# Clique-width and Well-Quasi-Ordering of Triangle-Free Graph Classes*

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**Abstract.** Daligault, Rao and Thomassé asked whether every hereditary graph class that is well-quasi-ordered by the induced subgraph relation has bounded clique-width. Lozin, Razgon and Zamaraev (WG 2015) gave a negative answer to this question, but their counterexample is a class that can only be characterised by infinitely many forbidden induced subgraphs. This raises the issue of whether their question has a positive answer for finitely defined hereditary graph classes. Apart from two stubborn cases, this has been confirmed when at most two induced subgraphs $H_1, H_2$ are forbidden. We confirm it for one of the two stubborn cases, namely for the case $(H_1, H_2) = (\text{triangle}, P_2 + P_4)$ by proving that the class of $(\text{triangle}, P_2 + P_4)$-free graphs has bounded clique-width and is well-quasi-ordered. Our technique is based on a special decomposition of 3-partite graphs. We also use this technique to completely determine which classes of $(\text{triangle}, H)$-free graphs are well-quasi-ordered.

## 1 Introduction

A graph class $\mathcal{G}$ is well-quasi-ordered by some containment relation if for any infinite sequence $G_0, G_1, \ldots$ of graphs in $\mathcal{G}$, there is a pair $i, j$ with $i < j$ such that $G_i$ is contained in $G_j$. A graph class $\mathcal{G}$ has bounded clique-width if there exists a constant $c$ such that every graph in $\mathcal{G}$ has clique-width at most $c$. Both being well-quasi-ordered and having bounded clique-width are highly desirable properties of graph classes in the area of theoretical computer science. To illustrate this, let us mention the seminal project of Robertson and Seymour on graph minors that culminated in 2004 in the proof of Wagner's conjecture, which states that the set of all finite graphs is well-quasi-ordered by the minor relation. As an algorithmic consequence, given a minor-closed graph class, it is possible to test in cubic time whether a given graph belongs to this class. The algorithmic importance of having bounded clique-width follows from the fact that many well-known

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NP-hard problems, such as GRAPH COLOURING and HAMILTON CYCLE, become polynomial-time solvable for graph classes of bounded clique-width (this follows from combining results from several papers [4,13,16,22] with a result of Oum and Seymour [21]).

Courcelle [3] proved that the class of graphs obtained from graphs of clique-width 3 via one or more edge contractions has unbounded clique-width. Hence the clique-width of a graph can be much smaller than the clique-width of its minors. On the other hand, the clique-width of a graph is at least the clique-width of any of its induced subgraphs (see, for example, [5]). We therefore focus on *hereditary* classes, that is, on graph classes that are closed under taking induced subgraphs. Our goal is to increase our understanding of the relation between well-quasi-orders and clique-width of hereditary graph classes.

It is readily seen that a class of graphs is hereditary if and only if it can be characterised by a unique set $\mathcal{F}$ of minimal forbidden induced subgraphs, which due to their minimality form an antichain, that is, no graph in $\mathcal{F}$ is an induced subgraph of another graph in $\mathcal{F}$. Note that the class of cycles is not well-quasi-ordered by the induced subgraph relation. As every cycle has clique-width at most 4, having bounded clique-width does not imply being well-quasi-ordered by the induced subgraph relation. In 2010, Daligault, Rao and Thomassé [10] asked about the reverse implication: does every hereditary graph class that is well-quasi-ordered by the induced subgraph relation have bounded clique-width? In 2015, Lozin, Razgon and Zamaraev [20] gave a negative answer. As the set $\mathcal{F}$ in their counter-example is infinite, the question of Daligault, Rao and Thomassé [10] remains open for *finitely defined* hereditary graph classes, that is, hereditary graph classes for which $\mathcal{F}$ is finite.

*Conjecture 1 ([20]).* If a finitely defined hereditary class of graphs $\mathcal{G}$ is well-quasi-ordered by the induced subgraph relation, then $\mathcal{G}$ has bounded clique-width.

If Conjecture 1 is true, then for finitely defined hereditary graph classes the aforementioned algorithmic consequences of having bounded clique-width also hold for the property of being well-quasi-ordered by the induced subgraph relation. A hereditary graph class defined by a single forbidden induced subgraph $H$ is well-quasi-ordered by the induced subgraph relation if and only if it has bounded clique-width if and only if $H$ is an induced subgraph of $P_4$ (see, for instance, [9,11,18]). Hence Conjecture 1 holds when $\mathcal{F}$ has size 1. We consider the case when $\mathcal{F}$ has size 2, say $\mathcal{F} = \{H_1, H_2\}$. Such graph classes are called *bigenic* or $(H_1, H_2)$*-free* graph classes. In this case Conjecture 1 is also known to be true except for two stubborn open cases, namely $(H_1, H_2) = (K_3, P_2 + P_4)$ and $(H_1, H_2) = (\overline{P_1 + P_4}, P_2 + P_3)$; see [7].

**Our Results.** We prove that the class of $(K_3, P_2 + P_4)$-free graphs has bounded clique-width and is well-quasi-ordered by the induced subgraph relation. We do this by using a general technique explained in Section 3. This technique is based on extending (a labelled version of) well-quasi-orderability or boundedness of clique-width of the bipartite graphs in a hereditary graph class $X$ to a special subclass of 3-partite graphs in $X$. The crucial property of these 3-partite graphs is that no three vertices from the three different partition classes form a clique or independent set. We call such 3-partite graphs *curious*. A more restricted version of this concept was used to prove that $(K_3, P_1 + P_5)$-free graphs have bounded clique-width [6]. In Section 4 we show how to generalise results for curious $(K_3, P_2 + P_4)$-free graphs to the whole class of $(K_3, P_2 + P_4)$-free graphs and that our technique can also be applied to prove that $(K_3, P_1 + P_5)$-free graphs are well-quasi-ordered.

