# Well-Quasi-Ordering versus Clique-Width: New Results on Bigenic Classes*

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**Abstract.** Daligault, Rao and Thomassé conjectured that if a hereditary class of graphs is well-quasi-ordered by the induced subgraph relation then it has bounded clique-width. Lozin, Razgon and Zamaraev recently showed that this conjecture is not true for infinitely defined classes. For finitely defined classes the conjecture is still open. It is known to hold for classes of graphs defined by a single forbidden induced subgraph $H$, as such graphs are well-quasi-ordered and are of bounded clique-width if and only if $H$ is an induced subgraph of $P_4$. For bigenic classes of graphs i.e. ones defined by two forbidden induced subgraphs there are several open cases in both classifications. We reduce the number of open cases for well-quasi-orderability of such classes from 12 to 9. Our results agree with the conjecture and imply that there are only two remaining cases to verify for bigenic classes.

## 1 Introduction

Well-quasi-ordering is a highly desirable property and frequently discovered concept in mathematics and theoretical computer science [16,20]. One of the most remarkable recent results in this area is Robertson and Seymour's proof of Wagner's conjecture, which states that the set of all finite graphs is well-quasi-ordered by the minor relation [25]. One of the first steps towards this result was the proof of the fact that graph classes of bounded treewidth are well-quasi-ordered by the minor relation [24] (a graph parameter $\pi$ is said to be bounded for some graph class $\mathcal{G}$ if there exists a constant $c$ such that $\pi(G) \leq c$ for each $G \in \mathcal{G}$).

The notion of clique-width generalizes that of treewidth in the sense that graph classes of bounded treewidth have bounded clique-width, but not necessarily vice versa. The importance of both notions is due to the fact that many algorithmic problems that are NP-hard on general graphs become polynomial-time solvable when restricted to graph classes of bounded treewidth or clique-width. For treewidth this follows from the meta-theorem of Courcelle [6], combined with

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a result of Bodlaender [2]. For clique-width this follows from combining results from several papers [8,15,18,23] with a result of Oum and Seymour [22].

In the study of graph classes of bounded treewidth, we can restrict ourselves to minor-closed graph classes, because from the definition of treewidth it immediately follows that the treewidth of a graph is never smaller than the treewidth of its minor. This restriction, however, is not justified when we study graph classes of bounded clique-width, as the clique-width of a graph can be much smaller than the clique-width of its minor. In particular, Courcelle [7] showed that if $\mathcal{G}$ is the class of graphs of clique-width 3 and $\mathcal{G}'$ is the class of graphs obtainable from graphs in $\mathcal{G}$ by applying one or more edge contraction operations, then $\mathcal{G}'$ has unbounded clique-width. On the other hand, the clique-width of a graph is never smaller than the clique-width of any of its induced subgraphs (see, for example, [9]). This allows us to restrict ourselves to classes of graphs closed under taking induced subgraphs. Such graph classes are also known as *hereditary* classes.

It is well-known (and not difficult to see) that a class of graphs is hereditary if and only if it can be characterized by a set of minimal forbidden induced subgraphs. Due to the minimality, the set $\mathcal{F}$ of forbidden induced subgraphs is always an antichain, that is, no graph in $\mathcal{F}$ is an induced subgraph of another graph in $\mathcal{F}$. For some hereditary classes this set is finite, in which case we say that the class is *finitely defined*, whereas for other hereditary classes (such as, for instance, bipartite graphs) the set of minimal forbidden induced subgraphs forms an infinite antichain. The presence of these infinite antichains immediately shows that the induced subgraph relation is not a well-quasi-order. In fact there even exist graph classes of bounded clique-width that are not well-quasi-ordered by the induced subgraph relation: take, for example, the class of cycles, which all have clique-width at most 4. What about the inverse implication: does well-quasi-ordering imply bounded clique-width? This was stated as an open problem by Daligault, Rao and Thomassé [13] and a negative answer to this question was recently given by Lozin, Razgon and Zamaraev [21]. However, the latter authors disproved the conjecture by giving a hereditary class of graphs whose set of minimal forbidden induced subgraphs is infinite. Hence, for finitely defined classes the question remains open.

*Conjecture 1.* If a finitely defined class of graphs $\mathcal{G}$ is well-quasi-ordered by the induced subgraph relation, then $\mathcal{G}$ has bounded clique-width.

We emphasize that our motivation for verifying Conjecture 1 is not only mathematical but also algorithmic. Should Conjecture 1 be true, then for finitely defined classes of graphs the aforementioned algorithmic consequences of having bounded clique-width also hold for the property of being well-quasi-ordered by the induced subgraph relation.

