

# Colouring Diamond-free Graphs\*

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## Abstract

The COLOURING problem is that of deciding, given a graph  $G$  and an integer  $k$ , whether  $G$  admits a (proper)  $k$ -colouring. For all graphs  $H$  up to five vertices, we classify the computational complexity of COLOURING for (diamond,  $H$ )-free graphs. Our proof is based on combining known results together with proving that the clique-width is bounded for (diamond,  $P_1 + 2P_2$ )-free graphs. Our technique for handling this case is to reduce the graph under consideration to a  $k$ -partite graph that has a very specific decomposition. As a by-product of this general technique we are also able to prove boundedness of clique-width for four other new classes of  $(H_1, H_2)$ -free graphs. As such, our work also continues a recent systematic study into the (un)boundedness of clique-width of  $(H_1, H_2)$ -free graphs, and our five new classes of bounded clique-width reduce the number of open cases from 13 to 8.

**1998 ACM Subject Classification** G.2.2 Graph Theory

**Keywords and phrases** colouring, clique-width, diamond-free, graph class, hereditary graph class

**Digital Object Identifier** 10.4230/LIPIcs.SWAT.2016.

## 1 Introduction

The COLOURING problem is that of testing whether a given graph can be coloured with at most  $k$  colours for some given integer  $k$ , such that any two adjacent vertices receive different colours. The complexity of COLOURING is fully understood for general graphs: it is NP-complete even if  $k = 3$  [35]. Therefore it is natural to study its complexity when the input is restricted. A classic result in this area is due to Grötschel, Lovász, and Schrijver [26], who proved that COLOURING is polynomial-time solvable for perfect graphs.

As surveyed in [14, 20, 25, 42], COLOURING has been well studied for hereditary graph classes, that is, classes that can be defined by a family  $\mathcal{H}$  of forbidden induced subgraphs. For a family  $\mathcal{H}$  consisting of one single forbidden induced subgraph  $H$ , the complexity of COLOURING is completely classified: the problem is polynomial-time solvable if  $H$  is an induced subgraph of  $P_4$  or  $P_1 + P_3$  and NP-complete otherwise [34]. Hence, many papers (e.g. [13, 18, 29, 34, 36, 39, 40, 44]) have considered the complexity of COLOURING for bigenic hereditary graph classes, that is, graph classes defined by families  $\mathcal{H}$  consisting of

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\* First and last author supported by EPSRC (EP/K025090/1).



two forbidden graphs  $H_1$  and  $H_2$ ; such classes of graphs are also called  $(H_1, H_2)$ -free. This classification is far from complete (see [25] for the state of art). In fact there are still an infinite number of open cases, including cases where both  $H_1$  and  $H_2$  are small. For instance, Lozin and Malyshev [36] determined the computational complexity of COLOURING for  $(H_1, H_2)$ -free graphs for all graphs  $H_1$  and  $H_2$  up to four vertices except when  $\mathcal{H} \in \{(K_{1,3}, 4P_1), (K_{1,3}, 2P_1 + P_2), (C_4, 4P_1)\}$  (we refer to Section 2 for notation and terminology).

The *diamond* is the graph  $\overline{2P_1 + P_2}$ , that is, the graph obtained from the clique on four vertices by removing an edge. Diamond-free graphs are well studied in the literature. For instance, Tucker [45] gave an  $O(kn^2)$  time algorithm for COLOURING for perfect diamond-free graphs. It is also known that COLOURING is polynomial-time solvable for diamond-free graphs that contain no even induced cycles [32] as well as for diamond-free graphs that contain no induced cycle of length at least 5 [8]. Diamond-free graphs also played an important role in proving that the class  $P_6$ -free graphs contains 24 minimal obstructions for 4-COLOURING [15].

## 1.1 Our Main Result

In this paper we focus on COLOURING for (diamond,  $H$ )-free graphs where  $H$  is a graph on at most five vertices. It is known that COLOURING is NP-complete for (diamond,  $H$ )-free graphs when  $H$  contains a cycle or a claw [34] and polynomial-time solvable for  $H = sP_1 + P_2$  ( $s \geq 0$ ) [18],  $H = 2P_1 + P_3$  [5],  $H = P_1 + P_4$  [11],  $H = P_2 + P_3$  [19] and  $H = P_5$  [1]. Hence, the only graph  $H$  on five vertices that remains is  $H = P_1 + 2P_2$ , for which we prove polynomial-time solvability in this paper. This leads to the following result.

► **Theorem 1.** *Let  $H$  be a graph on at most five vertices. Then COLOURING is polynomial-time solvable for (diamond,  $H$ )-free graphs if  $H$  is a linear forest and NP-complete otherwise.*

To solve the case  $H = P_1 + 2P_2$ , one could try to reduce to a subclass of diamond-free graphs, for which COLOURING is polynomial-time solvable, such as the aforementioned results of [8, 32, 45]. This would require us to deal with the presence of small cycles up to  $C_7$ , which may not be straightforward. Instead we aim to identify tractability from an underlying property: we show that the class of (diamond,  $P_1 + 2P_2$ )-free graphs has bounded clique-width. This approach has several advantages and will lead to a number of additional results, as we will discuss in the remainder of Section 1.

**Clique-width** is a graph decomposition that can be constructed via vertex labels and four specific graph operations, which ensure that vertices labelled alike will always keep the same label and thus behave identically. The clique-width of a graph  $G$  is the minimum number of different labels needed to construct  $G$  using these four operations (we refer to Section 2 for a precise definition). A graph class  $\mathcal{G}$  has *bounded* clique-width if there exists a constant  $c$  such that every graph from  $\mathcal{G}$  has clique-width at most  $c$ .

Clique-width is a well-studied graph parameter (see, for instance, the surveys [27, 31]). An important reason for the popularity of clique-width is that a number of classes of NP-complete problems, such as those that are definable in Monadic Second Order Logic using quantifiers on vertices but not on edges, become polynomial-time solvable on any graph class  $\mathcal{G}$  of bounded clique-width (this follows combining results from [16, 23, 33, 43] with a result from [41]). The COLOURING problem is one of the best-known NP-complete problems that is solvable in polynomial time on graph classes of bounded clique-width [33]; another well-known example of such a problem is HAMILTON PATH [23].