**Consequences of Our Results.** Previously, well-quasi-orderability was known for $(K_3, P_6)$-free graphs [1], $(P_2 + P_4)$-free bipartite graphs [17] and $(P_1 + P_5)$-free bipartite graphs [17]. It has also been shown that $H$-free bipartite graphs are not well-quasi-ordered if $H$ contains an induced $3P_1 + P_2$ [18], $3P_2$ [12] or $2P_3$ [17]. This leads to the following dichotomy.

**Theorem 1.** *Let $H$ be a graph. The class of $H$-free bipartite graphs is well-quasi-ordered by the induced subgraph relation if and only if $H = sP_1$ for some $s \geq 1$ or $H$ is an induced subgraph of $P_1 + P_5$, $P_2 + P_4$ or $P_6$.*

Now combining the aforementioned known results for $(K_3, H)$-free graphs and $H$-free bipartite graphs with our new results yields exactly the same dichotomy for $(K_3, H)$-free graphs as the one in Theorem 1.

**Theorem 2.** *Let $H$ be a graph. The class of $(K_3, H)$-free graphs is well-quasi-ordered by the induced subgraph relation if and only if $H = sP_1$ for some $s \geq 1$ or $H$ is an induced subgraph of $P_1 + P_5$, $P_2 + P_4$, or $P_6$.*

**Future Work.** The class of $(\overline{P_1 + P_4}, P_2 + P_3)$-free graphs is the only bigenic graph class left for which Conjecture 1 still needs to be verified. After updating the summaries in [7] with our new results, this class is also one of the six remaining bigenic graph classes for which well-quasi-orderability is still open. And it is one of the six remaining bigenic graph classes for which we do not know if their clique-width is bounded [2]. Hence, a new approach is required to solve this case.

Besides our technique based on curious graphs, we also expect that Theorem 2 will itself be a useful ingredient for showing results for other graph classes, just as Theorem 1 has already proven to be useful (see e.g. [17]).

For clique-width the following dichotomy is known for $H$-free bipartite graphs.

**Theorem 3 ([8]).** *Let $H$ be a graph. The class of $H$-free bipartite graphs has bounded clique-width if and only if $H = sP_1$ for some $s \geq 1$ or $H$ is an induced subgraph of $K_{1,3} + 3P_1$, $K_{1,3} + P_2$, $P_1 + S_{1,1,3}$ or $S_{1,2,3}$.*

It would be interesting to determine whether $(K_3, H)$-free graphs allow the same dichotomy with respect to the boundedness of their clique-width. The evidence so far is affirmative, but in order to answer this question two remaining cases need to be solved, namely $(H_1, H_2) = (K_3, P_1 + S_{1,1,3})$ and $(H_1, H_2) = (K_3, S_{1,2,3})$; see Section 2 for the definition of the graph $S_{h,i,j}$. Both cases turn out to be highly non-trivial; in particular, the class of $(K_3, P_1 + S_{1,1,3})$-free graphs contains the class of $(K_3, P_1 + P_5)$-free graphs, and the class of $(K_3, S_{1,2,3})$-free graphs contains both the classes of $(K_3, P_1 + P_5)$-free and $(K_3, P_2 + P_4)$-free graphs.

## 2 Preliminaries

We consider only finite, undirected graphs without multiple edges or self-loops. The *disjoint union* $(V(G) \cup V(H), E(G) \cup E(H))$ of two vertex-disjoint graphs $G$ and $H$ is denoted by $G + H$ and the disjoint union of $r$ copies of a graph $G$ is denoted by $rG$. The *complement* $\overline{G}$ of a graph $G$ has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices $u, v$ if and only if $uv \notin E(G)$. For a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of $G$ *induced* by $S$. If $S = \{s_1, \ldots, s_r\}$, we may also write $G[s_1, \ldots, s_r]$. We write $G' \subseteq_i G$ to indicate that $G'$ is an induced subgraph of $G$.

The graphs $C_r$, $K_r$, $K_{1,r-1}$ and $P_r$ denote the cycle, complete graph, star and path on $r$ vertices, respectively. The graphs $K_3$ and $K_{1,3}$ are also called the *triangle* and *claw*, respectively. The graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, denotes the *subdivided claw*, that is, the tree that has only one vertex $x$ of degree 3 and exactly three leaves, which are of distance $h$, $i$ and $j$ from $x$, respectively. Observe that $S_{1,1,1} = K_{1,3}$. We let $\mathcal{S}$ denote the class of graphs, each connected component of which is either a subdivided claw or a path. For a set of graphs $\{H_1, \ldots, H_p\}$, a graph $G$ is $(H_1, \ldots, H_p)$*-free* if it has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we may write $H_1$-free instead of $(H_1)$-free.

For a graph $G = (V, E)$, the set $N(u) = \{v \in V \mid uv \in E\}$ denotes the *neighbourhood* of $u \in V$. A graph is *k-partite* if its vertex can be partitioned into $k$ (possibly empty) independent sets; 2-partite graphs are also known as *bipartite* graphs.