A class of graphs is *monogenic* or *$H$-free* if it is characterized by a single forbidden induced subgraph $H$. For monogenic classes, the conjecture is true. In this case, the two notions even coincide: a class of graphs defined by a single forbidden induced subgraph $H$ is well-quasi-ordered if and only if it has bounded clique-width if and only if $H$ is an induced subgraph of $P_4$ (see, for instance, [12,14,19]). A class of graph is *bigenic* or *$(H_1, H_2)$-free* if it is characterized by two incomparable forbidden induced subgraphs $H_1$ and $H_2$. The family of bigenic classes is more diverse than the family of monogenic classes. The questions of well-quasi-orderability and having bounded clique-width still need to be resolved. Recently, considerable progress has been made towards answering the latter question for bigenic classes; see [10] for the most recent survey, which shows that there are currently eight (non-equivalent) open cases. With respect to well-quasi-orderability of bigenic classes, Korpelainen and Lozin [19] left all but 14 cases open. Since then, Atminas and Lozin [1] proved that the class of $(K_3, P_6)$-free graphs is well-quasi-ordered by the induced subgraph relation and that the class of $(\overline{2P_1 + P_2}, P_6)$-free graphs is not, reducing the number of remaining open cases to 12. All available results for bigenic classes verify Conjecture 1. Moreover, eight of the 12 open cases have bounded clique-width (and thus verify Conjecture 1) leaving four remaining open cases of bigenic classes for which we still need to verify Conjecture 1.

**Our Results.** Our first goal is to obtain more (bigenic) classes that are well-quasi-ordered by the induced subgraph relation and to support Conjecture 1 with further evidence. Our second goal is to increase our general knowledge on well-quasi-ordered graph classes and the relation to the possible boundedness of their clique-width.

Towards our first goal we prove in Section 4 that the class of $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs (which has bounded clique-width [11]) is well-quasi-ordered by the induced subgraph relation. We also determine, by giving infinite antichains, two bigenic classes that are not, namely the class of $(\overline{2P_1 + P_2}, P_2 + P_4)$-free graphs, which has unbounded clique-width [11], and the class of $(\overline{P_1 + P_4}, P_1 + 2P_2)$-free graphs, for which boundedness of the clique-width is unknown. Consequently, there are nine classes of $(H_1, H_2)$-free graphs for which we do not know whether they are well-quasi-ordered by the induced subgraph relation, and there are two open cases left for the verification of Conjecture 1 for bigenic classes; see Open Problems 1 and 2 below. See Fig. 1 for drawings of the forbidden induced subgraphs.

Towards our second goal, we aim to develop general techniques as opposed to tackling specific cases in an ad hoc fashion. Our starting point is a very fruitful technique used for determining (un)boundedness of the clique-width of a graph class $\mathcal{G}$. We transform a given graph from $\mathcal{G}$ via a number of elementary graph operations that do not modify the clique-width by "too much" into a graph from a class for which we do know whether or not its clique-width is bounded.

It is a natural question to research how the above modification technique can be used for well-quasi-orders. The permitted elementary graph operations are vertex deletion, subgraph complementation and bipartite complementation. As we will explain in Section 3, these three graph operations do not preserve well-quasi-ordering. We circumvent this by checking whether these three operations preserve boundedness of a graph parameter called uniformicity, which was introduced by Korpelainen and Lozin [19]. In their paper they proved that boundedness of uniformicity is preserved by vertex deletion. Here we prove this for the remaining two graph operations. Korpelainen and Lozin [19] also showed that every graph class $\mathcal{G}$ of bounded uniformicity is well-quasi-ordered by the so-called labelled induced subgraph relation (which in turn implies that $\mathcal{G}$ is well-quasi-ordered by the induced subgraph relation). As the reverse implication does not hold, we sometimes need to rely only on the labelled induced subgraph relation. Hence, in Section 3 we also show that the three permitted graph operations preserve well-quasi-orderability by the labelled induced subgraph relation. We believe that the graph modification technique will also be useful for proving well-quasi-orderability of other graph classes. As such, we view the results in Section 3 as our second main contribution.

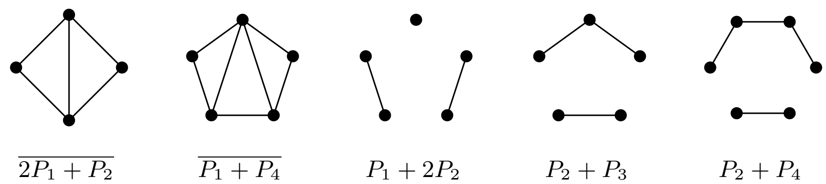


**Fig. 1.** The forbidden induced subgraphs considered in this paper.

**Future Work.** We identify several potential directions for future work starting with the two remaining bigenic classes for which Conjecture 1 must still be verified.

