Reducing the Chromatic Number by Vertex or Edge Deletions

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Abstract

A vertex or edge in a graph is critical if its deletion reduces the chromatic number of the graph by 1. We consider the problems of testing whether a graph has a critical vertex or edge, respectively. We give a complete classification of the complexity of both problems for H-free graphs, that is, graphs with no induced subgraph isomorphic to H. Moreover, we show that an edge is critical if and only if its contraction reduces the chromatic number by 1. Hence, we obtain the same classification for the problem of testing if a graph has an edge whose contraction reduces the chromatic number by 1. As a consequence of our results, we are also able to complete the complexity classification of the more general vertex deletion and edge contraction blocker problems for H-free graphs when the graph parameter is the chromatic number.

Keywords: edge contraction, vertex deletion, chromatic number.

1 Introduction

A vertex or edge of a graph G is *critical* if its removal reduces the chromatic number $\chi(G)$ by 1. An edge is *contraction-critical* if its contraction reduces $\chi(G)$ by 1. We call the problems of deciding if a graph has a critical vertex, critical edge or contraction-critical edge Critical Vertex, Critical Edge and Contraction-Critical Edge, respectively. It is not difficult to show that these three problems are computationally hard in general. Here we classify the computational complexity of them for graphs with no induced subgraph isomorphic to some specified graph H; we call such graphs H*-free*.

Each of the above decision problems can be generalized as follows. Let S be a fixed set of one or more graph operations, and let π be some fixed graph parameter. Given a graph G, and an integer k, we ask if G can be modified into a graph G' by using at most k operations from S so that $\pi(G') \leq \pi(G) - d$ for some given *threshold* $d \geq 0$. Such problems are called *blocker problems*, as the vertices or edges involved "block" some desirable graph property (such as being colorable with only a few colors). Over the last few years, blocker problems have been well studied, see for instance [1,2,3,4,5,9,10,11,12]. In these papers the set S consists of a single operation that is either a vertex deletion vd, edge deletion ed, or edge contraction ec. The decision problems are called Vertex Deletion Blocker(π) if $S = \{\mathsf{vd}\}$, Edge Deletion Blocker(π) if $S = \{\mathsf{ed}\}$ and Contraction Blocker(π) if $S = \{\mathsf{ec}\}$. By taking $d = k = 1$ and $\pi = \chi$ we obtain the problems Critical Vertex, Critical Edge and Contraction-Critical Edge. The complexities of Vertex Deletion Blocker(χ) and Contraction Blocker(χ) are known for H-free graphs if H is connected [10]. As a consequence of our results for $k = d = 1$, we can complete these two classifications for all graphs H, just as we did in a previous paper [11] for $\pi = \alpha$ (independence number) and $\pi = \omega$ (clique number), except for the case when $\pi = \omega$, $S = \{\mathsf{ec}\}$ and $H = C_3 + P_1$. The Edge Deletion Blocker(χ) problem is known [1] to be polynomial-time solvable for threshold graphs and NP-hard for cobipartite graphs. For this problem we obtain a partial classification that leaves exactly two cases open.

Terminology. The graph $G+G'$ is the disjoint union of the graphs G and G'. The graph pG is the disjoint union of p copies of G. We let K_n, P_n and C_n be the complete graph, path and cycle on n vertices, respectively. For a subset

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$S \subseteq V$ of a graph G, we let $G[S]$ be the subgraph of G *induced* by S. We write $H \subseteq_i G$ if H is an induced subgraph of G. The graph $\overline{G}$ is the complement of G. A graph G is $(H_1, \ldots, H_p)$*-free* if G is H-free for every $H \in \{H_1, \ldots, H_p\}$. The contraction of an edge uv removes u and v from V and replaces them by a new vertex made adjacent to precisely those vertices adjacent to u or v in G.

2 Critical Vertices and Edges

We start with the following result (proof omitted).

Proposition 2.1 *An edge is critical if and only if it is contraction-critical.*

Due to Proposition 2.1, Critical Edge and Contraction-Critical Edge have the same classification for H-free graphs.

Let Coloring be the problem of deciding whether $\chi(G) \leq k$ for some given integer $k \geq 1$. We need the following known result.

Theorem 2.2 ([7]) *If a graph $H \subseteq_i P_4$ or of $H \subseteq_i P_1 + P_3$, then* Coloring *is polynomial-time solvable for H-free graphs, otherwise it is* NP*-complete.*

We also need the following lemma.

Lemma 2.3 ([10]) *Let H be a graph. Then* Critical Vertex *and* Critical Edge *are* NP*-hard for H-free graphs if $H \supseteq_i K_{1,3}$ or $H \supseteq_i C_r$ for some $r \geq 3$.*

The *clique covering number* $\sigma(G)$ of a graph G is the smallest number of cliques in a graph so that each vertex belongs to exactly one clique. The hardness construction in the (omitted) proof of our next result uses clique covers and some other elements of the proof of Theorem 2 in [7] for showing that Coloring is NP-hard for $(C_5, 4P_1, P_1 + 2P_2, 2P_2)$-free graphs.

Theorem 2.4 Critical Vertex *and* Critical Edge *are both co-*NP*-hard for $(C_5, 4P_1, 2P_1 + P_2, 2P_2)$-free graphs.*

We can now prove the following dichotomies.