Let $X$ be a set of vertices in a graph $G = (V, E)$. A vertex $y \in V \setminus X$ is *complete* to $X$ if it is adjacent to every vertex of $X$ and *anti-complete* to $X$ if it is adjacent to no vertex of $X$. A set of vertices $Y \subseteq V \setminus X$ is *complete* (resp. *anti-complete*) to $X$ if every vertex in $Y$ is complete (resp. anti-complete) to $X$. A vertex $y \in V \setminus X$ *distinguishes* $X$ if $y$ has both a neighbour and a non-neighbour in $X$. The set $X$ is a *module* of $G$ if no vertex in $V \setminus X$ distinguishes $X$. A module $X$ is *non-trivial* if $1 < |X| < |V|$, otherwise it is *trivial*. A graph is *prime* if it has only trivial modules. Two (non-adjacent) vertices are *false twins* if they share the same neighbours. Prime graphs on at least three vertices contain no false twins, as any such pair of vertices would form a non-trivial module.

The *clique-width* $\mathrm{cw}(G)$ of a graph $G$ is the minimum number of labels needed to construct $G$ by using the following four operations:

1. $i(v)$: creating a new graph consisting of a single vertex $v$ with label $i$;
2. $G_1 \oplus G_2$: taking the disjoint union of two labelled graphs $G_1$ and $G_2$;
3. $\eta_{i,j}$: joining each vertex with label $i$ to each vertex with label $j$ $(i \neq j)$;
4. $\rho_{i \rightarrow j}$: renaming label $i$ to $j$.

A class of graphs $\mathcal{G}$ has *bounded* clique-width if there is a constant $c$ such that the clique-width of every graph in $\mathcal{G}$ is at most $c$; otherwise the clique-width is *unbounded*. For an induced subgraph $G'$ of a graph $G$, the *subgraph complementation* operation replaces every edge present in $G'$ by a non-edge, and vice versa. For two disjoint vertex subsets $S$ and $T$ in $G$, the *bipartite complementation* operation replaces every edge with one end-vertex in $S$ and the other one in $T$ by a non-edge and vice versa. Let $k \geq 0$ be a constant and let $\gamma$ be some graph operation. A class $\mathcal{G}'$ is $(k,\gamma)$-*obtained* from a class $\mathcal{G}$ if:

1. every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing $\gamma$ at most $k$ times, and
2. for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from $G$ by performing $\gamma$ at most $k$ times.

We say that $\gamma$ *preserves* boundedness of clique-width if for any finite constant $k$ and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is $(k,\gamma)$-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

**Fact 1.** Vertex deletion preserves boundedness of clique-width [19].
**Fact 2.** Subgraph complementation preserves boundedness of clique-width [15].
**Fact 3.** Bipartite complementation preserves boundedness of clique-width [15].

**Lemma 1 ([5]).** *Let $G$ be a graph and let $\mathcal{P}$ be the set of all induced subgraphs of $G$ that are prime. Then* $\mathrm{cw}(G) = \max_{H \in \mathcal{P}} \mathrm{cw}(H)$.

**Lemma 2 ([6]).** *Let $G$ be a connected $(K_3, C_5, S_{1,2,3})$-free graph that does not contain a pair of false twins. Then $G$ is either bipartite or a cycle.*

A *quasi order* $\leq$ on a set $X$ is a reflexive, transitive binary relation. Two elements $x, y \in X$ in this quasi-order are *comparable* if $x \leq y$ or $y \leq x$, otherwise they are *incomparable*. A set of elements in a quasi-order is a *chain* if every pair of elements is comparable and it is an *antichain* if every pair of elements is incomparable. The quasi-order $\leq$ is a *well-quasi-order* if any infinite sequence of elements $x_1, x_2, x_3, \ldots$ in $X$ contains a pair $(x_i, x_j)$ with $x_i \leq x_j$ and $i < j$. Equivalently, a quasi-order is a well-quasi-order if and only if it has no infinite strictly decreasing sequence $x_1 \gneq x_2 \gneq x_3 \gneq \cdots$ and no infinite antichain. For an arbitrary set $M$, let $M^*$ denote the set of finite sequences of elements of $M$. A quasi-order $\leq$ on $M$ defines a quasi-order $\leq^*$ on $M^*$ as follows: $(a_1, \ldots, a_m) \leq^* (b_1, \ldots, b_n)$ if and only if there is a sequence of integers $i_1, \ldots, i_m$ with $1 \leq i_1 < \cdots < i_m \leq n$ such that $a_j \leq b_{i_j}$ for $j \in \{1, \ldots, m\}$. We call $\leq^*$ the *subsequence relation.*

**Lemma 3 (Higman's Lemma [14]).** *If $(M, \leq)$ is a well-quasi-order then $(M^*, \leq^*)$ is a well-quasi-order.*

For a quasi-order $(W, \leq)$, a graph $G$ is a *labelled* graph if each vertex $v$ of $G$ is equipped with an element $l_G(v) \in W$ (the *label* of $v$). Given two labelled graphs $G$ and $H$, we say that $G$ is a *labelled induced subgraph* of $H$ if $G$ is isomorphic to an induced subgraph of $H$ and there is an isomorphism that maps each vertex $v$ of $G$ to a vertex $w$ of $H$ with $l_G(v) \leq l_H(w)$. Clearly, if $(W, \leq)$ is a well-quasi-order, then a class of graphs $X$ cannot contain an infinite sequence of labelled graphs that is strictly-decreasing with respect to the labelled induced subgraph relation. We therefore say that a graph class $X$ is well-quasi-ordered by the *labelled* induced subgraph relation if it contains no infinite antichains of labelled graphs whenever $(W, \leq)$ is a *well*-quasi-order. Such a class is readily seen to also be well-quasi-ordered by the induced subgraph relation. Similarly to the notion of preserving boundedness of clique-width, we say that a graph operation $\gamma$ *preserves* well-quasi-orderability by the labelled induced subgraph relation if for any finite constant $k$ and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is $(k,\gamma)$-obtained from $\mathcal{G}$ is well-quasi-ordered by this relation if and only if $\mathcal{G}$ is.

