEX DELETION carry over to their SECLUDED variants, but otherwise we obtain mostly absence of polynomial-size problem kernels unless the polynomial hierarchy collapses.

A summary of our results is given in Table 1.

Related work. SECLUDED PATH and SECLUDED STEINER TREE were introduced and proved NP-complete by Chechik et al. [5]. They obtained approximation algorithms for both problems with approximation factors related to the maximum degree. They also showed that SECLUDED PATH is fixed-parameter tractable with respect to the maximum vertex degree of the input graph, whereas vertex weights lead to NP-hardness for maximum degree four.

Fomin et al. [11] studied the parameterized complexity of SECLUDED PATH and SECLUDED STEINER TREE, showing that both are fixed-parameter tractable even in the vertex-weighted setting. Furthermore, they showed that SECLUDED STEINER TREE is fixed-parameter tractable with respect to $r+p$, where $r = k-s$, k is the desired size of the closed neighborhood of the solution, s is the size of an optimum Steiner tree, and p is the number of terminals. On the other hand this problem is co-W[1]-hard when parameterized by r only.

The small secluded concept can be found in the context of cut problems in graph [21, 12]. Fomin et al. [12] introduced the CUTTING AT MOST k VERTICES problem, which asks, given a graph $G = (V, E)$ and two integers $k \geq 1$ and $\ell \geq 0$, whether there is a non-empty set $S \subseteq V$ such that $|S| \leq k$, and $|N_G(X)| \leq \ell$, thus resembling our small secluded concept. Both works [12, 21] study the parameterized complexity of related cut problems in graphs.

The concept of isolation can be found in the context of cuts [7] and was thoroughly explored for finding dense subgraphs [14, 16, 17, 18]. Herein, chiefly the constraint that the vertices in the solution shall have maximum/minimum/average outdegree bounded by a parameter was considered [14, 17, 18], leading to various parameterized tractability and hardness results. Also the overall number of edges outgoing the solution has been studied recently [16].

Preliminary observations.   Concerning classical computational complexity, the SMALL (LARGE) SECLUDED variant of a problem is at least as hard as the nonsecluded problem, by a simple reduction in which we set $\ell = n$, where n denotes the number of vertices in the graph. Since this reduction is a parameterized reduction with respect to k, parameterized hardness results for this parameter transfer, too. Furthermore, observe that hardness also transfers from SECLUDED Π to SMALL SECLUDED Π for all problems Π, since SECLUDED Π allows for a parameterized Turing reduction to SMALL SECLUDED Π: try out all k' and ℓ' with $k = k' + \ell'$. Additionally, many tractability results (in particular polynomial time solvability and fixed-parameter tractability) transfer from SMALL SECLUDED Π parameterized by $(k+\ell)$ to SECLUDED Π parameterized by k.

▶ **Observation 1.1.** SECLUDED Π *parameterized by k is parameterized Turing reducible to* SMALL SECLUDED Π *parameterized by $(k + \ell)$ for all predicates Π.*

Therefore, for the SMALL (LARGE) SECLUDED variants of the problems the interesting cases are those where the base problem is tractable (deciding whether input graph G contains a vertex set S of size k that satisfies $\Pi(G, S)$) or where ℓ is a parameter.

Notation.   We use standard notation from parameterized complexity and graph theory. All graphs in this paper are undirected. We denote $d_G(u, v)$ the *distance* between vertices u and v in G, that is, the number of edges of a shortest u-v path in G. For a set V' of vertices and a vertex $v \in V$ we let the distance of v from V' be $d_G(v, V') := \min\{d_G(u, v) \mid u \in V'\}$. We use $N_G^d[V'] = \{v \mid d_G(v, V') \leq d\}$ and $N_G^d(V') = N_G^d[V'] \setminus V'$ for any $d \geq 0$ (hence $N_G^0(V') = \emptyset$). We omit the index if the graph is clear from context and also use $N[V']$ for $N^1[V']$ and $N(V')$ for $N^1(V')$. If $V' = \{v\}$, then we write $N^d[v]$ in place of $N^d[\{v\}]$.

Organization.   We dedicate one section to each studied problem. We study s-t-SEPARATOR in Section 2, q-DOMINATING SET in Section 3, $\mathcal{F}$-FREE VERTEX DELETION in Section 4, FEEDBACK VERTEX SET in Section 5, and INDEPENDENT SET in Section 6. Section 7 summarizes results and gives directions for future research. We remark that some proofs and proof details (marked with ($\star$)) are deferred to a full version of this paper.

2 s-t-**Separator**

In this section, we show that SECLUDED s-t-SEPARATOR is in P, while SMALL SECLUDED s-t-SEPARATOR is NP-hard and $W[1]$-hard with respect to the size of the open neighborhood, or the size of the solution. Moreover, we also exclude the existence of polynomial-size kernels for the latter problem with respect to the sum of the bounds.

2.1   Secluded s-t-Separator

In this subsection we show that the following problem can be solved in polynomial time.

SECLUDED s-t-SEPARATOR
Input: A graph $G = (V, E)$, two distinct vertices $s, t \in V$, and an integer k.
Question: Is there an s-t separator $S \subseteq V \setminus \{s, t\}$ such that $|N_G[S]| \leq k$?

▶ **Theorem 2.1.** SECLUDED s-t-SEPARATOR *can be solved in polynomial time.*

Proof. We reduce the problem to the problem of finding an ordinary s-t separator in an auxiliary graph. Let $(G = (V, E), s, t, k)$ be the input instance and G'' be a graph obtained

from G by adding two vertices s' and t' and making s' only adjacent to s and t' only adjacent to t'. Now let $G' = (V', E')$ be the third power of G'', that is, $V' = V(G') = V(G'') = V \cup \{s', t'\}$ and $\{u, v\} \in E'$ if and only if $d_{G''}(u, v) \leq 3$.

We claim that there is an s-t-separator S in G with $|N[S]| \leq k$ if and only if there is an s'-t'-separator S' in G' with $|S'| \leq k$. The theorem then follows as we can construct G' and find the minimum s'-t'-separator in G' in polynomial time using standard methods, for example, based on network flows.