## 1.2 Methodology

The key technique for proving that (diamond,  $P_1 + 2P_2$ )-free graphs have bounded clique-width is the use of a certain graph decomposition of  $k$ -partite graphs. We obtain this decomposition by generalizing the so-called canonical decomposition of bipartite graphs, which decomposes a bipartite graph into two smaller bipartite graphs such that edges between these two smaller bipartite graphs behave in a very restricted way. Fouquet, Giakoumakis and Vanherpe [24] introduced this decomposition and characterized exactly those bipartite graphs that can recursively be canonically decomposed into graphs isomorphic to  $K_1$ . Such bipartite graphs are said to be totally decomposable by canonical decomposition. We say that  $k$ -partite graphs are totally  $k$ -decomposable if they can be, according to our generalized definition, recursively  $k$ -decomposed into graphs isomorphic to  $K_1$ . We show that totally  $k$ -decomposable graphs have clique-width at most  $2k$ .

Our goal is to transform (diamond,  $P_1 + 2P_2$ )-free graphs into graphs in some class for which we already know that the clique-width is bounded. Besides the class of totally  $k$ -decomposable graphs, we will also reduce to other known graph classes of bounded clique-width, such as the class of (diamond,  $P_2 + P_3$ )-free graphs [19] and certain classes of  $H$ -free bipartite graphs [21]. Of course, our transformations must not change the clique-width by “too much”. We ensure this by using certain graph operations that are known to preserve (un)boundedness of clique-width [31, 37].

## 1.3 Consequences for Clique-Width

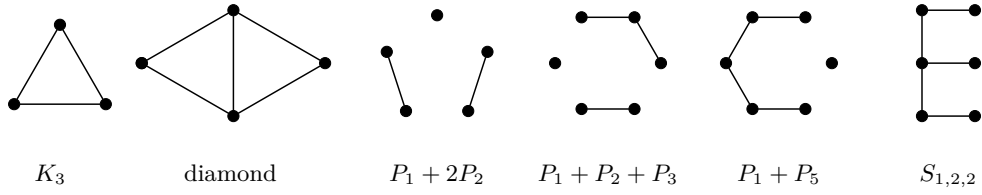
There are numerous papers (as listed in, for instance, [22, 27, 31]) that determine the (un)boundedness of the clique-width or variants of it (see e.g. [4, 28]) of special graph classes. Due to the complex nature of clique-width, proofs of these results are often long and technical, and there are still many open cases. In particular gaps exist in a number of dichotomies on the (un)boundedness of clique-width for graph classes defined by one or more forbidden induced subgraphs. As such our paper also continues a line of research [5, 6, 19, 21, 22] in which we focus on these gaps in a *systematic* way. It is known [22] that the class of  $H$ -free graphs has bounded clique-width if and only if  $H$  is an induced subgraph of  $P_4$ . Over the years many partial results [2, 7, 9, 10, 11, 12, 20, 38] on the (un)boundedness of clique-width appeared for classes of  $(H_1, H_2)$ -free graphs, but until recently [22] it was not even known whether the number of missing cases was bounded. Combining these older results with recent progress [5, 18, 19, 22] reduced the number of open cases to 13 (up to an equivalence relation) [22].

As a by-product of our methodology, we are able not only to settle the case  $(H_1, H_2) = (\text{diamond}, P_1 + 2P_2)$ , but in fact we solve **five** of the remaining 13 open cases by proving that the class of  $(H_1, H_2)$ -free graphs has bounded clique-width if

- 1–4:**  $H_1 = K_3$  and  $H_2 \in \{P_1 + 2P_2, P_1 + P_2 + P_3, P_1 + P_5, S_{1,2,2}\}$  or
- 5:**  $H_1 = \text{diamond}$  and  $H_2 = P_1 + 2P_2$ .

The above graphs are displayed in Figure 1. Note that the  $(K_3, P_1 + 2P_2)$  case is properly contained in all four of the other cases. These four other newly solved cases are pairwise incomparable.

Updating the classification (see [22]) with our five new results gives the following theorem. Here,  $\mathcal{S}$  is the class of graphs, each connected component of which is either a subdivided claw or a path, and we write  $H \subseteq_i G$  if  $H$  is an induced subgraph of  $G$ ; see Section 2 for notation that we have not formally defined yet.



■ **Figure 1** The forbidden graphs considered in this paper.

► **Theorem 2.** Let  $\mathcal{G}$  be a class of graphs defined by two forbidden induced subgraphs. Then:

1.  $\mathcal{G}$  has bounded clique-width if it is equivalent<sup>1</sup> to a class of  $(H_1, H_2)$ -free graphs such that one of the following holds:
  - a.  $H_1$  or  $H_2 \subseteq_i P_4$ ;
  - b.  $H_1 = sP_1$  and  $H_2 = K_t$  for some  $s, t$ ;
  - c.  $H_1 \subseteq_i P_1 + P_3$  and  $\overline{H_2} \subseteq_i K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + P_2 + P_3, P_1 + P_5, P_1 + S_{1,1,2}, P_6, S_{1,2,2}$  or  $S_{1,1,3}$ ;
  - d.  $H_1 \subseteq_i 2P_1 + P_2$  and  $\overline{H_2} \subseteq_i P_1 + 2P_2, 2P_1 + P_3, 3P_1 + P_2$  or  $P_2 + P_3$ ;
  - e.  $H_1 \subseteq_i P_1 + P_4$  and  $\overline{H_2} \subseteq_i P_1 + P_4$  or  $P_5$ ;
  - f.  $H_1 \subseteq_i 4P_1$  and  $\overline{H_2} \subseteq_i 2P_1 + P_3$ ;
  - g.  $H_1, \overline{H_2} \subseteq_i K_{1,3}$ .
2.  $\mathcal{G}$  has unbounded clique-width if it is equivalent to a class of  $(H_1, H_2)$ -free graphs such that one of the following holds:
  - a.  $H_1 \notin \mathcal{S}$  and  $H_2 \notin \mathcal{S}$ ;
  - b.  $\overline{H_1} \notin \mathcal{S}$  and  $\overline{H_2} \notin \mathcal{S}$ ;
  - c.  $H_1 \supseteq_i K_{1,3}$  or  $2P_2$  and  $\overline{H_2} \supseteq_i 4P_1$  or  $2P_2$ ;
  - d.  $H_1 \supseteq_i 2P_1 + P_2$  and  $\overline{H_2} \supseteq_i K_{1,3}, 5P_1, P_2 + P_4$  or  $P_6$ ;
  - e.  $H_1 \supseteq_i 3P_1$  and  $\overline{H_2} \supseteq_i 2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2$  or  $2P_3$ ;
  - f.  $H_1 \supseteq_i 4P_1$  and  $\overline{H_2} \supseteq_i P_1 + P_4$  or  $3P_1 + P_2$ .

## 1.4 Future Work

Naturally we would like to extend Theorem 1 and solve the following open problem.

► **Open Problem 1.** What is the computational complexity of COLOURING for  $(\text{diamond}, H)$ -free graphs when  $H$  is a graph on at least six vertices?