**Lemma 4 ([7]).** *Subgraph and bipartite complementations and vertex deletion preserve well-quasi-orderability by the labelled induced subgraph relation.*

**Lemma 5 ([1]).** *A hereditary class $X$ of graphs is well-quasi-ordered by the labelled induced subgraph relation if and only if the set of prime graphs in $X$ is. In particular, $X$ is well-quasi-ordered by the labelled induced subgraph relation if and only if the set of connected graphs in $X$ is.*

**Lemma 6 ([1,17]).** *$(P_7, S_{1,2,3})$-free bipartite graphs are well-quasi-ordered by the labelled induced subgraph relation.*

Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be well-quasi-orders. We define the *Cartesian Product* $(L_1, \leq_1) \times (L_2, \leq_2)$ of these well-quasi-orders as the order $(L, \leq_L)$ on the set $L := L_1 \times L_2$ where $(l_1, l_2) \leq_L (l_1', l_2')$ if and only if $l_1 \leq_1 l_1'$ and $l_2 \leq_2 l_2'$. Lemma 3 implies that $(L, \leq_L)$ is also a well-quasi-order. If $G$ has a labelling with elements of $L_1$ and of $L_2$, say $l_1 : V(G) \to L_1$ and $l_2 : V(G) \to L_2$, we can construct the *combined labelling* in $(L_1, \leq_1) \times (L_2, \leq_2)$ that labels each vertex $v$ of $G$ with the label $(l_1(v), l_2(v))$. We omit the proof of the next lemma.

**Lemma 7.** *Fix a well-quasi-order $(L_1, \leq_1)$ that has at least one element. Let $X$ be a class of graphs. For each $G \in X$ fix a labelling $l_G^1 : V(G) \to L_1$. Then $X$ is well-quasi-ordered by the labelled induced subgraph relation if and only if for every well-quasi-order $(L_2, \leq_2)$ and every labelling of the graphs in $X$ by this order, the combined labelling in $(L_1, \leq_1) \times (L_2, \leq_2)$ obtained from these labellings also results in a well-quasi-ordered set of labelled graphs.*

For an integer $k \geq 1$, a graph $G$ is *$k$-uniform* if there is a symmetric square $0, 1$ matrix $K$ of order $k$ and a graph $F_k$ on vertices $1, 2, \ldots, k$ such that $G \in \mathcal{P}(K, F_k)$, where $\mathcal{P}(K, F_k)$ is a graph class defined as follows. Let $H$ be the disjoint union of infinitely many copies of $F_k$. For $i = 1, \ldots, k$, let $V_i$ be the subset of $V(H)$ containing vertex $i$ from each copy of $F_k$. Construct from $H$ an infinite graph $H(K)$ on the same vertex set by applying a subgraph complementation to $V_i$ if and only if $K(i, i) = 1$ and by applying a bipartite complementation to a pair $V_i, V_j$ if and only if $K(i, j) = 1$. Thus, two vertices $u \in V_i$ and $v \in V_j$ are adjacent in $H(K)$ if and only if $uv \in E(H)$ and $K(i, j) = 0$ or $uv \notin E(H)$ and $K(i, j) = 1$. Then, $\mathcal{P}(K, F_k)$ is the hereditary class consisting of all the finite induced subgraphs of $H(K)$. The minimum $k$ such that $G$ is $k$-uniform is the *uniformicity* of $G$. The second of the next two lemmas follows directly from the above definitions.

**Lemma 8 ([18]).** *Any class of graphs of bounded uniformicity is well-quasi-ordered by the labelled induced subgraph relation.*

**Lemma 9.** *Every $k$-uniform graph has clique-width at most $2k$.*

## 3 Partitioning 3-Partite Graphs

Let $G$ be a 3-partite graph given with a partition of its vertex set into three independent sets $V_1$, $V_2$ and $V_3$. Suppose each $V_i$ can be partitioned into sets $V_i^0, \dots, V_i^\ell$ such that, taking subscripts modulo 3: for $i \in \{1,2,3\}$ if $j < k$ then $V_i^j$ is complete to $V_{i+1}^k$ and anti-complete to $V_{i+2}^k$. For $i \in \{0,\dots,\ell\}$ let $G^i = G[V_1^i \cup V_2^i \cup V_3^i]$. Then the graphs $G^i$ are the *slices* of $G$. If the slices belong to some class $X$, then $G$ can be *partitioned into slices from* $X$; see Fig. 1 for an example.

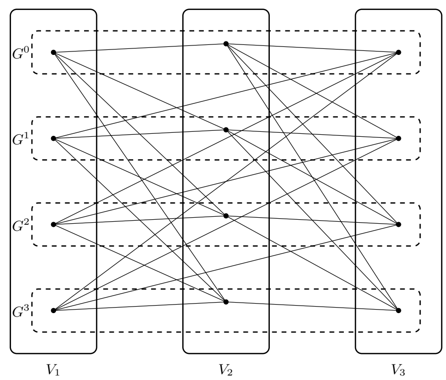


**Fig. 1.** A 3-partite graph that is partitioned into slices $G^0, \dots, G^3$ isomorphic to $P_3$.

**Lemma 10.** *If $G$ is a 3-partite graph that can be partitioned into slices of clique-width at most $k$ then $G$ has clique-width at most* $\max(3k,6)$.