"$\Rightarrow$": Let S be an s-t-separator in G with $|N[S]| \leq k$. Observe that S then also constitutes an s'-t'-separator in G'' as every path in G'' from s' must go through s and every path to t' must go through t. We claim that $S' = N[S]$ is an s'-t'-separator in G'. Suppose for contradiction that there is an s'-t' path $P = p_0, p_1, \ldots, p_q$ in $G' - S'$. Let A' be the set of vertices of the connected component of $G'' - S$ containing s' and let a be the last index such that $p_a \in A'$ (note that $p_0 = s' \in A'$ and $p_q = t' \notin A'$ by definition). It follows that $p_{a+1} \notin A'$ and, since $\{p_a, p_{a+1}\} \in E'$, there is a p_a-p_{a+1} path P' in G'' of length at most 3. As we have $p_a \in A'$ and $p_{a+1} \in V \setminus (A' \cup S')$ and $G[A']$ is a connected component of $G'' - S$, there must be a vertex x of S on P'. Since neither p_a nor p_{a+1} is in $S' = N[S]$, it follows that $d_G(p_a, x) \geq 2$ and $d_G(p_{a+1}, x) \geq 2$. This contradicts P' having length at most 3.

"$\Leftarrow$": Let S' be an s'-t'-separator in G' of size at most k. Let A' be the vertex set of the connected component of $G' - S'$ containing s'. Consider the set $S = \{v \in S' \mid d_{G''}(v, A') = 2\}$. We claim that S is an s-t-separator in G and, moreover, that $N[S] \subseteq S'$ and, hence, $|N[S]| \leq k$. As to the second part, we have $S \subseteq S'$ by definition. Suppose for contradiction that there was a vertex $u \in N(S) \setminus S'$ that is a neighbor of $v \in S$. Then, since $d_{G''}(v, A') = 2$, we have $d_{G''}(u, A') \leq 3$, u has a neighbor in A' in G', and, thus u is in A'. This implies that $d_{G''}(v, A') = 1$, a contradiction. Hence, $|N[S]| \leq k$.

It remains to show that S is an s-t-separator in G. For this, we prove that S is an s'-t'-separator in G''. Since it contains neither s nor t, it follows that it must be also an s-t-separator in G. Assume for contradiction that there is an s'-t' path in $G'' - S$. This implies that $d_{(G''-S)}(t', A')$ is well defined (and finite). Let $q := d_{(G''-S)}(t', A')$ and P be the corresponding shortest path in $G'' - S$. Let us denote $P = p_0, \ldots, p_q$ with $p_q = t'$ and $p_0 \in A'$. If $d_{G''}(t', A') \leq 3$, then t' has a neighbor in A' in G', and therefore it is in A' contradicting our assumption that S' is an s'-t'-separator in G'. As $t' = p_q$, we have $q > 3$. Since $d_{G''}(p_0, A') = 0$, $d_{G''}(p_q, A') > 3$, and $d_{G''}(p_{i+1}, A') \leq d_{G''}(p_i, A') + 1$ for every $i \in \{0, \ldots q - 1\}$, there is an a such that $d_{G''}(p_a, A') = 2$. If p_a is not in S', then p_a is in A', contradicting our assumptions on P and q as $a \geq 2$. Therefore we have $d_{G''}(p_a, A') = 2$ and p_a is in S'. It follows that p_a is in S, a contradiction.   ◀

2.2   Small Secluded s-t-Separator

In this subsection we prove hardness results for the following problem.

SMALL SECLUDED s-t-SEPARATOR
Input: A graph $G = (V, E)$, two distinct vertices $s, t \in V$, and two integers k, ℓ.
Question: Is there an s-t separator $S \subseteq V \setminus \{s, t\}$ such that $|S| \leq k$ and $|N_G(S)| \leq \ell$?

We show that, in contrast to SECLUDED s-t-SEPARATOR, the above problem is NP-hard. Moreover, at the same time, we show parameterized hardness with respect to k and with respect to ℓ.

▶ **Theorem 2.2.** SMALL SECLUDED s-t-SEPARATOR *is NP-hard and* $W[1]$-*hard when parameterized by* k *or by* ℓ.

In the proof of the theorem, we reduce from the CUTTING AT MOST k VERTICES WITH TERMINAL [12] problem, which asks, given a graph $G = (V, E)$, a vertex $s \in V$, and two integers $k \geq 1$, $\ell \geq 0$, whether there is a set $S \subseteq V$ such that $s \in S$, $|S| \leq k$, and $|N_G(X)| \leq \ell$. Fomin et al. [12] proved that CUTTING AT MOST k VERTICES WITH TERMINAL is NP-hard and W[1]-hard when parameterized by k or by ℓ.

Proof. We give a polynomial-parameter transformation from CUTTING AT MOST k VERTICES WITH TERMINAL to SMALL SECLUDED s-t-SEPARATOR.

Construction. Let $\mathcal{I} := (G = (V, E), s, k, \ell)$ be an instance of CUTTING AT MOST k VERTICES WITH TERMINAL. We construct an instance $\mathcal{I}' := (G', s', t', k', \ell')$ of SMALL SECLUDED s-t-SEPARATOR equivalent to $\mathcal{I}$ as follows. To obtain G' from G we add to G two vertices s' and t' and two edges $\{s', s\}$ and $\{s, t'\}$. Note that $G = G' - \{s', t'\}$. We set $k' = k$ and $\ell' = \ell + 2$. Hence, we ask for an s'-t' separator $S \subseteq V(G')$ in G' of size at most k' and $|N_{G'}(S)| \leq \ell'$. Clearly, the construction can be carried out in polynomial time.

Correctness. We show that $\mathcal{I}$ is a yes-instance of CUTTING AT MOST k VERTICES WITH TERMINAL if and only if $\mathcal{I}'$ is a yes-instance of SMALL SECLUDED s-t-SEPARATOR.

"$\Rightarrow$": Let $\mathcal{I}$ be a yes-instance and let $S \subseteq V(G)$ be a solution to $\mathcal{I}$, that is, $s \in S$, $|S| \leq k$, and $|N_G(S)| \leq \ell$. We claim that S is also a solution to $\mathcal{I}'$. Since $s \in S$ and s' and t' are both only adjacent to s, S separates s' from t' in G'. Moreover, $|S| \leq k = k'$ and, as $N_{G'}(S) = N_G(S) \cup \{s', t'\}$, we have $|N_{G'}(S)| \leq \ell + 2 = \ell'$. Hence, S' is a solution to $\mathcal{I}'$, and $\mathcal{I}'$ is a yes-instance.