Solving Open Problem 1 is highly non-trivial. It is known that 4-COLOURING is NP-complete for  $(C_3, P_2)$ -free graphs [30]. Hence, the polynomial-time results in Theorem 1 cannot be extended to all linear forests. The first open case to consider would be  $H = P_6$ , for which only partial results are known. Indeed, the COLOURING problem is polynomial-time solvable for  $(C_3, P_6)$ -free graphs [9], but its complexity is unknown for  $(C_3, P_7)$ -free graphs (on a side note, a recent result for the latter graph class is that 3-COLOURING is polynomial-time solvable [3]).

<sup>1</sup> Given four graphs  $H_1, H_2, H_3, H_4$ , the class of  $(H_1, H_2)$ -free graphs and the class of  $(H_3, H_4)$ -free graphs are *equivalent* if the unordered pair  $H_3, H_4$  can be obtained from the unordered pair  $H_1, H_2$  by some combination of the operations (i) complementing both graphs in the pair and (ii) if one of the graphs in the pair is  $K_3$ , replacing it with  $\overline{P_1 + P_3}$  or vice versa. If two classes are equivalent, then one of them has bounded clique-width if and only if the other one does (see [22]).

We observe that boundedness of the clique-width of (diamond,  $P_1 + 2P_2$ )-free graphs implies boundedness of the clique-width of  $(2P_1 + P_2, \overline{P_1 + 2P_2})$ -free graphs (recall that the diamond is the complement of the graph  $2P_1 + P_2$ ). Hence our results imply that COLOURING can also be solved in polynomial time for graphs in this class. In fact, COLOURING has been studied extensively for  $(H_1, H_2)$ -free graphs, and we refer to the survey of Golovach et al. [25] for a summary of known results. After incorporating the consequences of our new results, there are 13 classes of  $(H_1, H_2)$ -free graphs for which COLOURING could potentially still be solved in polynomial time by showing that their clique-width is bounded (see also [25]):

► **Open Problem 2.** Is COLOURING polynomial-time solvable for  $(H_1, H_2)$ -free graphs when:

1.  $\overline{H_1} \in \{3P_1, P_1 + P_3\}$  and  $H_2 \in \{P_1 + S_{1,1,3}, S_{1,2,3}\}$ ;
2.  $H_1 = 2P_1 + P_2$  and  $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + P_5\}$ ;
3.  $H_1 = \text{diamond}$  and  $H_2 \in \{P_1 + P_2 + P_3, P_1 + P_5\}$ ;
4.  $H_1 = P_1 + P_4$  and  $\overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$ ;
5.  $\overline{H_1} = P_1 + P_4$  and  $H_2 \in \{P_1 + 2P_2, P_2 + P_3\}$ ;
6.  $H_1 = \overline{H_2} = 2P_1 + P_3$ .

As mentioned in Section 1.3, after updating the list of remaining open cases for clique-width from [22], we find that eight non-equivalent open cases remain for clique-width. These are the following cases.

► **Open Problem 3.** Does the class of  $(H_1, H_2)$ -free graphs have bounded or unbounded clique-width when:

1.  $H_1 = 3P_1$  and  $\overline{H_2} \in \{P_1 + S_{1,1,3}, P_2 + P_4, S_{1,2,3}\}$ ;
2.  $H_1 = 2P_1 + P_2$  and  $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + P_5\}$ ;
3.  $H_1 = P_1 + P_4$  and  $\overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$  or
4.  $H_1 = \overline{H_2} = 2P_1 + P_3$ .

Bonomo, Grippo, Milanič and Safe [4] determined all pairs of connected graphs  $H_1, H_2$  for which the class of  $(H_1, H_2)$ -free graphs has power-bounded clique-width. In order to compare their result with our results for clique-width, we only need to solve the open case  $(H_1, H_2) = (K_3, S_{1,2,3})$ , which is equivalent to the (open) case  $(H_1, H_2) = (3P_1, \overline{S_{1,2,3}})$  mentioned in Open Problem 3, as our new result for the case  $(H_1, H_2) = (K_3, S_{1,2,2})$  has reduced the number of open cases  $(H_1, H_2)$  with  $H_1, H_2$  both connected from two to one.

## 2 Preliminaries

Below we define further graph terminology used throughout our paper. The *disjoint union*  $(V(G) \cup V(H), E(G) \cup E(H))$  of two vertex-disjoint graphs  $G$  and  $H$  is denoted by  $G + H$  and the disjoint union of  $r$  copies of a graph  $G$  is denoted by  $rG$ . The *complement* of a graph  $G$ , denoted by  $\overline{G}$ , has vertex set  $V(\overline{G}) = V(G)$  and an edge between two distinct vertices if and only if these vertices are not adjacent in  $G$ . For a subset  $S \subseteq V(G)$ , we let  $G[S]$  denote the subgraph of  $G$  induced by  $S$ , which has vertex set  $S$  and edge set  $\{uv \mid u, v \in S, uv \in E(G)\}$ . If  $S = \{s_1, \dots, s_r\}$  then, to simplify notation, we may also write  $G[s_1, \dots, s_r]$  instead of  $G[\{s_1, \dots, s_r\}]$ . We use  $G \setminus S$  to denote the graph obtained from  $G$  by deleting every vertex in  $S$ , i.e.  $G \setminus S = G[V(G) \setminus S]$ . Let  $H$  be another graph. We write  $H \subseteq_i G$  to indicate that  $H$  is an induced subgraph of  $G$ .

The graphs  $C_r, K_r, K_{1,r-1}$  and  $P_r$  denote the cycle, complete graph, star and path on  $r$  vertices, respectively. The graph  $K_{1,3}$  is also called the *claw*. The graph  $S_{h,i,j}$ , for  $1 \leq h \leq i \leq j$ , denotes the *subdivided claw*, that is, the tree that has only one vertex  $x$

of degree 3 and exactly three leaves, which are of distance  $h$ ,  $i$  and  $j$  from  $x$ , respectively. Observe that  $S_{1,1,1} = K_{1,3}$ . The graph  $S_{1,2,2}$  is also known as the E, since it can be drawn like a capital letter E (see Figure 1). Recall that the graph  $\overline{P_1} + 2\overline{P_2}$  is known as the *diamond*. The graphs  $K_3$  and  $\overline{P_1} + 2\overline{P_2}$  are also known as the *triangle* and the 5-vertex *wheel*, respectively. For a set of graphs  $\{H_1, \dots, H_p\}$ , a graph  $G$  is  $(H_1, \dots, H_p)$ -free if it has no induced subgraph isomorphic to a graph in  $\{H_1, \dots, H_p\}$ ; if  $p = 1$ , we may write  $H_1$ -free instead of  $(H_1)$ -free.