*Proof.* Since every slice $G^j$ of $G$ has clique-width at most $k$, it can be constructed using the labels $1,\dots,k$. Applying relabelling operations if necessary, we may assume that at the end of this construction, every vertex receives the label 1. We can modify this construction so that we use the labels $1_1,\dots,k_1,1_2,\dots,k_2,1_3,\dots,k_3$ instead, in such a way that at all points in the construction, for each $i \in \{1,2,3\}$ every constructed vertex in $V_i$ has a label in $\{1_i,\dots,k_i\}$. To do this we replace:

- creation operations $i(v)$ by $i_j(v)$ if $v \in V_j$,
- relabel operations $\rho_{j\to k}()$ by $\rho_{j_1\to k_1}(\rho_{j_2\to k_2}(\rho_{j_3\to k_3}()))$ and
- join operations $\eta_{j,k}()$ by
$$\eta_{j_1,k_1}(\eta_{j_1,k_2}(\eta_{j_1,k_3}(\eta_{j_2,k_1}(\eta_{j_2,k_2}(\eta_{j_2,k_3}(\eta_{j_3,k_1}(\eta_{j_3,k_2}(\eta_{j_3,k_3}()))))))))).$$

This modified construction uses $3k$ labels and at the end of it, every vertex in $V_i$ is labelled with label $1_i$. We may do this for every slice $G^j$ of $G$ independently. We now show how to use these constructed slices to construct $G[V(G^0)\cup\cdots\cup V(G^j)]$ using six labels in such a way that every vertex in $V_i$ is labelled with label $1_i$. We do this by induction. If $j = 0$ then $G[V(G^0)] = G^0$, so we are done. If $j > 0$ then by the induction hypothesis, we can construct $G[V(G^0) \cup \cdots \cup V(G^{j-1})]$ in this way. Consider the copy of $G^j$ constructed earlier and relabel its vertices using the operations $\rho_{1_1 \to 2_1}$, $\rho_{1_2 \to 2_2}$ and $\rho_{1_3 \to 2_3}$ so that in this copy of $G^j$, every vertex in $V_i$ is labelled $2_i$. Next take the disjoint union of the obtained graph with the constructed $G[V(G^0) \cup \cdots \cup V(G^{j-1})]$. Then, apply join operations $\eta_{1_1,2_2}$, $\eta_{1_2,2_3}$ and $\eta_{1_3,2_1}$. Finally, apply the relabelling operations $\rho_{2_1 \to 1_1}$, $\rho_{2_2 \to 1_2}$ and $\rho_{2_3 \to 1_3}$. This constructs $G[V(G^0) \cup \cdots \cup V(G^j)]$ in such a way that every vertex in $V_i$ is labelled with $1_i$. By induction, $G$ has clique-width at most $\max(3k, 6)$. □

**Lemma 11.** *Let $X$ be a hereditary graph class containing a class $Z$. Let $Y$ be the set of 3-partite graphs in $X$ that can be partitioned into slices from $Z$. If $Z$ is well-quasi-ordered by the labelled induced subgraph relation then so is $Y$.*

*Proof.* For each graph $G$ in $Y$, we may fix a partition into independent sets $(V_1, V_2, V_3)$ with respect to which the graph can be partitioned into slices from $Z$. Let $(L_1, \leq_1)$ be the well-quasi-order with $L_1 = \{1, 2, 3\}$ in which every pair of distinct elements is incomparable. By Lemma 7, we need only consider labellings of graphs in $G$ of the form $(i, l(v))$ where $v \in V_i$ and $l(v)$ belongs to an arbitrary well-quasi-order $L$. Suppose $G$ can be partitioned into slices $G^1, \ldots, G^k$, with vertices labelled as in $G$. The slices along with the labellings completely describe the edges in $G$. Suppose $H$ is another such graph, partitioned into slices $H^1, \ldots, H^k$. If $(H^1, \ldots, H^\ell)$ is smaller than $(G^1, \ldots, G^k)$ under the subsequence relation, then $H$ is an induced subgraph of $G$. The result follows by Lemma 3. □

We will now introduce curious graphs. Let $G$ be a 3-partite graph given together with a partition of its vertex set into three independent sets $V_1$, $V_2$ and $V_3$. An induced $K_3$ or $3P_1$ in $G$ is *rainbow* if it has exactly one vertex in each set $V_i$. We say that $G$ is *curious with respect to the partition* $(V_1, V_2, V_3)$ if it contains no rainbow $K_3$ or $3P_1$ when its vertex set is partitioned in this way. We say that $G$ is *curious* if there is a partition $(V_1, V_2, V_3)$ with respect to which $G$ is curious. We will prove that given a hereditary class $X$, if the bipartite graphs in $X$ are well-quasi-ordered by the labelled induced subgraph relation or have bounded clique-width, then the same is true for the curious graphs in $X$. A linear order $(x_1, x_2, \ldots, x_k)$ of the vertices of an independent set $I$ is

- *increasing* if $i < j$ implies $N(x_i) \subseteq N(x_j)$,
- *decreasing* if $i < j$ implies $N(x_i) \supseteq N(x_j)$,
- *monotone* if it is either increasing or decreasing.