Theorem 2.5 *Let H be a graph. Then* Critical Vertex*,* Critical Edge *and* Contraction-Critical Edge *restricted to H-free graphs are polynomial-time solvable if $H \subseteq_i P_1 + P_3$ or $H \subseteq_i P_4$, and* NP*-hard or co-*NP*-hard otherwise.*

Proof. Let $H \subseteq_i P_1 + P_3$ or $H \subseteq_i P_4$. Let G be an H-free graph. By Theorem 2.2 we can compute $\chi(G)$ in polynomial time. We note that any vertex deletion results in a graph that is H-free as well. Hence in order to solve Critical Vertex we can compute the chromatic number of $G - v$ for each vertex v in polynomial time and compare it with $\chi(G)$. As $(P_1 + P_3)$-free graphs and P_4-free graphs are closed under edge contraction as well, we can follow the same approach for solving Contraction-Critical Edge. By Proposition 2.1 we obtain the same result for Critical Edge.

Now suppose that neither $H \subseteq_i P_1 + P_3$ nor $H \subseteq_i P_4$. By Proposition 2.1 it suffices to consider Critical Vertex and Critical Edge. If H has a cycle or an induced claw, then we use Lemma 2.3. Assume not. Then H is a disjoint union of r paths for some $r \geq 1$. If $r \geq 4$ we use Theorem 2.4. If $r = 3$ then either $H = 3P_1 \subseteq_i P_1 + P_3$, which is not possible, or $H \supseteq_i 2P_1 + P_2$ meaning that we can apply Theorem 2.4 again. Suppose $r = 2$. If both paths contain an edge, then $2P_2 \subseteq_i H$. If at most one path has edges, then it must have at least four vertices, as otherwise $H \subseteq_i P_1 + P_3$. This means that $2P_1 + P_2 \subseteq_i H$. In both cases we apply Theorem 2.4. If $r = 1$, then H is a path on at least five vertices, which means $2P_2 \subseteq_i H$. We apply Theorem 2.4 again. □

3 Vertex Deletion and Contraction Blocker Problems

We need the following two results from our previous papers as lemmas.

Lemma 3.1 ([5]) Contraction Blocker(χ) *and* Vertex Deletion Blocker(χ) *are polynomial-time solvable for P_4-free graphs.*

Lemma 3.2 ([10]) *For $3P_1$-free graphs,* Contraction Blocker(χ) *is* NP*-hard, but* Vertex Deletion Blocker(χ) *problem is polynomial-time solvable.*

A graph G is *complete multipartite* if $V(G)$ can be partitioned into k independent sets $V_1, \ldots, V_k$ for some integer k, such that two vertices are adjacent if and only if they belong to two different sets V_i and V_j. The graph $\overline{P_1 + P_3}$ is also known as the *paw*. We need a result of Olariu on paw-free graphs.

Lemma 3.3 ([8]) *Every connected $\overline{P_1 + P_3}$-free graph is either triangle-free or complete multipartite.*

Two disjoint subsets of vertices in a graph are *complete* if there is an edge between every vertex of A and every vertex of B. Lemma 3.3 implies the following lemma, which we use to prove Proposition 3.5 (proofs omitted).

Lemma 3.4 *The vertex set of every (P_1+P_3)-free graph G can be decomposed into two disjoint sets A and B such that $G[A]$ is $3P_1$-free, $G[B]$ is P_4-free and A and B are complete to each other.*

Proposition 3.5 Vertex Deletion Blocker(χ) *problem is polynomial-time solvable for $(P_1 + P_3)$-free graphs.*

We can now state the following two dichotomies. The first dichotomy was shown in [10] already. The tractable cases of the second dichotomy follow from Lemmas 3.1 and Proposition 3.5, and the hard cases from Theorem 2.5. Note that due to Lemma 3.2 the two dichotomies are not the same.

Theorem 3.6 *Let H be a graph. Then the following holds:*

- *If $H \subseteq_i P_4$, then* Contraction Blocker(χ) *for H-free graphs is polynomial-time solvable for H-free graphs, and it is* NP*-hard otherwise.*
- *If $H \subseteq_i P_1 + P_3$ or P_4, then* Vertex Deletion Blocker(χ) *for H-free graphs is polynomial-time solvable, and it is* NP*-hard or co-*NP*-hard otherwise.*

4 Conclusions

We gave complete complexity classifications of five problems Critical Vertex, Critical Edge, Contraction-Critical Edge, Vertex Deletion Blocker(χ) and Contraction Blocker(χ) for H-free graphs. The classifications for the first four problems coincide with the known classification of Coloring for H-free graphs (see Theorem 2.2), whereas the case $H = P_1 + P_3$ is no longer tractable for the latter problem. We finish our paper with two directions for future work.

First, we have no full complexity classification for the Edge Deletion Blocker(χ) problem for H-free graphs. This problem has been less studied than the vertex deletion and edge contraction variant. The reason for this is that H-free graphs are not closed under taking edge deletions, whereas they are closed under vertex deletions and in case when H is a linear forest under edge contractions as well. Proposition 2.1 combined with Lemma 2.3 shows that the problem is NP-hard if H is not the disjoint union of paths. Bazgan et al. [1] showed that Edge Deletion Blocker(χ) is polynomial-time solvable for threshold graphs, that is, for $(C_4, 2P_2, P_4)$-free graphs, and NP-hard for cobipartite graphs, that is, for complements of bipartite graphs. The latter class is a subclass of the class of $3P_1$-free graphs. The above combined with Theorem 2.4 leads to a classification of Edge Deletion Blocker(χ) for H-free graphs up to two open cases, namely when $H = P_1 + P_2$ or $H = P_4$.

The second research direction is to classify the complexity of these six problems for graphs classes characterized by a family $\{H_1, \ldots, H_p\}$ of graphs for any $p \geq 2$. We note that such a classification for Coloring is still wide open even for $p = 2$ (see [6]). Hence, research in this direction might lead to an increased understanding of the complexity of the Coloring problem.

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