**Open Problem 1** *Is Conjecture 1 true for the class of* $(H_1, H_2)$*-free graphs when:* $H_1 = K_3$ *and* $H_2 = P_2 + P_4$ *or when* $H_1 = \overline{P_1 + P_4}$ *and* $H_2 = P_2 + P_3$*?*

For both classes we know neither whether they are well-quasi-ordered by the induced subgraph relation nor whether their clique-width is bounded. Below we list all seven classes of $(H_1, H_2)$-free graphs for which we do not know whether they are well-quasi-ordered by the induced subgraph relation.

**Open Problem 2** *Is the class of* $(H_1, H_2)$*-free graphs well-quasi-ordered by the induced subgraph relation when:*

*(i)* $H_1 = \overline{3P_1}$ *and* $H_2 \in \{P_1 + 2P_2, P_1 + P_5, P_2 + P_4\}$*;*
*(ii)* $H_1 = \overline{2P_1 + P_2}$ *and* $H_2 \in \{P_1 + 2P_2, P_1 + P_4\}$*;*
*(iii)* $H_1 = \overline{P_1 + P_4}$ *and* $H_2 \in \{P_1 + P_4, 2P_2, P_2 + P_3, P_5\}$.

In relation to this, we mention that the infinite antichain for $(\overline{P_1 + P_4}, P_1 + 2P_2)$-free graphs was initially found by a computer search. This computer search also showed that similar antichains do not exist for any of the remaining nine open cases. As such, constructing antichains for these cases is likely to be a challenging problem and this suggests that many of these cases may in fact be well-quasi-ordered. Some of these remaining classes have been shown to have bounded clique-width [3,4,5,10]. We believe that some of the structural characterizations for proving these results may be useful for showing well-quasi-orderability. Indeed, we are currently trying to prove that the class of $(K_3, P_1 + P_5)$-free graphs is well-quasi-ordered via the technique of bounding the so-called lettericity for graphs in these classes. Again, applying complementations and vertex deletions does not change the lettericity of a graph by "too much".

Another potential direction for future research is investigating linear clique-width for classes defined by two forbidden induced subgraphs. Indeed, it is not hard to show that $k$-uniform graphs have bounded linear clique-width. Again, we can use complementations and vertex deletions when dealing with this parameter.

## 2 Preliminaries

The *disjoint union* $(V(G) \cup V(H), E(G) \cup E(H))$ of two vertex-disjoint graphs $G$ and $H$ is denoted by $G + H$ and the disjoint union of $r$ copies of a graph $G$ is denoted by $rG$. The *complement* of a graph $G$, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and only if these vertices are not adjacent in $G$. For a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of $G$ *induced* by $S$, which has vertex set $S$ and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. If $S = \{s_1, \ldots, s_r\}$ then, to simplify notation, we may also write $G[s_1, \ldots, s_r]$ instead of $G[\{s_1, \ldots, s_r\}]$. We use $G \setminus S$ to denote the graph obtained from $G$ by deleting every vertex in $S$, i.e. $G \setminus S = G[V(G) \setminus S]$.

The graphs $C_r, K_r, K_{1,r-1}$ and $P_r$ denote the cycle, complete graph, star and path on $r$ vertices, respectively. For a set of graphs $\{H_1, \ldots, H_p\}$, a graph $G$ is $(H_1, \ldots, H_p)$-*free* if it has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we may write $H_1$-free instead of $(H_1)$-free.

For a graph $G = (V, E)$, the set $N(u) = \{v \in V \mid uv \in E\}$ denotes the neighbourhood of $u \in V$. A graph is *bipartite* if its vertex set can be partitioned into (at most) two independent sets. The *biclique* $K_{r,s}$ is the bipartite graph with sets in the partition of size $r$ and $s$ respectively, such that every vertex in one set is adjacent to every vertex in the other set. Let $X$ be a set of vertices of a graph $G = (V, E)$. A vertex $y \in V \setminus X$ is *complete* to $X$ if it is adjacent to every vertex of $X$ and *anti-complete* to $X$ if it is non-adjacent to every vertex of $X$. Similarly, a set of vertices $Y \subseteq V \setminus X$ is *complete* (resp. *anti-complete*) to $X$ if every vertex in $Y$ is complete (resp. anti-complete) to $X$. A vertex $y \in V \setminus X$ *distinguishes* $X$ if $y$ has both a neighbour and a non-neighbour in $X$. The set $X$ is a *module* of $G$ if no vertex in $V \setminus X$ distinguishes $X$. A module $U$ is *non-trivial* if $1 < |U| < |V|$, otherwise it is *trivial*. A graph is *prime* if it has only trivial modules.