Bipartite graphs that are $2P_2$-free are also known as bipartite *chain* graphs. It is readily seen that a bipartite graph $G$ is $2P_2$-free if and only if the vertices in each independent set of the bipartition admit a monotone ordering. Suppose $G$ is a curious graph with respect to some partition $(V_1, V_2, V_3)$. We say that (with respect to this partition) the graph $G$ is a curious graph of *type $t$* if exactly $t$ of the graphs $G[V_1 \cup V_2]$, $G[V_1 \cup V_3]$ and $G[V_2 \cup V_3]$ contain an induced $2P_2$. If $G$ is a curious graph of type 0 or 1 with respect to the partition $(V_1, V_2, V_3)$ then without loss of generality, we may assume that $G[V_1 \cup V_2]$ and $G[V_1 \cup V_3]$ are both $2P_2$-free. We omit the proof of the next lemma.

**Lemma 12.** *Let $G$ be a curious graph with respect to $(V_1, V_2, V_3)$, such that $G[V_1 \cup V_2]$ and $G[V_1 \cup V_3]$ are both $2P_2$-free. Then the vertices of $V_1$ admit a linear ordering which is decreasing in $G[V_1 \cup V_2]$ and increasing in $G[V_1 \cup V_3]$.*

**Lemma 13.** *If $G$ is a curious graph of type 0 or 1 with respect to a partition $(V_1, V_2, V_3)$ then $G$ can be partitioned into slices that are bipartite.*

*Proof.* Let $x_1, \ldots, x_\ell$ be a linear order on $V_1$ satisfying Lemma 12. Let $V_1^0 = \emptyset$ and for $i \in \{1, \ldots, \ell\}$, let $V_1^i = \{x_i\}$. We partition $V_2$ and $V_3$ as follows. For $i \in \{0, \ldots, \ell\}$, let $V_2^i = \{y \in V_2 \mid x_j y \in E(G) \text{ if and only if } j \leq i\}$. For $i \in \{0, \ldots, \ell\}$, let $V_3^i = \{z \in V_3 \mid x_j z \notin E(G) \text{ if and only if } j \leq i\}$. In particular, note that the vertices of $V_2^\ell \cup V_3^0$ and $V_2^0 \cup V_3^\ell$ are complete and anti-complete to $V_1$, respectively. The following properties hold: if $j < k$ then $V_1^j$ is complete to $V_2^k$ and anti-complete to $V_3^k$, and if $j > k$ then $V_1^j$ is anti-complete to $V_2^k$ and complete to $V_3^k$. If $j < k$ and $y \in V_2^j$ is non-adjacent to $z \in V_3^k$ then $G[x_k, y, z]$ is a rainbow $3P_1$, a contradiction. If $j > k$ and $y \in V_2^j$ is adjacent to $z \in V_3^k$ then $G[x_j, y, z]$ is a rainbow $K_3$, a contradiction. It follows that: if $j < k$ then $V_2^j$ is complete to $V_3^k$ and if $j > k$ then $V_2^j$ is anti-complete to $V_3^k$.

For $i \in \{0, \ldots, \ell\}$, let $G^i = G[V_1^i \cup V_2^i \cup V_3^i]$. The above properties about the edges between the sets $V_j^i$ show that $G$ can be partitioned into the slices $G^0, \ldots, G^\ell$. Now, for each $i \in \{0, \ldots, \ell\}$, $V_1^i$ is anti-complete to $V_3^i$, so every slice $G^i$ is bipartite. This completes the proof. □

**Lemma 14.** *Fix $t \in \{2, 3\}$. If $G$ is a curious graph of type $t$ with respect to a partition $(V_1, V_2, V_3)$ then $G$ can be partitioned into slices of type at most $t - 1$.*

*Proof Sketch.* Fix $t \in \{2, 3\}$ and let $G$ be a curious graph of type $t$ with respect to a partition $(V_1, V_2, V_3)$. We may assume that $G[V_1 \cup V_2]$ contains an induced $2P_2$.

We start with the following claim (we omit the proof).

*Claim 1. Given a $2P_2$ in $G[V_1 \cup V_2]$, every vertex of $V_3$ has exactly two neighbours in the $2P_2$ and these neighbours either both lie in $V_1$ or both lie in $V_2$.*

Consider a maximal set $\{H^1, \ldots, H^q\}$ of vertex-disjoint sets that induce copies of $2P_2$ in $G[V_1 \cup V_2]$. We say that a vertex of $V_3$ *distinguishes* two graphs $G[H^i]$ and $G[H^j]$ if its neighbours in $H^i$ and $H^j$ do not belong to the same set $V_k$. We group these sets $H^i$ into *blocks* $B^1, \ldots, B^p$ that are not distinguished by any vertex of $V_3$. In other words, every $B^i$ contains at least one $2P_2$ and every vertex of $V_3$ is complete to one of the sets $B^i \cap V_1$ and $B^i \cap V_2$ and anti-complete to the other. For $j \in \{1, 2\}$, let $B_j^i = B^i \cap V_j$. We define a relation $<_B$ on the blocks as follows: $B^i <_B B^j$ holds if $B_1^i$ is complete to $B_2^j$, while $B_2^i$ is anti-complete to $B_1^j$. For distinct blocks $B^i$, $B^j$ at most one of $B^i <_B B^j$ and $B^j <_B B^i$ can hold.

We need the following two claims (we omit their proofs).