"$\Leftarrow$": Let $\mathcal{I}'$ be a yes-instance and let $S' \subseteq V(G')$ be an s'-t' separator in G' with $|S'| \leq k'$ and $|N_{G'}(S')| \leq \ell'$. We claim that S' is also a solution to $\mathcal{I}$. Note that $|S'| \leq k' = k$. Since S' is an s'-t' separator in G' and s' and t' are both adjacent to s, it follows that $s \in S'$ and $s', t' \in N_{G'}(S')$. Thus, we have $s \in S'$ and $|N_G(S')| = |N_{G' - \{s', t'\}}(S')| = |N_{G'}(S')| - 2 \leq \ell' - 2 = \ell$. Hence, S' is a solution to $\mathcal{I}$ and $\mathcal{I}$ is a yes-instance.

Note that, in the reduction, k' and ℓ' only depend on k and ℓ, respectively. Since CUTTING AT MOST k VERTICES WITH TERMINAL parameterized by k or by ℓ is W[1]-hard [12], it follows that SMALL SECLUDED s-t-SEPARATOR parameterized by k or by ℓ is W[1]-hard. ◄

It would be interesting to know whether SMALL SECLUDED s-t-SEPARATOR is FPT when parameterized by $k + \ell$. We conjecture that this is the case. However, under standard assumptions, the problem does not admit a polynomial-size kernel with respect to this parameter:

▶ **Theorem 2.3** (⋆)**.** *Unless $NP \subseteq coNP/poly$,* SMALL SECLUDED s-t-SEPARATOR *parameterized by $k + \ell$ does not admit a polynomial kernel.*

3 q-Dominating Set

In this section, for two constants $p, q \in \mathbb{N}$ with $0 \leq p < q$, we study the following problems:

p-SECLUDED q-DOMINATING SET
Input: A graph $G = (V, E)$ and an integer k.
Question: Is there a set $S \subseteq V$ such that $V = N_G^q[S]$ and $|N_G^p[S]| \leq k$?

SMALL p-SECLUDED q-DOMINATING SET
Input: A graph $G = (V, E)$ and two integers k, ℓ.
Question: Is there a set $S \subseteq V$ such that $V = N_G^q[S]$, $|S| \leq k$, and $|N_G^p(S)| \leq \ell$?

For $p = 0$, the size restrictions in both cases boil down to $|S| \leq k$. This is the well-known case of q-Dominating Set (also known as q-Center) which is NP-hard and W[2]-hard with respect to k (see Lokshtanov et al. [20], for example). Therefore, for the rest of the section we focus on the case $p > 0$. Additionally, by a simple reduction from q-Dominating Set, letting $\ell = |V(G)|$, we arrive at the following observation.

▶ **Observation 3.1.** *For any* $0 < p < q$, Small p-Secluded q-Dominating Set *is* W[2]*-hard with respect to* k.

We now go on to show NP-hardness and W[2]-hardness with respect to k for p-Secluded q-Dominating Set. We reduce from the following problem:

Set Cover
Input: A finite universe U, a family $F \subseteq 2^U$, and an integer k.
Question: Is there a subset $X \subseteq F$ such that $|X| \leq k$ and $\bigcup_{x \in X} x = U$?

We write $\bigcup X$ short for $\bigcup_{x \in X} x$. It is known that Set Cover is NP-complete, W[2]-hard with respect to k, and admits no polynomial kernel with respect to $|F|$, unless NP $\subseteq$ coNP/poly [6].

▶ **Theorem 3.2.** *For any* $0 < p < q$, p-Secluded q-Dominating Set *is NP-hard. Moreover, it does not admit a polynomial kernel with respect to* k, *unless NP* $\subseteq$ *coNP/poly.*

Proof. We give a polynomial-parameter transformation from Set Cover parameterized by $|F|$. Let (U, F, k) be an instance of Set Cover. Without loss of generality we assume that $0 \leq k < |F|$.

Construction. Let $k' = p + 1 + |F| \cdot p + k$. We construct the graph G of a p-Secluded q-Dominating Set instance (G, k') as follows. We start the construction by taking two vertices s and r and three vertex sets $V_U = \{u \mid u \in U\}$, $V_F = \{v_A \mid A \in F\}$, and $V_F' = \{v_A' \mid A \in F\}$. We connect vertex r with vertex s by a path of length exactly q. For each $A \in F$ we connect vertices v_A and r by an edge and vertices v_A and v_A' by a path $t_0^A, t_1^A, \ldots, t_p^A$ of length exactly p, where $t_0^A = v_A$ and $t_p^A = v_A'$. All introduced paths are internally disjoint and the internal vertices are all new. We connect a vertex $v_A' \in V_F'$ with a vertex $u \in V_U$ by an edge if and only if $u \in A$. Furthermore, we introduce a clique C_U of size k' and make all its vertices adjacent to each vertex in $V_F' \cup V_U$.

If $q - p \geq 2$, then for each $u \in U$, we create a path $b_0^u, b_1^u, \ldots, b_{q-p-2}^u$ of length exactly $q - p - 2$ such that $b_0^u = u$ and the other vertices are new. Furthermore, in this case, for each $h \in \{0, \ldots, q - p - 2\}$ we introduce a clique C_h^u of size k' and make all its vertices adjacent to vertex b_h^u. If $q - p = 1$ we do not introduce any new vertices.

One can show that the original instance of Set Cover is a yes-instance if and only if the constructed instance of p-Secluded q-Dominating Set is. We defer the proof of the equivalence to a full version of the paper.   ◀

In the following, we observe that the parameterized complexity of both problems varies for different choices of p and q.

▶ **Theorem 3.3** (⋆). *For any* $0 < p \leq \frac{1}{2}q$, p-Secluded q-Dominating Set *is* W[2]*-hard with respect to* k.

For Small p-Secluded q-Dominating Set, we remark that we can adapt the reduction for Theorem 3.2: instead of restricting the size of the closed neighborhood of the q-dominating set to at most $p + 1 + |F| \cdot p + k$, we restrict the size of the q-dominating set to at most $k + 1$ and the size of its open neighborhood to at most $p + |F| \cdot p$. Analogously, we can adapt the reduction for Theorem 3.3. This yields the following hardness results.