For a graph  $G = (V, E)$ , the set  $N(u) = \{v \in V \mid uv \in E\}$  denotes the neighbourhood of  $u \in V$ . A graph is  $k$ -partite if its vertex set can be partitioned into  $k$  independent sets (some of which may be empty). A graph is *bipartite* if it is 2-partite. The *bipartite complement* of a bipartite graph  $G$  with bipartition  $(X, Y)$  is the graph obtained from  $G$  by replacing every edge from a vertex in  $X$  to a vertex in  $Y$  by a non-edge and vice versa. The *biclique*  $K_{r,s}$  is the bipartite graph with sets in the partition of size  $r$  and  $s$  respectively, such that every vertex in one set is adjacent to every vertex in the other set.

Let  $X$  be a set of vertices in a graph  $G = (V, E)$ . A vertex  $y \in V \setminus X$  is *complete* to  $X$  if it is adjacent to every vertex of  $X$  and *anti-complete* to  $X$  if it is non-adjacent to every vertex of  $X$ . Similarly, a set of vertices  $Y \subseteq V \setminus X$  is *complete* (resp. *anti-complete*) to  $X$  if every vertex in  $Y$  is complete (resp. anti-complete) to  $X$ . A vertex  $y$  or a set  $Y$  is *trivial* to  $X$  if it is either complete or anti-complete to  $X$ . Note that if  $Y$  contains both vertices complete to  $X$  and vertices not complete to  $X$ , we may have a situation in which every vertex in  $Y$  is trivial to  $X$ , but  $Y$  itself is not trivial to  $X$ .

**Clique-Width.** The *clique-width* of a graph  $G$ , denoted  $\text{cw}(G)$ , is the minimum number of labels needed to construct  $G$  by using the following four operations:

1. creating a new graph consisting of a single vertex  $v$  with label  $i$ ;
2. taking the disjoint union of two labelled graphs  $G_1$  and  $G_2$ ;
3. joining each vertex with label  $i$  to each vertex with label  $j$  ( $i \neq j$ );
4. renaming label  $i$  to  $j$ .

An algebraic term that represents such a construction of  $G$  and uses at most  $k$  labels is said to be a  $k$ -expression of  $G$  (i.e. the clique-width of  $G$  is the minimum  $k$  for which  $G$  has a  $k$ -expression). Recall that a class of graphs  $\mathcal{G}$  has bounded clique-width if there is a constant  $c$  such that the clique-width of every graph in  $\mathcal{G}$  is at most  $c$ ; otherwise the clique-width of  $\mathcal{G}$  is *unbounded*.

Let  $G$  be a graph. We define the following operations. For an induced subgraph  $G' \subseteq_i G$ , the *subgraph complementation* operation (acting on  $G$  with respect to  $G'$ ) replaces every edge present in  $G'$  by a non-edge, and vice versa. Similarly, for two disjoint vertex subsets  $S$  and  $T$  in  $G$ , the *bipartite complementation* operation with respect to  $S$  and  $T$  acts on  $G$  by replacing every edge with one end-vertex in  $S$  and the other one in  $T$  by a non-edge and vice versa.

We now state some useful facts about how the above operations (and some other ones) influence the clique-width of a graph. We will use these facts throughout the paper. Let  $k \geq 0$  be a constant and let  $\gamma$  be some graph operation. We say that a graph class  $\mathcal{G}'$  is  $(k, \gamma)$ -obtained from a graph class  $\mathcal{G}$  if the following two conditions hold:

1. every graph in  $\mathcal{G}'$  is obtained from a graph in  $\mathcal{G}$  by performing  $\gamma$  at most  $k$  times, and
2. for every  $G \in \mathcal{G}$  there exists at least one graph in  $\mathcal{G}'$  obtained from  $G$  by performing  $\gamma$  at most  $k$  times.

We say that  $\gamma$  *preserves* boundedness of clique-width if for any finite constant  $k$  and any graph class  $\mathcal{G}$ , any graph class  $\mathcal{G}'$  that is  $(k, \gamma)$ -obtained from  $\mathcal{G}$  has bounded clique-width if

and only if  $\mathcal{G}$  has bounded clique-width.

**Fact 1** Vertex deletion preserves boundedness of clique-width [37].

**Fact 2** Subgraph complementation preserves boundedness of clique-width [31].

**Fact 3** Bipartite complementation preserves boundedness of clique-width [31].

Two vertices are *false twins* if they have the same neighbourhood. (note that such vertices must be non-adjacent). The following lemma follows immediately from the definition of clique-width.

► **Lemma 3.** *If a vertex  $x$  in a graph  $G$  has a false twin then  $\text{cw}(G) = \text{cw}(G \setminus \{x\})$ .*

We will also make use of the following two results.

► **Lemma 4** ([19]). *The class of (diamond,  $P_2 + P_3$ )-free graphs has bounded clique-width.*

► **Lemma 5** ([21]). *Let  $H$  be a graph. The class of  $H$ -free bipartite graphs has bounded clique-width if and only if  $H = sP_1$  for some  $s \geq 1$ ;  $H \subseteq_i K_{1,3} + 3P_1$ ;  $H \subseteq_i K_{1,3} + P_2$ ;  $H \subseteq_i P_1 + S_{1,1,3}$ ; or  $H \subseteq_i S_{1,2,3}$ .*

### 3 Totally $k$ -Decomposable Graphs

In this section we describe our key technique, which is based on the following notion introduced by Fouquet, Giakoumakis and Vanherpe [24]. A bipartite graph  $G$  is *totally decomposable by canonical decomposition* if it can be recursively decomposed into graphs isomorphic to  $K_1$  by decomposition of a bipartite graph  $G$  with bipartition  $(V_1, V_2)$  into two non-empty graphs  $G[V'_1 \cup V'_2]$  and  $G[V''_1 \cup V''_2]$  where  $V'_i$  and  $V''_i$  form a partition of  $V_i$  for  $i \in \{1, 2\}$  such that each of  $G[V'_1 \cup V'_2]$  and  $G[V''_1 \cup V''_2]$  is either an independent set or a biclique.