A *quasi order* $\leq$ on a set $X$ is a reflexive, transitive binary relation. Two elements $x, y \in X$ in this quasi-order are *comparable* if $x \leq y$ or $y \leq x$, otherwise they are *incomparable*. A set of elements in a quasi-order is a *chain* if every pair of elements is comparable and it is an *antichain* if every pair of elements is incomparable. The quasi-order $\leq$ is a *well-quasi-order* if any infinite sequence of elements $x_1, x_2, x_3, \ldots$ in $X$ contains a pair $(x_i, x_j)$ with $x_i \leq x_j$ and $i < j$. Equivalently, a quasi-order is a well-quasi-order if and only if it has no infinite strictly decreasing sequence $x_1 \gneq x_2 \gneq x_3 \gneq \cdots$ and no infinite antichain. For an arbitrary set $M$, let $M^*$ denote the set of finite sequences of elements of $M$. Any quasi-order $\leq$ on $M$ defines a quasi-order $\leq^*$ on $M^*$ as follows: $(a_1, \ldots, a_m) \leq^* (b_1, \ldots, b_n)$ if and only if there is a sequence of integers $i_1, \ldots, i_m$ with $1 \leq i_1 < \cdots < i_m \leq n$ such that $a_j \leq b_{i_j}$ for $j \in \{1, \ldots, m\}$. We call $\leq^*$ the *subsequence relation.*

**Lemma 1 (Higman's Lemma [17]).** *If $(M, \leq)$ is a well-quasi-order then $(M^*, \leq^*)$ is a well-quasi-order.*

**Labelled Induced Subgraphs and Uniformicity.** To define the notion of labelled induced subgraphs, let us consider an arbitrary quasi-order $(W, \leq)$. We say that $G$ is a *labelled* graph if each vertex $v$ of $G$ is equipped with an element $l_G(v) \in W$ (the *label* of $v$). Given two labelled graphs $G$ and $H$, we say that $G$ is a *labelled induced subgraph* of $H$ if $G$ is isomorphic to an induced subgraph of $H$ and there is an isomorphism that maps each vertex $v$ of $G$ to a vertex $w$ of $H$ with $l_G(v) \leq l_H(w)$. Clearly, if $(W, \leq)$ is a well-quasi-order then a class of graphs $X$ cannot contain an infinite sequence of labelled graphs that is strictly-decreasing with respect to the labelled induced subgraph relation. We therefore say that a class of graphs $X$ is well-quasi-ordered by the *labelled* induced subgraph relation if it contains no infinite antichains of labelled graphs whenever $(W, \leq)$ is a *well*-quasi-order. Such a class is readily seen to be well-quasi-ordered by the induced subgraph relation as well. We will use the following three results.

**Lemma 2 ([1]).** *The class of $P_6$-free bipartite graphs is well-quasi-ordered by the labelled induced subgraph relation.*

**Lemma 3 ([1]).** *Let $k, \ell, m$ be positive integers. Then the class of $(P_k, K_\ell, K_{m,m})$-free graphs is well-quasi-ordered by the labelled induced subgraph relation.*

**Lemma 4 ([1]).** *Let $X$ be a hereditary class of graphs. Then $X$ is well-quasi-ordered by the labelled induced subgraph relation if and only if the set of prime graphs in $X$ is. In particular, $X$ is well-quasi-ordered by the labelled induced subgraph relation if and only if the set of connected graphs in $X$ is.*

Let $k$ be a natural number, let $K$ be a symmetric square $0, 1$ matrix of order $k$, and let $F_k$ be a graph on the vertex set $\{1, 2, \ldots, k\}$. Let $H$ be the disjoint union of infinitely many copies of $F_k$, and for $i = 1, \ldots, k$, let $V_i$ be the subset of $V(H)$ containing vertex $i$ from each copy of $F_k$. Now we construct from $H$ an infinite graph $H(K)$ on the same vertex set by applying a subgraph complementation to $V_i$ if and only if $K(i, i) = 1$ and by applying bipartite complementation to a pair $V_i, V_j$ if and only if $K(i, j) = 1$. In other words, two vertices $u \in V_i$ and $v \in V_j$ are adjacent in $H(K)$ if and only if $uv \in E(H)$ and $K(i, j) = 0$ or $uv \notin E(H)$ and $K(i, j) = 1$. Finally, let $\mathcal{P}(K, F_k)$ be the hereditary class consisting of all the finite induced subgraphs of $H(K)$.