▶ **Corollary 3.4.** *For any $0 < p < q$, SMALL p-SECLUDED q-DOMINATING SET is NP-hard. Moreover, it does not admit a polynomial kernel with respect to $(k+\ell)$ unless NP $\subseteq$ coNP/poly. For any $0 < p \le \frac{1}{2}q$, SMALL p-SECLUDED q-DOMINATING SET is W[2]-hard with respect to $(k + \ell)$.*

Now we look at the remaining choices for p and q, that is all p, q with $p > \frac{1}{2}q$. In these cases we can show fixed-parameter tractability.

▶ **Theorem 3.5.** *For any $p > \frac{1}{2}q$, SMALL p-SECLUDED q-DOMINATING SET can be solved in $O(mk^{k+2}(k + \ell)^{qk})$ time and, hence, it is fixed-parameter tractable with respect to $k + \ell$.*

Proof. Consider a solution S for an instance (G, k, ℓ) of SMALL p-SECLUDED q-DOMINATING SET. If $x \in S$, then $|N^p[x]| \le k+\ell$, since $|S| \le k$ and $|N^p(S)| \le \ell$. Moreover, $|N^p[x]| \le k+\ell$ implies $|N[v]| \le k + \ell$ for every $v \in N^{p-1}[x]$. It follows that, if $|N^p[y]| \le k + \ell$ and $y \notin S$, then for each $x \in N^q[y] \cap S$ every vertex on every x-y path of length at most $2p - 1 \ge q$ has degree at most $k + \ell - 1$, since each such vertex has distance at most $p - 1$ to x or y.

If $k + \ell = 1$, then either G has at most one vertex or (G, k, ℓ) is a no-instance. Hence, in the following, we assume $k + \ell \ge 2$. We call vertices u and v *linked*, if there is a path of length at most q between u and v in G such that the degree of every vertex on the path is at most $k + \ell - 1$. Let $B[u] = \{v \mid u \text{ and } v \text{ are linked}\}$. One can show $|B[v]| \le (k + \ell)^q$ for any v (we defer the proof to a full version of the paper).

Let $Y = \{y \mid |N^p[y]| \le k + \ell\}$. Obviously, we have $S \subseteq Y$, since $|N^p[S]| \le k + \ell$. If $y \in Y \setminus S$, then there is $x \in S$ such that x and y are linked. It follows that $y \in B[x]$ and, thus, $Y \subseteq \bigcup_{x \in S} B[x]$. Hence, $|Y| \le k \cdot (k + \ell)^q \le (k + \ell)^{q+1}$.

This suggests the following algorithm for SMALL p-SECLUDED q-DOMINATING SET: Find the set Y. If $|Y| > k \cdot (k + \ell)^q$, then answer "no". Otherwise, for each $k' \le k$ and each size-k' subset S' of Y, check whether S' is a p-secluded q-dominating set in G. If any such set is found, return it. Otherwise, answer "no". Since $S \subseteq Y$, this check is exhaustive.

As to the running time, the set Y can be determined in $O(n(k + \ell))$ time by running a BFS from each vertex and stopping it after it discovers $k + \ell$ vertices or all vertices in distance at most p, whichever occurs earlier. Then, there are $k \cdot \binom{k \cdot (k+\ell)^q}{k} \le k^{k+1}(k + \ell)^{qk}$ candidate subsets of Y. For each such set S' we can check whether it is a p-secluded q-dominating set in G by running a BFS from each vertex of S' and marking the vertices which are in distance at most p and at most q, respectively. This takes $O(mk)$ time. Hence, in total, the algorithm runs in $O(mk^{k+2}(k + \ell)^{qk})$ time. ◀

By Observation 1.1, the previous result transfers to p-SECLUDED q-DOMINATING SET parameterized by k.

▶ **Corollary 3.6.** *For any $p > \frac{1}{2}q$, p-SECLUDED q-DOMINATING SET is fixed-parameter tractable with respect to k.*

4 $\mathcal{F}$-free Vertex Deletion

In this section, we study the $\mathcal{F}$-FREE VERTEX DELETION ($\mathcal{F}$-FVD) problem for families $\mathcal{F}$ of graphs with at most a constant number c of vertices, that is, the problem of destroying all induced subgraphs isomorphic to graphs in $\mathcal{F}$ by at most k vertex deletions. The problem can, in particular, model various graph clustering tasks [3, 15], where the secluded variants can be naturally interpreted as removing a small set of outliers that are weakly connected to the clusters.

4.1   Secluded $\mathcal{F}$-free Vertex Deletion

In this section, we prove a polynomial-size problem kernel for Secluded $\mathcal{F}$-free Vertex Deletion, where $\mathcal{F}$ is a family of graphs with at most a constant number c of vertices:

Secluded $\mathcal{F}$-free Vertex Deletion
Input: A graph $G = (V, E)$ and an integer k.
Question: Is there a set $S \subseteq V$ such that $G - S$ is $\mathcal{F}$-free and $|N_G[S]| \leq k$?

Henceforth, we call a set $S \subseteq V$ such that $G - S$ is $\mathcal{F}$-free an $\mathcal{F}$-*free vertex deletion set*.

Note that Secluded $\mathcal{F}$-free Vertex Deletion can be polynomial-time solvable for some families $\mathcal{F}$ for which $\mathcal{F}$-FVD is NP-hard: Vertex Cover (where $\mathcal{F}$ contains only the graph consisting of a single edge) is NP-hard, yet any vertex cover S satisfies $N[S] = V$. Therefore, an instance to Secluded Vertex Cover is a yes-instance if and only if $k \geq n$. In general, however, one can show that Secluded $\mathcal{F}$-Free Vertex Deletion is NP-complete for every family $\mathcal{F}$ that includes only graphs of minimum vertex degree two (Theorem 4.1). We mention in passing that, from this peculiar difference of the complexity of Vertex Cover and Secluded Vertex Cover, it would be interesting to find properties of $\mathcal{F}$ which govern whether Secluded $\mathcal{F}$-free Vertex Deletion is NP-hard or polynomial-time solvable along the lines of the well-known dichotomy results [9, 19].