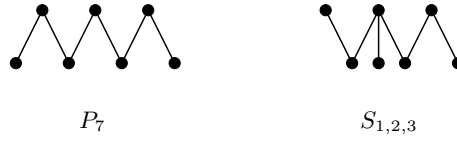
For our purposes we need to generalize the above notion to  $k$ -partite graphs. Let  $G$  be a  $k$ -partite graph with a fixed vertex  $k$ -partition  $(V_1, V_2, \dots, V_k)$ . We say that a  *$k$ -decomposition of  $G$*  with respect to this partition consists of two non-empty graphs, each with their own partition:  $G[V'_1 \cup V'_2 \cup \dots \cup V'_k]$  with partition  $(V'_1, V'_2, \dots, V'_k)$  and  $G[V''_1 \cup V''_2 \cup \dots \cup V''_k]$  with partition  $(V''_1, V''_2, \dots, V''_k)$ , such that the following two conditions hold:

1. for every  $i \in \{1, \dots, k\}$ ,  $V'_i$  and  $V''_i$  form a partition of  $V_i$ , and
2. for every  $i, j \in \{1, \dots, k\}$  with  $i \neq j$ , the set  $V'_i$  is either complete or anti-complete to  $V''_j$  in  $G$  (note that  $V_i$  is an independent set for every  $i \in \{1, \dots, k\}$ , so  $V'_i$  will automatically be anti-complete to  $V''_i$ ).

We say that  $G$  is *totally  $k$ -decomposable* if it can be recursively  $k$ -decomposed into graphs isomorphic to  $K_1$ . Note that every connected bipartite graph has a unique bipartition (up to isomorphism). If a graph is totally decomposable by canonical decomposition then this can recursively be done component-wise. Thus the definition of total canonical decomposability is indeed the same as total 2-decomposability. Fouquet, Giakoumakis and Vanherpe proved the following characterization, which we will need for our proofs (see Figure 2 for pictures of  $P_7$  and  $S_{1,2,3}$ ).

► **Lemma 6** ([24]). *A bipartite graph is totally decomposable by canonical decomposition if and only if it is  $(P_7, S_{1,2,3})$ -free.*

It seems difficult to generalize Lemma 6 to give a full characterization for totally  $k$ -decomposable graphs for  $k \geq 3$ . However, the following lemma is sufficient for our purposes.



■ **Figure 2** The forbidden graphs from Lemma 6.

► **Lemma 7.** *A 3-partite graph is totally 3-decomposable with respect to a 3-partition  $(V_1, V_2, V_3)$  if the following two conditions are both satisfied:*

- *$G[V_1 \cup V_2], G[V_1 \cup V_3]$  and  $G[V_2 \cup V_3]$  are all  $(P_7, S_{1,2,3})$ -free, and*
- *for every  $v_1 \in V_1$ , every  $v_2 \in V_2$  and every  $v_3 \in V_3$ , the graph  $G[v_1, v_2, v_3]$  is isomorphic neither to  $K_3$  nor to  $3P_1$ .*

**Proof.** Let  $G$  be such a graph. Note that any induced subgraph  $H$  of  $G$  also satisfies the hypotheses of the lemma, with partition  $(V(H) \cap V_1, V(H) \cap V_2, V(H) \cap V_3)$ . It is therefore sufficient to show that  $G$  has a 3-decomposition.

If  $V_i$  is empty for some  $i \in \{1, 2, 3\}$  then  $G$  is a  $(P_7, S_{1,2,3})$ -free bipartite graph and is therefore totally 2-decomposable with respect to the given partition by Lemma 6. We may therefore assume that every set  $V_i$  is non-empty.

Now  $G[V_1, V_2]$  is a bipartite  $(P_7, S_{1,2,3})$ -free graph, so by Lemma 6,  $G[V_1 \cup V_2]$  is totally 2-decomposable. Since  $V_1$  and  $V_2$  are both non-empty, it follows that  $V_1$  can be partitioned into two sets  $V'_1$  and  $V''_1$  and  $V_2$  can be partitioned into two sets  $V'_2$  and  $V''_2$  such that  $V'_1$  is either complete or anti-complete to  $V''_2$  and  $V'_2$  is either complete or anti-complete to  $V''_1$ . Furthermore, we may assume  $V'_1 \cup V'_2 \neq \emptyset$  and  $V''_1 \cup V''_2 \neq \emptyset$ .

Since  $V_1, V_2, V'_1 \cup V'_2$  and  $V''_1 \cup V''_2$  are non-empty, we may assume without loss of generality that  $V'_1$  and  $V'_2$  are non-empty. Assume that these sets are maximal i.e. no vertex of  $V''_1$  (respectively  $V'_2$ ) can be moved to  $V'_1$  (respectively  $V'_2$ ). Note that  $V''_1$  or  $V'_2$  may be empty.

We will prove that we can partition  $V_3$  into sets  $V'_3$  and  $V''_3$ , such that for all  $i, j \in \{1, 2, 3\}$  with  $j \neq i$ ,  $V'_i$  is complete or anti-complete to  $V''_j$ . Note that we already know that  $V'_1$  (respectively  $V'_2$ ) is complete or anti-complete to  $V''_2$  (respectively  $V''_1$ ).

First suppose that  $V'_1$  is complete to  $V''_2$ . If a vertex of  $V_3$  has a neighbour in both  $V'_1$  and  $V'_2$  then these three vertices would form a forbidden  $K_3$ , so every vertex in  $V_3$  is anti-complete to  $V'_1$  or  $V'_2$ . Let  $V'_3$  be the set of vertices in  $V_3$  that are anti-complete to  $V'_2$  and let  $V''_3 = V_3 \setminus V'_3$ . Note that every vertex of  $V''_3$  must be anti-complete to  $V'_1$ . Suppose, for contradiction, that  $z \in V'_3$  has a non-neighbour  $v \in V'_1$ . Since  $V'_1$  is maximal,  $v$  must have a non-neighbour  $w \in V''_2$ . This means that  $G[v, w, z]$  is a  $3P_1$ . This contradiction means that  $V'_1$  is complete to  $V'_3$ . Similarly,  $V'_2$  is complete to  $V''_3$ . Therefore  $G[V'_1 \cup V'_2 \cup V'_3]$  and  $G[V''_1 \cup V''_2 \cup V''_3]$  form the required 3-decomposition of  $G$ .

Similarly, if  $V'_1$  is anti-complete to  $V''_2$  then  $V_3$  can be partitioned into sets  $V'_3$  and  $V''_3$  that are complete to  $V''_2$  and  $V'_1$ , respectively. By analogous arguments, we find that  $V''_1$  is anti-complete to  $V'_3$  and  $V'_2$  is anti-complete to  $V''_3$ . We then proceed as in the previous case. This completes the proof. ◀

We also need the following lemma.