Let $k$ be a natural number. A graph $G$ is *$k$-uniform* if there is a matrix $K$ and a graph $F_k$ such that $G \in \mathcal{P}(K, F_k)$. The minimum $k$ such that $G$ is $k$-uniform is the *uniformicity* of $G$.

The following result was proved by Korpelainen and Lozin. The class of disjoint unions of cliques is a counterexample for the reverse implication.

**Theorem 1 ([19]).** *Any class of graphs of bounded uniformicity is well-quasi-ordered by the labelled induced subgraph relation.*

## 3 Permitted Graph Operations

It is not difficult to see that if $G$ is an induced subgraph of $H$, then $\overline{G}$ is an induced subgraph of $\overline{H}$. Therefore, a graph class $X$ is well-quasi-ordered by the induced subgraph relation if and only if the set of complements of graphs in $X$ is. In this section, we strengthen this observation in several ways. *Subgraph complementation* in a graph $G$ is the operation of complementing a subgraph of $G$ induced by a subset of its vertices. Applied to the entire vertex set of $G$, this operation coincides with the usual complementation of $G$. However, applied to a pair of vertices, it changes the adjacency of these vertices only. Clearly, repeated applications of this operation can transform $G$ into any other graph on the same vertex set. Therefore, unrestricted applications of subgraph complementation may transform a well-quasi-ordered class $X$ into a class containing infinite antichains. However, if we bound the number of applications of this operation by a constant, we preserve many nice properties of $X$, including well-quasi-orderability with respect to the labelled induced subgraph relation. Next, we introduce the following operations. *Bipartite complementation* in a graph $G$ is the operation of complementing the edges between two disjoint subsets $X, Y \subseteq V(G)$. Note that applying a bipartite complementation between $X$ and $Y$ has the same effect as applying a sequence of three complementations: with respect to $X$, $Y$ and $X \cup Y$. Finally, we define the following operation: *Vertex deletion* in a graph $G$ is the operation of removing a single vertex $v$ from a graph, together with any edges incident to $v$.

Let $k \geq 0$ be a constant and let $\gamma$ be a graph operation. A graph class $\mathcal{G}'$ is $(k, \gamma)$*-obtained* from a graph class $\mathcal{G}$ if (i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing $\gamma$ at most $k$ times, and (ii) for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from $G$ by performing $\gamma$ at most $k$ times. We say that $\gamma$ *preserves* well-quasi-orderability by the labelled induced subgraph relation if for any finite constant $k$ and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is $(k, \gamma)$-obtained from $\mathcal{G}$ is well-quasi-ordered by this relation if and only if $\mathcal{G}$ is.

**Lemma 5.** *The following operations preserve well-quasi-orderability by the labelled induced subgraph relation:*

*(i) Subgraph complementation,*
*(ii) Bipartite complementation and*
*(iii) Vertex deletion.*

*Proof.* We start by proving the lemma for subgraph complementations. Let $X$ be a class of graphs and $Y$ be a set of graphs obtained from $X$ by applying a subgraph complementation to each graph in $X$. More precisely, for each graph

$G \in X$ we choose a set $Z_G$ of vertices in $G$; we let $G'$ be the graph obtained from $G$ by applying a complementation with respect to the subgraph induced by $Z_G$ and we let $Y$ be the set of graphs $G'$ obtained in this way. Clearly it is sufficient to show that $X$ is well-quasi-ordered by the labelled induced subgraph relation if and only if $Y$ is.

Suppose that $X$ is not well-quasi-ordered under the labelled induced subgraph relation. Then there must be a well-quasi-order $(L, \leq)$ and an infinite sequence of graphs $G_1, G_2, \dots$ in $\mathcal{X}$ with vertices labelled with elements of $L$, such that these graphs form an infinite antichain under the labelled induced subgraph relation. Let $(L', \leq')$ be the quasi-order with $L' = \{(k, l) : k \in \{0, 1\}, l \in L\}$ and $(k, l) \leq' (k', l')$ if and only if $k = k'$ and $l \leq l'$ (so $L'$ is the disjoint union of two copies of $L$, where elements of one copy are incomparable with elements in the other copy). Note that $(L', \leq')$ is a well-quasi-order since $(L, \leq)$ is a well-quasi-order.