▶ **Theorem 4.1 ($\star$).** *For each family $\mathcal{F}$ containing only graphs of minimum vertex degree two,* Secluded $\mathcal{F}$-Free Vertex Deletion *is NP-complete.*

It is easy to see that Secluded $\mathcal{F}$-free Vertex Deletion is fixed-parameter tractable. More specifically, it is solvable in $c^k \cdot \text{poly}(n)$ time: simply enumerate all inclusion-minimal $\mathcal{F}$-free vertex deletion sets S of size at most k using the standard search tree algorithm described by Cai [4] and check $|N[S]| \leq k$ for each of them. This works because, for any $\mathcal{F}$-free vertex deletion set S with $|N[S]| \leq k$, we can assume that S is an inclusion-minimal $\mathcal{F}$-free vertex deletion set since $|N[S']| \leq |N[S]|$ for every $S' \subsetneq S$.

We complement this observation of fixed-parameter tractability by the following kernelization result.

▶ **Theorem 4.2.** Secluded $\mathcal{F}$-free Vertex Deletion *has a problem kernel comprising $O(k^{c+1})$ vertices, where c is the maximum number of vertices in any graph of $\mathcal{F}$.*

Our proof of Theorem 4.2 exploits *expressive* kernelization algorithms for d-Hitting Set [1, 2, 8], which preserve inclusion-minimal solutions and that return subgraphs of the input hypergraph as kernels: Herein, given a hypergraph $H = (U, \mathcal{C})$ with $|C| \leq d$ for each $C \in \mathcal{C}$, and an integer k, d-Hitting Set asks whether there is a *hitting set* $S \subseteq U$ with $|S| \leq k$, that is, $C \cap S \neq \emptyset$ for each $C \in \mathcal{C}$. Our kernelization for Secluded $\mathcal{F}$-free Vertex Deletion is based on transforming the input instance (G, k) to a d-Hitting Set instance (H, k), computing an expressive d-Hitting Set problem kernel (H', k), and outputting a Secluded $\mathcal{F}$-free Vertex Deletion instance (G', k), where G' is the graph induced by the vertices remaining in H' together with at most $k + 1$ additional neighbors for each vertex in G.

▶ **Definition 4.3.** Let $(G = (V, E), k)$ be an instance of Secluded $\mathcal{F}$-free Vertex Deletion. For a vertex $v \in V$, let $N_j(v) \subseteq N_G(v)$ be j arbitrary neighbors of v, or $N_j(v) := N_G(v)$ if v has degree less than j. For a subset $S \subseteq V$, let $N_j(S) := \bigcup_{v \in S} N_j(v)$. Moreover, let

- $c := \max_{F \in \mathcal{F}} |V(F)|$ be the maximum number of vertices in any graph in $\mathcal{F}$,
- $H = (U, \mathcal{C})$ be the hypergraph with $U := V$ and $\mathcal{C} := \{S \subseteq V \mid G[S] \in \mathcal{F}\}$,

- $H' = (U', C')$ be a subgraph of H with $|U'| \in O(k^c)$ such that each set $S \subseteq U$ with $|S| \leq k$ is an inclusion-minimal hitting set for H if and only if it is for H', and
- $G' = (V', E')$ be the subgraph of G induced by $U' \cup N_{k+1}(U')$.

To prove Theorem 4.2, we show that (G', k) is a problem kernel for the input instance (G, k). The subgraph H' exists and is computable in linear time from H [2, 8]. Moreover, for constant c, one can compute H from G and G' from H' in polynomial time. It is obvious that the number of vertices of G' is $O(k^{c+1})$. Hence, it remains to show that (G', k) is a yes-instance if and only if (G, k) is. This is achieved by the following two lemmas.

▶ **Lemma 4.4.** *For any $S \subseteq U'$ with $|N_{G'}[S]| \leq k$, it holds that $N_G[S] = N_{G'}[S]$.*

Proof. Since $S \subseteq U' \subseteq V' \cap V$ and since G' is a subgraph of G, it is clear that $N_G[S] \supseteq N_{G'}[S]$. For the opposite direction, observe that each $v \in S$ has degree at most k in G'. Thus, v has degree at most k in G since, otherwise, $k + 1$ of its neighbors would be in G' by construction. Thus, $N_{G'}(v) \supseteq N_{k+1}(v) = N_G(v)$ for all $v \in S$ and, thus, $N_{G'}[S] \supseteq N_G[S]$. ◄

▶ **Lemma 4.5** (⋆). *Graph G allows for an $\mathcal{F}$-free vertex deletion set S with $|N_G[S]| \leq k$ if and only if G' allows for an $\mathcal{F}$-free vertex deletion set S with $|N_{G'}[S]| \leq k$.*

4.2 Small Secluded $\mathcal{F}$-free Vertex Deletion

In this subsection, we present a fixed-parameter algorithm for the following problem parameterized by $\ell + k$.

SMALL SECLUDED $\mathcal{F}$-FREE VERTEX DELETION
Input: A graph $G = (V, E)$ and two integers k, ℓ.
Question: Is there a subset $S \subseteq V$ such that $G - S$ is $\mathcal{F}$-free, $|S| \leq k$, and $|N_G(S)| \leq \ell$?

As before, we call a set $S \subseteq V$ such that $G - S$ is $\mathcal{F}$-free an $\mathcal{F}$-*free vertex deletion set*.

In the previous section, we discussed a simple search tree algorithm for SECLUDED $\mathcal{F}$-FREE VERTEX DELETION that was based on the fact that we could assume that our solution is an inclusion-minimal $\mathcal{F}$-free vertex deletion set. However, an $\mathcal{F}$-free vertex deletion set S with $|S| \leq k$ and $|N_G(S)| \leq \ell$ is not necessarily inclusion-minimal: some vertices may have been added to S just in order to shrink its open neighborhood. However, the following simple lemma limits the number of possible candidate vertices that can be used to enlarge S in order to shrink $N(S)$, which we will use in a branching algorithm.