► **Lemma 8.** *Let  $G$  be a  $k$ -partite graph with vertex partition  $(V_1, \dots, V_k)$ . If  $G$  is totally  $k$ -decomposable with respect to this partition then the clique-width of  $G$  is at most  $2k$ . Moreover, there is a  $2k$ -expression for  $G$  that assigns, for  $i \in \{1, \dots, k\}$ , label  $i$  to every vertex of  $V_i$ .*



**Proof.** We prove the lemma by induction. Clearly, if  $G$  contains only one vertex then the lemma holds. Suppose that the lemma is true for all such graphs on at most  $n$  vertices. Let  $G$  be a totally  $k$ -decomposable graph on  $n + 1$  vertices with vertex partition  $(V_1, \dots, V_k)$ . Since  $G$  has a  $k$ -decomposition, we can partition every set  $V_i$  into two sets  $V_i'$  and  $V_i''$  such that each set  $V_i'$  is either complete or anti-complete to each set  $V_j''$  for  $i, j \in \{1, \dots, k\}$ . By the induction hypothesis, we can find a  $2k$ -expression that constructs the non-empty graph  $G[V_1' \cup V_2' \cup \dots \cup V_k']$  such that the vertices in each set  $V_i'$  have label  $i$  for  $i \in \{1, \dots, k\}$ . Similarly, we can find a  $2k$ -expression that constructs the non-empty graph  $G[V_1'' \cup V_2'' \cup \dots \cup V_k'']$  such that the vertices in each set  $V_j''$  have label  $k + j$  for  $j \in \{1, \dots, k\}$ . We take the disjoint union of these two constructions. Next, for  $i, j \in \{1, \dots, k\}$ , we join the vertices label  $i$  to the vertices label  $k + j$  if  $V_i'$  is complete to  $V_j''$  in  $G$ . Finally, for  $i \in \{1, \dots, k\}$ , we relabel the vertices with label  $k + i$  to have label  $i$ . By induction, this completes the proof of the lemma.  $\blacktriangleleft$

#### 4 Bounding the Clique-Width

To prove our results on clique-width we need two more lemmas. The first lemma (we omit the proof due to space restrictions<sup>2</sup>) implies that the four triangle-free cases in our new results hold when the graph under consideration is  $C_5$ -free. In the second lemma we state a number of sufficient conditions for a graph class to be of bounded clique-width when  $C_5$  is no longer a forbidden induced subgraph. While we will not use these lemmas directly in the proof of the diamond-free case, that result also relies on these two lemmas, as it depends on the  $(K_3, P_1 + 2P_2)$ -free case.

► **Lemma 9.** *The class of  $(K_3, C_5, S_{1,2,3})$ -free graphs has bounded clique-width.*

► **Lemma 10.** *A  $(K_3, S_{1,2,3})$ -free graph has bounded clique-width if its vertices can be partitioned into ten independent sets  $V_1, \dots, V_5, W_1, \dots, W_5$  such that the following conditions hold (we interpret subscripts modulo 5):*

1. for all  $i$ ,  $V_i$  is anti-complete to  $V_{i-2} \cup V_{i+2} \cup W_{i-1} \cup W_{i+1}$ ;
2. for all  $i$ ,  $W_i$  is complete to  $W_{i-1} \cup W_{i+1}$ ;
3. for all  $i$ , each vertex of  $V_i$  is either trivial to  $V_{i+1}$  or trivial to  $V_{i-1}$ ;
4. for all  $i$ , every vertex in  $V_i$  is trivial to  $W_i$ ;
5. for all  $i$ ,  $W_i$  is trivial to  $W_{i-2}$  and to  $W_{i+2}$ ;
6. for all  $i, j$ , the graphs induced by  $V_i \cup V_j$  and  $V_i \cup W_j$  are  $P_7$ -free;
7. for all  $i$ , there are no three vertices  $v \in V_i$ ,  $w \in V_{i+1}$  and  $x \in W_{i+3}$  such that  $v, w$  and  $x$  are pairwise non-adjacent.

**Proof.** Let  $G$  be a  $(K_3, S_{1,2,3})$ -free graph with such a partition that satisfies Conditions 1–7 of the lemma. Note that for all  $i$ , every vertex  $v \in V_i$  is trivial to  $V_{i+2}, V_{i-2}, W_{i-1}, W_{i+1}, W_i$  and either trivial to  $V_{i+1}$  or trivial to  $V_{i-1}$ . Therefore a vertex  $v \in V_i$  can only be non-trivial to  $W_{i-2}, W_{i+2}$  and at most one of  $V_{i-1}$  and  $V_{i+1}$ . Likewise, every vertex  $w \in W_i$  is trivial to  $W_{i-1}, W_{i+1}, W_{i-2}, W_{i+2}, V_{i-1}$  and  $V_{i+1}$ . Therefore, a vertex  $w \in W_i$  can only be non-trivial to  $V_i, V_{i-2}$  and  $V_{i+2}$  (and every vertex in  $V_i$  is trivial to  $W_i$ ).

For  $i \in \{1, \dots, 5\}$ , let  $W_i'$  be the set of elements of  $W_i$  that are non-trivial to both  $V_{i-2}$  and  $V_{i+2}$ , let  $V_i'$  be the set of elements of  $V_i$  that are non-trivial to both  $V_{i+1}$  and  $W_{i-2}$  and

<sup>2</sup> Omitted proofs can be found in the arXiv preprint of this paper [17].

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let  $V_i''$  be the set of elements of  $V_i$  that are non-trivial to both  $V_{i-1}$  and  $W_{i+2}$ . Note that  $V_i' \cap V_i'' = \emptyset$  by Condition 3.

We say that an edge is *irrelevant* if one of its end-points is in a set  $V_i, V_i', V_i'', W_i$  or  $W_i'$ , and its other end-point is complete to this set, otherwise we say that the edge is *relevant*. We will now show that for  $i \in \{1, \dots, 5\}$ , the graph  $G[V_i' \cup V_{i+1}'' \cup W_{i-2}']$  can be separated from the rest of  $G$  by using a bounded number of bipartite complementations. To do this, we first prove the following claim.

**Claim 1.** *If  $u \in V_i' \cup V_{i+1}'' \cup W_{i-2}'$  and  $v \notin V_i' \cup V_{i+1}'' \cup W_{i-2}'$  are adjacent then  $uv$  is an irrelevant edge.*

We split the proof of Claim 1 into the following cases.

**Case 1:**  $u \in V_i'$ .

Since  $u$  is in  $V_i'$ ,  $v$  must be in  $V_{i-1} \cup V_{i+1} \cup W_{i-2} \cup W_{i+2}$ , otherwise  $uv$  would be irrelevant by Condition 1 or 4. We consider the possible cases for  $v$ .

**Case 1a:**  $v \in V_{i-1}$ .

Since  $u$  is in  $V_i'$ , it is non-trivial to  $V_{i+1}$ , so by Condition 3,  $u$  is trivial to  $V_{i-1}$ . Therefore  $uv$  is irrelevant.