For each graph $G_i$ in this sequence, with labelling $l_i$, we construct the graph $G'_i$ (recall that $G'_i$ is obtained from $G_i$ by applying a complementation on the vertex set $Z_{G_i}$). We label the vertices of $V(G'_i)$ with a labelling $l'_i$ as follows: set $l'_i(v) = (1, l_i(v))$ if $v \in Z_{G_i}$ and set $l'_i(v) = (0, l_i(v))$ otherwise.

We claim that when $G'_1, G'_2, \dots$ are labelled in this way they form an infinite antichain with respect to the labelled induced subgraph relation. Indeed, suppose for contradiction that $G'_i$ is a labelled induced subgraph of $G'_j$ for some $i \neq j$. This means that there is a injective map $f : V(G'_i) \to V(G'_j)$ such that $l'_i(v) \leq' l'_j(f(v))$ for all $v \in V(G'_i)$ and $v, w \in V(G'_i)$ are adjacent in $G'_i$ if and only if $f(v)$ and $f(w)$ are adjacent in $G'_j$. Now since $l'_i(v) \leq' l'_j(f(v))$ for all $v \in V(G'_i)$, by the definition of $\leq'$ we conclude the following: $l_i(v) \leq l_j(f(v))$ for all $v \in V(G'_i)$ and $v \in Z_{G_i}$ if and only if $f(v) \in Z_{G_j}$.

Suppose $v, w \in V(G_i)$ with $w \notin Z_{G_i}$ ($v$ may or may not belong to $Z_{G_i}$) and note that this implies $f(w) \notin Z_{G_j}$. Then $v$ and $w$ are adjacent in $G_i$ if and only if $v$ and $w$ are adjacent in $G'_i$ if and only if $f(v)$ and $f(w)$ are adjacent in $G'_j$ if and only if $f(v)$ and $f(w)$ are adjacent in $G_j$.

Next suppose $v, w \in Z_{G_i}$, in which case $f(v), f(w) \in Z_{G_j}$. Then $v$ and $w$ are adjacent in $G_i$ if and only if $v$ and $w$ are non-adjacent in $G'_i$ if and only if $f(v)$ and $f(w)$ are non-adjacent in $G'_j$ if and only if $f(v)$ and $f(w)$ are adjacent in $G_j$.

It follows that $f$ is an injective map $f : V(G_i) \to V(G_j)$ such that $l_i(v) \leq l_j(f(v))$ for all $v \in V(G_i)$ and $v, w \in V(G_i)$ are adjacent in $G_i$ if and only if $f(v)$ and $f(w)$ are adjacent in $G_j$. In other words $G_i$ is a labelled induced subgraph of $G_j$. This contradiction means that if $G_1, G_2, \dots$ is an infinite antichain then $G'_1, G'_2, \dots$ must also be an infinite antichain.

Therefore, if the class $X$ is not well-quasi-ordered by the labelled induced subgraph relation then neither is $Y$. Repeating the argument with the roles of $G_1, G_2, \dots$ and $G'_1, G'_2, \dots$ reversed shows that if $Y$ is not well-quasi-ordered under the labelled induced subgraph relation then neither is $X$. This completes the proof for subgraph complementations.

Since a bipartite complementation is equivalent to doing three subgraph complementations one after another, the result for bipartite complementations follows. Hence it remains to prove the result for vertex deletions. Let $X$ be a class of graphs and let $Y$ be a set of graphs obtained from $X$ by deleting exactly one vertex $z_G$ from each graph $G$ in $X$. We denote the obtained graph by $G - z_G$. Clearly it is sufficient to show that $X$ is well-quasi-ordered by the labelled induced subgraph relation if and only if $Y$ is.

Suppose that $Y$ is well-quasi-ordered by the labelled induced subgraph relation. We will show that $X$ is also a well-quasi-order by this relation. For each graph $G \in X$, let $G'$ be the graph obtained from $G$ by applying a bipartite complementation between $\{z_G\}$ and $N(z_G)$, so $z_G$ is an isolated vertex in $G'$. Let $Z$ be the set of graphs obtained in this way. By Lemma 5.(ii), $Z$ is a well-quasi-order by the labelled induced subgraph relation if and only if $X$ is. Suppose $G_1, G_2$ are graphs in $Z$ with vertices labelled from some well-quasi-order $(L, \leq)$. Then for $i \in \{1, 2\}$ the vertex $z_{G_i}$ has a label from $L$ and the graph $G_i - z_{G_i}$ belongs to $Y$. Furthermore if $G_1 - z_{G_1}$ is a labelled induced subgraph of $G_2 - z_{G_2}$ and $l_{G_1}(z_{G_1}) \leq l_{G_2}(z_{G_2})$ then $G_1$ is a labelled induced subgraph of $G_2$. Now by Lemma 1 it follows that $Z$ is well-quasi-ordered by the labelled induced subgraph relation. Therefore $X$ is also well-quasi-ordered by this relation.