▶ **Lemma 4.6.** *Let S be an $\mathcal{F}$-free vertex deletion set and $S' \supseteq S$ such that $|S'| \leq k$ and $|N_G(S')| \leq \ell$, then $|N_G(S)| \leq \ell + k$.*

Proof. $|N_G(S)| = |N_G[S] \setminus S| \leq |N_G[S'] \setminus S| \leq |N_G[S']| \leq |N_G(S') \cup S'| \leq \ell + k$. ◄

▶ **Theorem 4.7.** SMALL SECLUDED $\mathcal{F}$-FREE VERTEX DELETION *can be solved in* $\max\{c, k + \ell\}^k \cdot \text{poly}(n)$*-time, where c is the maximum number of vertices in any graph of $\mathcal{F}$.*

Proof. First, enumerate all inclusion-minimal $\mathcal{F}$-free vertex deletion sets S with $|S| \leq k$. This is possible in $c^k \cdot \text{poly}(n)$ time using the generic search tree algorithm described by Cai [4]. For each $k' \leq k$, this search tree algorithm generates at most $c^{k'}$ sets of size k'. For each enumerated set S of k' elements, do the following:
1. If $|N_G(S)| \leq \ell$, then output S as our solution.
2. If $|N_G(S)| > \ell + k$, then S cannot be part of a solution S' with $N_G(S') \leq \ell$ by Lemma 4.6, we proceed with the next set.

3. Otherwise, initiate a recursive branching: recursively branch into at most $\ell+k$ possibilities of adding a vertex from $N_G(S)$ to S as long as $|S| \leq k$.

The recursive branching initiated at step 3 stops at depth $k - k'$ since, after adding $k - k'$ vertices to S, one obtains a set of size k. Hence, the total running time of our algorithm is

$$\text{poly}(n) \cdot \sum_{k'=1}^{k} c^{k'} (\ell+k)^{k-k'} = \text{poly}(n) \cdot \sum_{k'=1}^{k} \max\{c, \ell+k\}^k = \text{poly}(n) \cdot \max\{c, \ell+k\}^k. \qquad \blacktriangleleft$$

Given Theorem 4.7, a natural question is whether the problem allows for a polynomial kernel.

5   Feedback Vertex Set

In this section, we study secluded versions of the FEEDBACK VERTEX SET (FVS) problem, which asks, given a graph G and an integer k, whether there is a set $W \subseteq V(G)$, $|W| \leq k$, such that $G - W$ is cycle-free.

5.1   Secluded Feedback Vertex Set

We show in this subsection that the problem below is NP-hard and admits a polynomial kernel.

SECLUDED FEEDBACK VERTEX SET (SFVS)
Input: A graph $G = (V, E)$ and an integer k.
Question: Is there a set $S \subseteq V$ such that $G - S$ is cycle-free and $|N_G[S]| \leq k$?

▶ **Theorem 5.1** (⋆)**.** SECLUDED FEEDBACK VERTEX SET *is NP-hard.*

The proof is by a reduction from the FVS problem and works by attaching to each vertex in the original graph a large set of new degree-one neighbors. On the positive side, SFVS remains fixed-parameter tractable with respect to k:

▶ **Theorem 5.2.** SECLUDED FEEDBACK VERTEX SET *admits a kernel with $O(k^5)$ vertices.*

In the remainder of this section, we describe the data reduction rules that yield the polynomial-size problem kernel. The running time and correctness proofs, as well as the kernel-size bound of Theorem 5.2 is deferred to a full version of this paper. The reduction rules are inspired by the kernelization algorithm for the TREE DELETION SET problem given by Giannopoulou et al. [13].

We start by introducing the following notation. A *2-core* [22] of a graph G is a maximum subgraph H of G such that, for each $v \in V(H)$, we have $\deg_H(v) \geq 2$. Note that a 2-core H of a given graph G is unique and can be found in polynomial time [22]. If H is a 2-core of G, then we use $\deg_{H|0}(v)$ to denote $\deg_H(v)$ if $v \in V(H)$ and $\deg_{H|0}(v) = 0$ if $v \notin V(H)$.

▶ **Observation 5.3.** *Let G be a graph, H its 2-core, and C a connected component of $G - V(H)$. Then $|N(C) \cap V(H)| \leq 1$ and $|N(H) \cap V(C)| \leq 1$.*

Proof. We only show the first statement. The second statement follows analogously. Towards a contradiction, assume that $|N(C) \cap V(H)| \geq 2$. Then, there are vertices $x, y \in V(H)$ with $x \neq y$ such that x and y have neighbors $a, b \in V(C)$. If $a = b$, then $G' = G[V(H) \cup \{a\}]$ is a subgraph of G such that $\deg_{G'}(v) \geq 2$ for every $v \in V(G')$, contradicting the choice of H as the 2-core of G. If $a \neq b$, then, since C is connected, there is a path P_C in C connecting a and b. Thus, $G' = G[V(H) \cup V(P_C)]$ is a subgraph of G such that $\deg_{G'}(v) \geq 2$ for every $v \in V(G')$, again contradicting the choice of H as the 2-core of G. ◀

Note that only the vertices in the 2-core are involved in cycles of G. However, the vertices outside the 2-core can influence the size of the closed neighborhood of the feedback vertex set. Next, we apply the following reduction rules to our input instance with G given its 2-core H.

▶ **Reduction Rule 1.** *If* $\deg_{H|0}(v) = 0$ *for every* $v \in N[u]$, *then delete* u.

Note that, if Reduction Rule 1 has been exhaustively applied, then $\deg_{H|0}(v) = 0$ implies that v has exactly one neighbor, which is in the 2-core of the graph.

▶ **Reduction Rule 2.** *If* $v_0, v_1, \ldots, v_\ell, v_{\ell+1}$ *is a path in the input graph such that* $\ell \geq 3$, $\deg_{H|0}(v_i) = 2$ *for every* $i \in \{1, \ldots, \ell\}$, $\deg_{H|0}(v_0) \geq 2$, *and* $\deg_{H|0}(v_{\ell+1}) \geq 2$, *then let* $r = \min\{\deg_G(v_i) \mid i \in \{1, \ldots, \ell\}\} - 2$ *and remove vertices* $v_1, \ldots, v_\ell$ *and their neighbors not in the 2-core. Then introduce two new vertices* u_1 *and* u_2 *with edges* $\{v_0, u_1\}$, $\{u_1, u_2\}$, *and* $\{u_2, v_{l+1}\}$ *and* $2r$ *further new vertices and connect* u_1 *with* r *of them and* u_2 *with another* r *of them.*

For $x \in V(G)$, we denote by petal(x) the maximum number of cycles only intersecting in x.