**Case 1b:**  $v \in V_{i+1}$ .

Suppose, for contradiction, that  $v$  is complete to  $W_{i-2}$ . Let  $w \in W_{i-2}$  be a neighbour of  $u$  (such a vertex  $w$  exists, since  $u$  is non-trivial to  $W_{i-2}$ ). Then  $G[u, v, w]$  is a  $K_3$ , a contradiction, so  $v$  cannot be complete to  $W_{i-2}$ . Now suppose, for contradiction that  $v$  is anti-complete to  $W_{i-2}$ . We may assume that  $v$  has a non-neighbour  $u' \in V_i'$ , otherwise  $v$  would be trivial to  $V_i'$ , in which case  $uv$  would be irrelevant. Since  $u' \in V_i'$ ,  $u'$  is non-trivial to  $W_{i-2}$ , so it must have a non-neighbour  $w \in W_{i-2}$ . Then, since  $v$  is anti-complete to  $W_{i-2}$ , it follows that  $G[u, v, w]$  is a  $3P_1$ , contradicting Condition 7. We may therefore assume that  $v$  is non-trivial to  $W_{i-2}$ . We know that  $v \notin V_{i+1}''$ . Therefore  $v$  must be trivial to  $V_i$ , so  $uv$  is irrelevant.

**Case 1c:**  $v \in W_{i-2}$ .

Reasoning as in the previous case, we find that  $v$  cannot be complete or anti-complete to  $V_{i+1}$ . Hence, as  $v \notin W_{i-2}'$ ,  $v$  must be trivial to  $V_i$ , so  $uv$  is irrelevant.

**Case 1d:**  $v \in W_{i+2}$ .

Since  $u$  is non-trivial to  $W_{i-2}$  (by definition of  $V_i'$ ), there is a vertex  $w \in W_{i-2}$  that is adjacent to  $u$ . By Condition 2,  $w$  is adjacent to  $v$ . Therefore  $G[u, v, w]$  is a  $K_3$ . This contradiction implies that  $v \notin W_{i+2}$ . This completes Case 1.

Now assume that  $u \notin V_i'$ . Then, by symmetry,  $u \notin V_{i+1}''$ . This means that the following case holds.

**Case 2:**  $u \in W_{i-2}'$ .

We argue similarly to Case 1b. We may assume that  $v$  is non-trivial to  $W_{i-2}'$ , otherwise  $uv$  would be irrelevant. By Conditions 1, 2 and 5, it follows that  $v \in V_i \cup V_{i+1}$ . Without loss of generality assume that  $v \in V_i$ . Since  $v \notin V_i'$  and  $v$  is non-trivial to  $W_{i-2}$ , it follows that  $v$  is trivial to  $V_{i+1}$ . If  $v$  is complete to  $V_{i+1}$  then since  $u$  is non-trivial to  $V_{i+1}$ , there must be a vertex  $w \in V_{i+1}$  adjacent to  $u$ , in which case  $G[u, v, w]$  is a  $K_3$ , a contradiction. Therefore  $v$  must be anti-complete to  $V_{i+1}$ . Since  $v$  is non-trivial to  $W_{i-2}'$ , there must be a vertex  $u' \in W_{i-2}'$  that is non-adjacent to  $v$ . Since  $u' \in W_{i-2}'$ ,  $u'$  must have a non-neighbour  $w \in V_{i+1}$ . Then  $G[u', v, w]$  is a  $3P_1$ , contradicting Condition 7. This completes Case 2.

We conclude that, if  $u \in V'_i \cup V''_{i+1} \cup W'_{i-2}$  and  $v \notin V'_i \cup V''_{i+1} \cup W'_{i-2}$  are adjacent, then  $uv$  is an irrelevant edge. Hence we have proven Claim 1.

By Claim 1 we find that if  $u \in V'_i \cup V''_{i+1} \cup W'_{i-2}$  and  $v \notin V'_i \cup V''_{i+1} \cup W'_{i-2}$  are adjacent then  $u$  or  $v$  is complete to some set  $V_j, V'_j, V''_j, W_j$  or  $W'_j$  that contains  $v$  or  $u$ , respectively. Applying a bounded number of bipartite complements (which we may do by Fact 3), we can separate  $G[V'_i \cup V''_{i+1} \cup W'_{i-2}]$  from the rest of  $G$ . By Conditions 6 and 7 and the fact that  $G$  is  $(K_3, S_{1,2,3})$ -free, Lemmas 7 and 8 imply that  $G[V'_i \cup V''_{i+1} \cup W'_{i-2}]$  has clique-width at most 6. Repeating this argument for each  $i$ , we may assume that  $V'_i \cup V''_{i+1} \cup W'_{i-2} = \emptyset$  for every  $i$ .

For  $i \in \{1, \dots, 5\}$  let  $V_i^*$  be the set of vertices in  $V_i$  that are either non-trivial to  $V_{i+1}$  or non-trivial to  $W_{i+2}$  and let  $V_i^{**}$  be the set of the remaining vertices in  $V_i$ . For  $i \in \{1, \dots, 5\}$ , let  $W_i^*$  be the set of vertices that are non-trivial to  $V_{i+2}$  and let  $W_i^{**}$  be the set of the remaining vertices in  $W_i$ .

We claim that every vertex in  $V_i$  that is non-trivial to  $V_{i-1}$  or that is non-trivial to  $W_{i-2}$  is in  $V_i^{**}$ . Indeed, if  $v \in V_i$  is non-trivial to  $V_{i-1}$  then by Condition 3,  $v$  is trivial to  $V_{i+1}$  and since  $V''_i$  is empty,  $v$  must be trivial to  $W_{i+2}$ . If  $v \in V_i$  is non-trivial to  $W_{i-2}$  then  $v$  must be trivial to  $V_{i+1}$  since  $V'_i$  is empty. Moreover, in this case  $v$  must also be trivial to  $W_{i+2}$ , otherwise, by Condition 2 the vertex  $v$ , together with a neighbour of  $v$  in each of  $W_{i+2}$  and  $W_{i-2}$ , would induce a  $K_3$  in  $G$ . It follows that every vertex in  $V_i$  that is non-trivial to  $V_{i-1}$  or that is non-trivial to  $W_{i-2}$  is indeed in  $V_i^{**}$ . Similarly, for all  $i$ , since  $W'_i$  is empty, every vertex in  $W_i$  that is non-trivial to  $V_{i-2}$  is in  $W_i^{**}$ .

We say that an edge  $uv$  is *insignificant* if  $u$  or  $v$  is in some set  $V_i^*, V_i^{**}, W_i^*$  or  $W_i^{**}$  and the other vertex is trivial to this set; all other edges are said to be *significant*. We prove the following claim.