Now suppose that $Y$ is not well-quasi-ordered by the labelled induced subgraph relation. Then $Y$ contains an infinite antichain $G_1, G_2, \ldots$ with the vertices of $G_i$ labelled by functions $l_i$ which takes values in some well-quasi-order $(L, \leq)$. For each $G_i$, let $G'_i$ be a corresponding graph in $X$, so $G_i = G'_i - z_{G'_i}$. Then in $G'_i$ we label $z_{G'_i}$ with a new label $*$ and label all other vertices $v \in V(G'_i)$ with the same label as that used in $G_i$. We make this new label $*$ incomparable to all the other labels in $L$ and note that the obtained quasi order $(L \cup \{*\}, \leq)$ is also a well-quasi-order. It follows that $G'_1, G'_2, \ldots$ is an antichain in $X$ when labelled in this way. Therefore, if $Y$ is not well-quasi-ordered by the labelled induced subgraph relation then $X$ is not either. This completes the proof. □

The above lemmas only apply to well-quasi-ordering with respect to the *labelled* induced subgraph relation. Indeed, if we take a cycle and delete a vertex, complement the subgraph induced by an edge or apply a bipartite complementation to two adjacent vertices, we obtain a path. However, while the set of cycles is an infinite antichain with respect to the induced subgraph relation, the set of paths is not.

We now show that our graph operations do not change uniformicity by "too much" either. The result for vertex deletion this was proved by Korpelainen and Lozin. We omit the proof of the remaining two operations.

**Lemma 6.** *Let $G$ be a graph of uniformicity $k$. Let $G'$, $G''$ and $G'''$ be graphs obtained from $G$ by applying one vertex deletion, subgraph complementation or bipartite complementation, respectively. Let $\ell'$, $\ell''$ and $\ell'''$ be the uniformicities of $G$, $G'$ and $G''$, respectively. Then the following three statements hold:*

*(i)* $\ell' < k < 2\ell' + 1$ [19];
*(ii)* $\frac{k}{2} \leq \ell'' \leq 2k$;
*(iii)* $\frac{k}{3} \leq \ell''' \leq 3k$.

## 4 One New WQO Class and Two New Non-WQO Classes

In this section we show that $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs are well-quasi-ordered by the labelled induced subgraph relation. We divide the proof into several sections, depending on whether or not the graphs under consideration contain certain induced subgraphs or not. We follow the general scheme that Dabrowski, Huang and Paulusma [11] used to prove that this class has bounded clique-width, but we will also need a number of new arguments. We first consider graphs containing a $K_5$ and state the following lemma (proof omitted).

**Lemma 7.** *The class of $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs that contain a $K_5$ is well-quasi-ordered by the labelled induced subgraph relation.*

By Lemma 7, we may restrict ourselves to looking at $K_5$-free graphs in our class. We now consider the case where these graphs have an induced $C_5$ (proof omitted).

**Lemma 8.** *The class of $(\overline{2P_1 + P_2}, P_2 + P_3, K_5)$-free graphs that contain an induced $C_5$ has bounded uniformicity.*

By Lemmas 7 and 8, we may restrict ourselves to looking at $(K_5, C_5)$-free graphs in our class. We need the following structural result (proof omitted).

**Lemma 9.** *Let $G$ be a $(\overline{2P_1 + P_2}, P_2 + P_3, K_5, C_5)$-free graph containing an induced $C_4$. Then by deleting at most 17 vertices and applying at most two bipartite complementations, we can modify $G$ into the disjoint union of a $P_2+P_3$-free bipartite graph and a 3-uniform graph.*

Since $P_2 + P_3$ is an induced subgraph of $P_6$, it follows that every $P_2 + P_3$-free graph is $P_6$-free. Combining Lemma 9 with Theorem 1 and Lemmas 2, 4, 5.(ii) and 5.(iii) we therefore obtain the following corollary.