*Claim 2. Let $B^i$ and $B^j$ be distinct blocks. There is a vertex $z \in V_3$ that differentiates $B^i$ and $B^j$. If $z$ is complete to $B_2^i \cup B_1^j$ and anti-complete to $B_1^i \cup B_2^j$ then $B^i <_B B^j$ (see also Fig. 2). If $z$ is complete to $B_2^j \cup B_1^i$ and anti-complete to $B_1^j \cup B_2^i$ then $B^j <_B B^i$.*

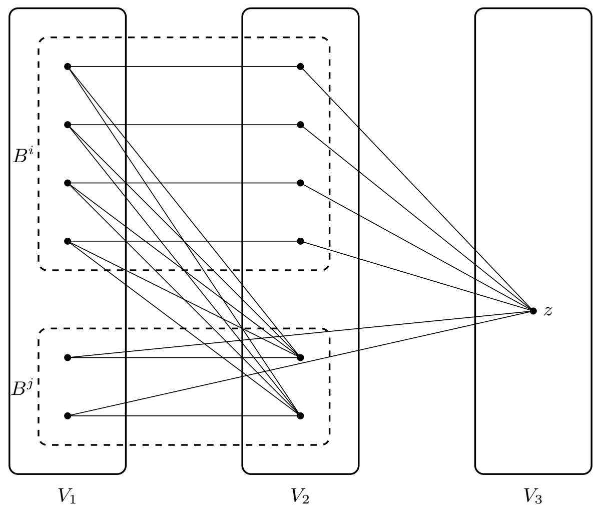


**Fig. 2.** Two blocks $B^i$ and $B^j$ with $B^i <_B B^j$ and a vertex $z \in V_3$ differentiating them.

*Claim 3. The relation $<_B$ is transitive.*

Combining Claims 1–3, we find that $<_B$ is a linear order on the blocks. We obtain the following conclusion, which we call the *chain property*.

*Claim 4. The set of blocks has a linear order $B^1 <_B B^2 <_B \cdots <_B B^p$ so that*

*(i) if $i < j$ then $B_1^i$ is complete to $B_2^j$, while $B_2^i$ is anti-complete to $B_1^j$ and*
*(ii) for each $z \in V_3$ there exists an $i \in \{0, \dots, p\}$ such that if $j \leq i$ then $z$ is complete to $B_2^j$ and anti-complete to $B_1^j$ and if $j > i$ then $z$ is anti-complete to $B_2^j$ and complete to $B_1^j$.*

We now consider the set of vertices in $V_1 \cup V_2$ that do not belong to any set $B^i$. Let $R$ denote this set and note that $G[R]$ is $2P_2$-free by maximality of the set $\{H^1, \dots, H^q\}$. For $i \in \{1, 2\}$ let $R_i = R \cap V_i$. We make the following claim (we omit its proof).

*Claim 5. If $x \in R_1$ has a neighbour in $B_2^i$, then $x$ is complete to $B_2^{i+1}$, and if $x$ has a non-neighbour in $B_2^i$, then $x$ is anti-complete to $B_2^{i-1}$. If $x \in R_2$ has a non-neighbour in $B_1^i$, then $x$ is anti-complete to $B_1^{i+1}$, and if $x$ has a neighbour in $B_1^i$, then $x$ is complete to $B_1^{i-1}$.*

Claim 5 allows us to update the sequence of blocks as follows:

**Update Procedure.** *For $i \in \{1, 2\}$, if $R_i$ contains a vertex $x$ that has both a neighbour $y$ and a non-neighbour $y'$ in $B_{3-i}^j$ for some $j$, we add $x$ to the sets $B_i^j$ and $B^j$ and remove it from $R_i$.*

We make the following claim (we omit its proof).

*Claim 6. Applying the Update Procedure preserves the chain property of the blocks $B^i$.*

By Claim 6 we may apply the Update Procedure exhaustively, after which the chain property will continue to hold. Once this procedure is complete, every remaining vertex of $R_1$ will be either complete or anti-complete to each set $B_2^j$. In fact, by Claim 5, we know that for every vertex $x \in R_1$, there is an $i \in \{0, \ldots, p\}$ such that $x$ has a neighbour in all $B_2^j$ with $j > i$ (if such a $j$ exists) and $x$ has a non-neighbour in all $B_2^j$ with $j \leq i$ (if any such $j$ exists). Since $x$ is complete or anti-complete to each set $B_2^j$, we obtain the following conclusion:

- for every vertex $x \in R_1$, there is an $i \in \{0, \ldots, p\}$ such that $x$ is complete to all $B_2^j$ with $j > i$ (if such a $j$ exists) and $x$ is anti-complete to all $B_2^j$ with $j \leq i$ (if any such $j$ exists). We denote the corresponding subset of $R_1$ by $Y_1^i$.

By symmetry, we also obtain the following:

- for every vertex $x \in R_2$, there is an $i \in \{0, \ldots, p\}$ such that $x$ is complete to all $B_1^j$ with $j \leq i$ (if such a $j$ exists) and $x$ is anti-complete to all $B_1^j$ with $j > i$ (if any such $j$ exists). We denote the corresponding subset of $R_2$ by $Y_2^i$.

We also partition the vertices of $V_3$ into $p+1$ subsets $V_3^0, \ldots, V_3^p$ such that the vertices of $V_3^j$ are complete to $B_2^i$ and anti-complete to $B_1^i$ for $i \leq j$ and complete to $B_1^i$ and anti-complete to $B_2^i$ for $i > j$. (So $V_3^0$ is complete to $B_i^1$ for all $i$ and $V_3^p$ is complete to $B_i^2$ for all $i$).