▶ **Reduction Rule 3.** *If there is a vertex* $x \in V(G)$ *such that* petal(x) $\geq \lceil \frac{k}{2} \rceil$, *then output that* (G, k) *is a no-instance of SFVS.*

▶ **Reduction Rule 4.** *If* $v \in V(G)$ *is a vertex such that* $\deg_G(v) > k$, *but* $\deg_{H|0}(v) < \deg_G(v)$, *then remove one of its neighbors not in the 2-core.*

▶ **Reduction Rule 5.** *Let* x, y *be two vertices of* G. *If there are at least* k *internally vertex disjoint paths of length at least 2 between* x *and* y *in* G, *then output that* (G, k) *is a no-instance of SFVS.*

Note that Reduction Rules 1, 2, 4, and 5 can be applied trivially in polynomial time. Reduction Rule 3 can be applied exhaustively in polynomial time due to Thomassé [23].

After applying the reduction rules above exhaustively, one can show that the resulting instance is either a no-instance, or the number of vertices is $O(k^5)$.

5.2   Small Secluded Feedback Vertex Set

In contrast to restricting the closed neighborhood of a feedback vertex set, restricting the open neighborhood by a parameter yields a W[1]-hard problem.

SMALL SECLUDED FEEDBACK VERTEX SET
Input: A graph $G = (V, E)$ and two integers k, ℓ.
Question: Is there a set $S \subseteq V$ such that $G - S$ is cycle-free, $|S| \leq k$, and $|N_G(S)| \leq \ell$?

▶ **Theorem 5.4.** SMALL SECLUDED FEEDBACK VERTEX SET *is W[1]-hard with respect to* ℓ.

Proof. We provide a parameterized reduction from MULTICOLORED INDEPENDENT SET (MIS): given a k-partite graph $G = (V, E)$ and its partite sets $V_1 \cup \ldots \cup V_k = V$, the question is whether there is an independent set I of size k such that $I \cap V_i \neq \emptyset$ for each $i \in \{1, \ldots, k\}$. MIS is W[1]-hard when parameterized by the size k of the independent set [10].

Let $G = (V, E)$ with partite sets $V_1 \cup V_2 \cup \ldots \cup V_k = V$ be an instance of MIS. We can assume that for each $i \in \{1, \ldots, k\}$ we have $|V_i| \geq 2$ and there is no edge $\{v, w\} \in E$ with $v, w \in V_i$. We create an instance (G', k', ℓ) of SMALL SECLUDED FEEDBACK VERTEX SET (SSFVS) with $k' = |V| - k$ and $\ell = k + 1$ as follows.

Construction: Refer to Figure 1 for an sketch of the following construction. Initially, let $G' := G$. For each $i \in \{1, \ldots, k\}$ turn V_i into a clique, that is, add the edge sets
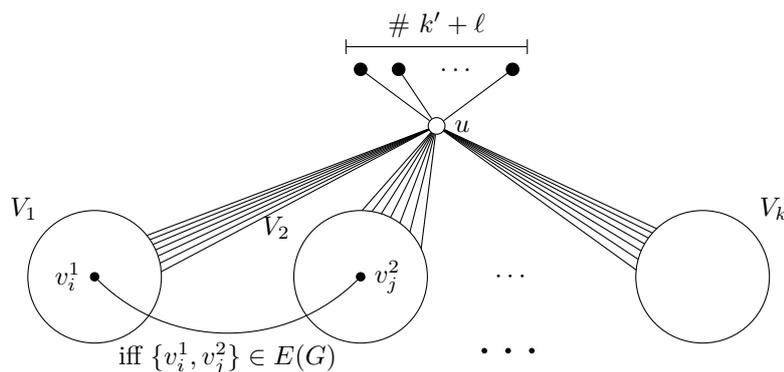
Figure 1 Sketch of the construction in proof of Theorem 5.4. The circles refer to cliques with vertex set V_i, $i \in \{1, \ldots, k\}$.

$\{\{a, b\} \mid a, b \in V_i, a \neq b\}$. Next, add to G' a vertex u and a set L of $k' + \ell$ vertices. Finally, connect each vertex in $V \cup L$ to u by an edge.

Correctness: We show that (G, k) is a yes-instance of MIS if and only if (G', k', ℓ) is a yes-instance of SSFVS.

"$\Rightarrow$": Let (G, k) be a yes-instance of MIS and let $I \subseteq V$ with $|I| = k$ be a multicolored independent set in G. We delete all vertices in $S := V(G') \setminus (I \cup L \cup \{u\})$ from G'. Observe that $|S| = |V| - k = k'$. Moreover, $N_{G'}(S) = k + 1 = \ell$. Since there is no edge between any two vertices in I, $G - S$ forms a star with center u and $k' + \ell + 1 + k$ vertices. Since every star is acyclic, (G', k', ℓ) is a yes-instance of SSFVS.

"$\Leftarrow$": Let (G', k', ℓ) be a yes-instance of SSFVS and let $S \subseteq V(G')$ be a solution. Observe that $G'[V_i \cup \{u\}]$ forms a clique of size $|V_i| + 1$ for each $i \in \{1, \ldots, k\}$. Since the budget does not allow for deleting the vertex u (i.e. $u \notin S$), all but one vertex in each V_i must be deleted. Since $k' = |V| - k$ and $|V_i| \geq 2$ for all $i \in \{1, \ldots, k\}$, S contains exactly $|V_i| - 1$ vertices of V_i for each $i \in \{1, \ldots, k\}$. Hence, $|S| = |V| - k$ and $N_{G'}(S) = k + 1 = \ell$. Let $F := V \setminus S$ denote the set of vertices in V not contained in S. Recall that $|F| = k$ and $|F \cap V_i| = 1$ for all $i \in \{1, \ldots, k\}$. Next, suppose there is an edge between two vertices $v, w \in F$. Since $u \notin S$ and u is incident to all vertices in V, the vertices u, v, w form a triangle in G'. This contradicts the fact that S is a solution for (G', k', ℓ), that is, that $G' - S$ is acyclic. It follows that $E(G'[F]) = \emptyset$, that is, no two vertices in F are connected by an edge. Together with $|F| = k$ and $|F \cap V_i| = 1$ for all $i \in \{1, \ldots, k\}$, it follows that F forms a multicolored independent set in G. Thus, (G, k) is a yes-instance of MIS. ◀

6   Independent Set

For INDEPENDENT SET, it makes little sense to bound the size of the closed neighborhood from above, as in this case the empty set always constitutes a solution. One might ask for an independent set with closed neighborhood as large as possible. However, for any inclusion-wise maximal independent set S, one has $N[S] = V$. Hence, this question is also trivial. Therefore, in this section we only consider the following problem.