**Corollary 1.** *The class of connected $(\overline{2P_1 + P_2}, P_2+P_3, K_5, C_5)$-free graphs with an induced $C_4$ is well-quasi-ordered by the labelled induced subgraph relation.*

**Theorem 2.** *The class of $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs is well-quasi-ordered by the labelled induced subgraph relation.*

*Proof.* Graphs in the class under consideration containing an induced subgraph isomorphic to $K_5$, $C_5$ or $C_4$ are well-quasi-ordered by the labelled induced subgraph relation by Lemmas 7 and 8 and Corollary 1, respectively. The remaining graphs form a subclass of $(P_6, K_5, K_{2,2})$-free graphs, since $C_4 = K_{2,2}$ and $P_2 + P_3$ is an induced subgraph of $P_6$. By Lemma 3, this class of graphs is well-quasi-ordered by the labelled induced subgraph relation. Therefore, the class of $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs is well-quasi-ordered by the labelled induced subgraph relation. □

Our final two results show that the classes of $(\overline{2P_1 + P_2}, P_2 + P_4)$-free graphs and $(\overline{P_1 + P_4}, P_1+2P_2)$-free graphs are not well-quasi-ordered by the induced subgraph relation. The antichain used to prove the first of these cases was previously used by Atminas and Lozin to show that the class of $(\overline{2P_1 + P_2}, P_6)$-free graphs is not well-quasi-ordered with respect to the induced subgraph relation. Because of this, we can show show a stronger result for the first case (proof omitted).

**Theorem 3.** *The class of* $(\overline{2P_1+P_2}, P_2+P_4, P_6)$*-free graphs is not well-quasi-ordered by the induced subgraph relation.*

**Theorem 4.** *The class of* $(\overline{P_1+P_4}, P_1+2P_2)$*-free graphs is not well-quasi-ordered by the induced subgraph relation.*

*Proof.* Let $n \geq 3$ be an integer. Consider a cycle $C_{4n}$, say $x_1 - x_2 - \cdots - x_{4n} - x_1$. We partition the vertices of $C_{4n}$ into the set $X = \{x_i \mid i \equiv 0 \text{ or } 1 \mod 4\}$ and $Y = \{x_i \mid i \equiv 2 \text{ or } 3 \mod 4\}$. Next we apply a complementation to each of $X$ and $Y$, so that in the resulting graph $X$ and $Y$ each induce a clique on $2n$ vertices with a perfect matching removed. Let $G_{4n}$ be the resulting graph.

Suppose, for contradiction that $G_{4n}$ contains an induced $P_1+2P_2$. Without loss of generality, the set $X$ must contain three of the vertices $v_1, v_2, v_3$ of the $P_1+2P_2$. Since every component of $P_1+2P_2$ contains at most two vertices, without loss of generality we may assume $v_1$ is non-adjacent to both $v_2$ and $v_3$. However, every every vertex of $G_{4n}[X]$ has exactly one non-neighbour in $X$. This contradiction shows that $G_{4n}$ is indeed $(P_1+2P_2)$-free.

Every vertex in $X$ has exactly one neighbour in $Y$ and vice versa. This means that any $K_3$ in $G_{4n}$ must lie entirely in $G_{4n}[X]$ or $G_{4n}[Y]$. Since $G_{4n}[X]$ or $G_{4n}[Y]$ are both complements of perfect matchings and every vertex of $\overline{P_1+P_4}$ lies in one of three induced $K_3$'s, which are pairwise non-disjoint, it follows that $G_{4n}$ is $\overline{P_1+P_4}$-free.

It remains to show that the graphs $G_{4n}$ form an infinite antichain with respect to the induced subgraph relation. Since $n \geq 3$, every vertex in $X$ (resp. $Y$) has at least two neighbours in $X$ (resp. $Y$) that are pairwise adjacent. Therefore, given $x_1$, we can determine which vertices lie in $X$ and which lie in $Y$. Every vertex in $X$ (resp. $Y$) has a unique neighbour in $Y$ (resp. $X$) and a unique non-neighbour in $X$ (resp. $Y$). Therefore, by specifying which vertex in $G_{4n}$ is $x_1$, we uniquely determine $x_2, \ldots, x_{4n}$. Suppose $G_{4n}$ is an induced subgraph of $G_{4m}$ for some $m \geq 3$. Then $n \leq m$ due to the number of vertices. By symmetry, we may assume that the induced copy of $G_{4n}$ in $G_{4m}$ has vertex $x_1$ of $G_{4n}$ in the position of vertex $x_1$ in $G_{4m}$. Then the induced copy of $G_{4n}$ must have vertices $x_2, \ldots, x_{4n}$ in the same position as $x_2, \ldots, x_{4n}$ in $G_{4m}$, respectively. Now $x_1$ and $x_{4n}$ are non-adjacent in $G_{4n}$. If $n < m$ then $x_1$ and $x_{4n}$ are adjacent in $G_{4m}$, a contradiction. We conclude that if $G_{4n}$ is an induced subgraph of $G_{4m}$ then $n = m$. In other words $\{G_{4n} \mid n \geq 3\}$ is an infinite antichain with respect to the induced subgraph relation. □

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