*Claim 7. For each $i$, if $j < i$ then $V_3^i$ is anti-complete to $Y_1^j$ and complete to $Y_2^j$, and if $j > i$ then $V_3^i$ is complete to $Y_1^j$ and anti-complete to $Y_2^j$.*

Suppose that $z \in V_3^i$ and $x \in Y_1^j$ and $y \in Y_2^j$ (note that such vertices $x$ and $y$ do not exist if $Y_1^j$ or $Y_2^j$, respectively, is empty). First suppose that $j < i$ and choose arbitrary vertices $x' \in B_1^i$, $y' \in B_2^i$. Note that $x$ and $z$ are both complete to $B_2^i$ and $y$ and $z$ are both anti-complete to $B_1^i$. Then $z$ cannot be adjacent to $x$ otherwise $G[x, y', z]$ would be a rainbow $K_3$ and $z$ must be adjacent to $y$, otherwise $G[x', y, z]$ would be a rainbow $3P_1$. Now suppose $i < j$ and choose arbitrary vertices $x' \in B_1^{i+1}$, $y' \in B_2^{i+1}$. Note that $x$ and $z$ are both anti-complete to $B_2^{i+1}$ and $y$ and $z$ are both complete to $B_1^{i+1}$. Then $z$ must be adjacent to $x$ otherwise $G[x, y', z]$ would be a rainbow $3P_1$ and $z$ must be non-adjacent to $y$, otherwise $G[x', y, z]$ would be a rainbow $K_3$. This completes the proof of Claim 7.

Let $G^i$ denote the subgraph of $G$ induced by $Y_1^i \cap Y_2^i \cap V_3^i$. By Claims 4, 6 and 7 the graph $G$ can be partitioned into slices: $G^0, G[B^1], G^1, G[B^2], \ldots, G[B^p], G^p$. Recall that the graph $G$ is of type $t$ and $G[V_1 \cup V_2]$ contains an induced $2P_2$. Since $G[Y_1^i \cup Y_2^i]$ is $2P_2$-free (by construction, since the original sequence $H^1, H^2, \ldots, H^q$ of $2P_2$s was maximal), it follows that each $G^i$ is of type at most $t-1$. Furthermore, since each $G[B_i]$ is bipartite, it forms a curious graph in which the set $V_3$ is empty, so it has type at most 1. This completes the proof. □

We are now ready to state the main result of this section.

**Theorem 4.** *Let $X$ be a hereditary class of graphs. If the set of bipartite graphs in $X$ is well-quasi-ordered by the labelled induced subgraph relation or has bounded clique-width, then this property also holds for the set of curious graphs in $X$.*

*Proof.* Suppose that the class of bipartite graphs in $X$ is well-quasi-ordered by the labelled induced subgraph relation or has bounded clique-width. By Lemmas 10, 11 and 13, the curious graphs of type at most 1 also have this property. Applying Lemmas 10, 11 and 14 once, we obtain the same conclusion for curious graphs of type at most 2. Applying Lemmas 10, 11 and 14 again, we obtain the same conclusion for curious graphs of type at most 3, that is, all curious graphs. □

## 4 Applications of Our Technique

We start with two lemmas. The first is implicit in the proofs of Lemma 9 and Theorem 3 in [6]; we omit the proof of the second.

**Lemma 15 ([6]).** *There is a constant c, such that given any $(K_3, P_1 + P_5)$-free graph G that contains an induced $C_5$, we can apply at most c vertex deletions and at most c bipartite complementation operations to obtain a graph H that is the disjoint union of $(K_3, P_1 + P_5)$-free curious graphs.*

**Lemma 16.** *There is a constant c, such that given any prime $(K_3, P_2 + P_4)$-free graph G that contains an induced $C_5$, we can apply at most c vertex deletions and at most c bipartite complementation operations to obtain a graph H that is the disjoint union of $(K_3, P_2 + P_4)$-free curious graphs and 3-uniform graphs.*

We can now prove the following theorem.[3]

**Theorem 5.** *For $H \in \{P_2 + P_4, P_1 + P_5\}$ the class of $(K_3, H)$-free graphs is well-quasi-ordered by the labelled induced subgraph relation and has bounded clique-width.*

*Proof.* Let $H \in \{P_2 + P_4, P_1 + P_5\}$. By Lemmas 1 and 5, we need only consider prime graphs in this class. Recall that a prime graph on at least three vertices cannot contain two vertices that are false twins, otherwise these two vertices would form a non-trivial module. Therefore, by Lemma 2, and since $H \subseteq_i S_{1,2,3}$, the classes of prime $(K_3, H)$-free graphs containing an induced $C_7$ is precisely the graph $C_7$. We may therefore restrict ourselves to $C_7$-free graphs. Since the graphs in the class are $H$-free, it follows they contain no induced cycles on eight or more vertices. We may therefore restrict ourselves to prime $(K_3, C_7, H)$-free graphs that either contain an induced $C_5$ or are bipartite. By Lemmas 15 or 16, given any prime $(K_3, C_7, H)$-free that contains an induced $C_5$, we can apply at most a constant number of vertex deletions and bipartite complementation operations to obtain a graph that is a disjoint union of $(K_3, H)$-free curious graphs and (in the $H = P_2 + P_4$ case) 3-uniform graphs. By Lemmas 4, 8 and 9, Facts 1 and 3, and Theorem 4, it is sufficient to only consider bipartite $(K_3, C_7, H)$-free graphs. These graphs are $H$-free bipartite graphs. Furthermore, they form a subclass of the class of $(P_7, S_{1,2,3})$-free bipartite graphs, since $H \subseteq_i P_7, S_{1,2,3}$. $(P_7, S_{1,2,3})$-free bipartite graphs are well-quasi-ordered by the labelled induced subgraph relation by Lemma 6 and have bounded clique-width by Theorem 3. □

[3] It was already known [6] that the class of $(K_3, P_1 + P_5)$-free graphs has bounded clique-width but it was not known that it is well-quasi-ordered.

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