LARGE SECLUDED INDEPENDENT SET (LSIS)
Input: A graph $G = (V, E)$ and two integers k, ℓ.
Question: Is there an independent set $S \subseteq V$ such that $|S| \geq k$ and $|N_G(S)| \leq \ell$?
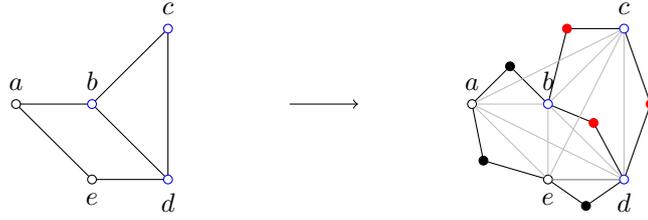
■ **Figure 2** Example of the construction in Theorem 6.1. The left-hand side shows the original graph, the right-hand side the graph constructed by the reduction, where the newly introduced edges between each pair of vertices of the original graph are drawn in grey. The vertices introduced for each edge of the original graph are filled red and black, the corresponding new edges are drawn in black. Note that the blue vertices of the original graph form a clique and that the vertices corresponding to the edges of said clique (filled red) form an independent set in the new graph.

The case $\ell = |V|$ equals INDEPENDENT SET and, thus, LSIS is $W[1]$-hard with respect to k. We show that LSIS is also W[1]-hard when parameterized by $k + \ell$.

▶ **Theorem 6.1.** LARGE SECLUDED INDEPENDENT SET *is $W[1]$-hard with respect to $k + \ell$.*

Proof. We provide a polynomial-parameter transformation from CLIQUE parameterized by the solution size k.

Construction. Let (G, k) be an instance of CLIQUE and assume without loss of generality that $k < |V(G)| - 1$ (otherwise, solve the instance in polynomial time). We construct an equivalent instance (G', k', ℓ') of LARGE SECLUDED INDEPENDENT SET as follows (see Figure 2 for an example).

Initially, let G' be an empty graph. Add all vertices of G to G'. Denote the vertex set by V. If two vertices of G are adjacent, we add a vertex to G', that is, G' additionally to V contains the vertex set $X := \{x_{uv} \mid \{u, v\} \in E\}$. Next, connect x_{uv} to u and v, that is, add the edge set $E' = \{\{u, x_{uv}\}, \{v, x_{uv}\} \mid \{u, v\} \in E\}$. Finally, connect any two vertices in V by an edge. Graph G' consists of the vertex set $V \cup X$ and of the edge set $E' \cup \binom{V}{2}$. Observe that X forms an independent set in G'. Set $k' := \binom{k}{2}$ and $\ell' := k$. We claim that (G, k) is a yes-instance of CLIQUE if and only if (G', k', ℓ') is a yes-instance of LARGE SECLUDED INDEPENDENT SET.

"⇒": Let $C \subseteq V(G)$ be a clique of size $k = |C|$ in G. We claim that $X' := \{x_{u,v} \mid u, v \in C\}$ forms an independent set of size $\binom{k}{2}$ with $|N(X')| = k = \ell'$ in G'. Since $X' \subseteq X$, X' forms an independent set. Moreover, since C is a clique of size k, there are $\binom{k}{2}$ edges in $G[C]$, and thus $|X'| = \binom{k}{2}$. By construction, each vertex in X is only adjacent to vertices in C. Hence, $|N(X')| = |C| = k$. Therefore, X' witnesses that (G', k', ℓ') is a yes-instance of LARGE SECLUDED INDEPENDENT SET.

"⇐": Let $U \subseteq V(G')$ form an independent set of size k' with open neighborhood of size upper-bounded by ℓ'. Suppose that $v \in V$ is contained in U (observe that U contains at most one vertex of V, as otherwise it would not be independent). Then $|N(U)| \geq |V| - 1 > k = \ell'$, which contradicts the choice of U. It follows that $U \cap V = \emptyset$, and hence $U \subseteq X$. By construction, for each $x_{uv} \in U$, the vertices u, v are contained in $N(U)$. Since each vertex in U corresponds to an edge in G, we have $\binom{k}{2}$ edges incident with at most k vertices. The only graph that fulfills this property is the complete graph on k vertices. Hence, G contains a clique of size k, and thus (G, k) is a yes-instance of CLIQUE(k). ◀

We remark that the proof above is similar to the W[1]-hardness proof for CUTTING ℓ VERTICES [21].

7   Summary and Future Work

In this paper, we studied the problem of finding sets of vertices in a graph that fulfill certain properties and have a small neighborhood. We presented computational complexity results for secluded and small secluded variants of s-t-Separator, q-Dominating Set, Feedback Vertex Set, $\mathcal{F}$-free Vertex Deletion, and for the large secluded variant of Independent Set. In the case of s-t-Separator, we leave as an open question the parameterized complexity of Small Secluded s-t-Separator with respect to $k + \ell$. We conjecture that it is fixed-parameter tractable. Concerning Secluded $\mathcal{F}$-free Vertex Deletion, we would like to point out that it is an interesting question which families $\mathcal{F}$ exactly yield NP-hardness as opposed to polynomial-time solvability.

A natural way to generalize our results would be to consider vertex-weighted graphs and directed graphs. This generalization was already investigated by Chechik et al. [5] for Secluded Path and Secluded Steiner Tree. Furthermore, we would like to mention that replacing the bound on the open neighborhood in the case of small secludedness by a bound on the outgoing edges of a solution would be an interesting modification of the problem. The variation follows the idea of the concept of *isolation* as used, e.g., in [14, 16, 17, 18]. As the number of outgoing edges is at least as large as the open neighborhood, this might offer new possibilities for fixed-parameter algorithms.

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