**Claim 2.** *If  $u \in W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}$  and  $v \notin W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}$  are adjacent then the edge  $uv$  is insignificant.*

To prove this claim suppose, for contradiction, that  $uv$  is a significant edge. We split the proof into two cases.

**Case 1:**  $u \in W_i$ .

We will show that  $v \in V_{i+2}^{**}$  or  $v \in V_{i-2}^*$  if  $u \in W_i^*$  or  $u \in W_i^{**}$ , respectively. By Conditions 1, 2, 4 and 5 we know that  $u$  is trivial to  $V_{i-1}, V_{i+1}, W_{i-1}, W_{i+1}, W_{i-2}$  and  $W_{i+2}$ , and that every vertex of  $V_i$  is trivial to  $W_i$ . Furthermore,  $u$  is trivial to  $W_i^{**} \setminus \{u\}$  since  $W_i$  is independent. Therefore  $v \in V_{i-2} \cup V_{i+2}$ . Note that  $v$  is non-trivial to  $W_i$  (by choice of  $v$ ). If  $u \in W_i^*$  then  $u$  must be trivial to  $V_{i-2}$ , since  $W'_i$  is empty. Therefore  $v \in V_{i+2}$ . Now if  $v \in V_{i+2}^*$  then  $v$  is non-trivial to  $V_{i-2}$  or non-trivial to  $W_{i-1}$ . In the first case  $v$  is non-trivial to both  $V_{i-2}$  and  $W_i$ , contradicting the fact that  $V'_{i+2}$  is empty. In the second case  $v$  has a neighbour  $w \in W_{i-1}$ . By Condition 2,  $w$  is adjacent to  $u$ , so  $G[u, v, w]$  is a  $K_3$ . This contradiction implies that if  $u \in W_i^*$  then  $v \in V_{i+2}^{**}$ , contradicting the choice of  $v$ . Now suppose  $u \in W_i^{**}$ . Then  $u$  is trivial to  $V_{i+2}$ , so  $v \in V_{i-2}$ . If  $v \in V_{i-2}^{**}$  then  $v$  is trivial to  $W_i$  (by definition of  $V_{i-2}^{**}$ ). Therefore if  $u \in W_i^{**}$  then  $v \in V_{i-2}^*$ , contradicting the choice of  $v$ .

We conclude that for every  $i \in \{1, \dots, 5\}$  the vertex  $u$  is not in  $W_i$ . Similarly, we may assume  $v \notin W_i$ . This means that the following case holds.

**Case 2:**  $u \in V_i, v \in V_j$  for some  $i, j$ .

Then  $i \neq j$ , since  $V_i$  is an independent set. By Condition 1,  $j \notin \{i-2, i+2\}$ . Without loss of generality, we may therefore assume that  $j = i+1$ . If  $u \in V_i^{**}$  then  $u$  is trivial to  $V_{i+1}$ , so we may assume that  $u \in V_i^*$ . If  $v \in V_{i+1}^*$  then  $v$  is non-trivial to  $V_{i+2}$ , so by Condition 3 it is

trivial to  $V_i$ , contradicting the fact that  $uv$  is significant. Therefore  $v \in V_{i+1}^{**}$ , contradicting the choice of  $v$ .

We conclude that if for some  $i$ ,  $u \in W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}$  and  $v \notin W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}$  are adjacent then the edge  $uv$  is insignificant. Hence we have proven Claim 2.

Note that  $W_i^*$ ,  $V_{i+2}^{**}$ ,  $V_{i+1}^*$  and  $W_{i-2}^{**}$  are independent sets. By Condition 1,  $W_i^*$  is anti-complete to  $V_{i+1}^*$  and  $V_{i+2}^{**}$  is anti-complete to  $W_{i-2}^{**}$ . Therefore  $W_i^* \cup V_{i+1}^*$  and  $V_{i+2}^{**} \cup W_{i-2}^{**}$  are independent sets. Thus  $G[W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}]$  is an  $S_{1,2,3}$ -free bipartite graph, which has bounded clique-width by Lemma 5. Applying a bounded number of bipartite complementations (which we may do by Fact 3), we can separate  $G[W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}]$  from the rest of the graph. We may thus assume that  $W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**} = \emptyset$ . Repeating this process for each  $i$  we obtain the empty graph. This completes the proof. ◀

We can now give the following result, which also implies the  $(K_3, P_1 + 2P_2)$ -free case.

► **Theorem 11.** *For  $H \in \{P_1 + P_5, S_{1,2,2}, P_1 + P_2 + P_3\}$ , the class of  $(K_3, H)$ -free graphs has bounded clique-width.*

**Proof Sketch.** Let  $H \in \{P_1 + P_5, S_{1,2,2}, P_1 + P_2 + P_3\}$  and consider a  $(K_3, H)$ -free graph  $G$ . We may assume that  $G$  is connected, and by Lemma 9, that  $G$  contains an induced cycle on five vertices, say  $C = v_1 - v_2 - \dots - v_5 - v_1$ . Since  $G$  is  $K_3$ -free, no vertex  $v$  is adjacent to two consecutive vertices of  $C$ . Therefore every vertex  $x$  of  $G$  has at most two neighbours on  $C$ , and if  $x$  has two neighbours, then they must be non-consecutive vertices of the cycle. We partition the vertices of  $G$  that are not on  $C$  into a set  $U$  of vertices adjacent to no vertices of  $C$ , sets  $W_i$  of vertices whose unique neighbour in  $C$  is  $v_i$  and sets  $V_i$  of vertices adjacent to  $v_{i-1}$  and  $v_{i+1}$ . Then, what is left to show is how to modify the graph using operations that preserve boundedness of clique-width, such that in the resulting graph the set  $U$  is empty and the partition  $V_1, \dots, V_5, W_1, \dots, W_5$  satisfies Conditions 1–7 of Lemma 10. For full proof details we refer to [17].

To prove our main result, we first consider the case where the graph contains a clique on at least four vertices and show that such graphs have bounded clique-width. Theorem 11 implies that  $(K_3, P_1 + 2P_2)$ -free graphs have bounded clique-width. It is therefore sufficient to consider graphs in the class that contain a  $K_3$ , but not a  $K_4$ . We show that we can either use operations that preserve boundedness of clique-width to modify the graph into one known to have bounded clique-width or else the graph has a very specific structure, in which case we can show that it has bounded clique-width directly. See [17] for details.

► **Theorem 12.** *The class of  $(\text{diamond}, P_1 + 2P_2)$ -free graphs has bounded clique-width.*

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