Contracting Bipartite Graphs to Paths and Cycles

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Abstract

Testing if a given graph G contains the k-vertex path P_k as a minor or as an induced minor is trivial for every fixed integer $k \geq 1$. The situation changes for the problem of checking if a graph can be modified into P_k by using only edge contractions. In this case the problem is known to be NP-complete even if $k = 4$. This led to an intensive investigation for testing contractibility on restricted graph classes. We focus on bipartite graphs. Heggernes, van 't Hof, Lévêque and Paul proved that the problem stays NP-complete for bipartite graphs if $k = 6$. We strengthen their result from $k = 6$ to $k = 5$. We also show that the problem of contracting a bipartite graph to the 6-vertex cycle C_6 is NP-complete. The cyclicity of a graph is the length of the longest cycle the graph can be contracted to. As a consequence of our second result, determining the cyclicity of a bipartite graph is NP-hard.

Keywords: edge contraction, bipartite graph, path.

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1 Introduction

Algorithmic problems for deciding whether the structure of a graph H appears as a "pattern" within the structure of another graph G are well-studied. Here, the definition of a pattern depends on the set of S of graph operations that we are allowed to use. Basic graph operations include vertex deletion vd, edge deletion ed and edge contraction ec. Contracting an edge uv means that we delete the vertices u and v and introduce a new vertex with neighbourhood $(N(u) \cup N(v)) \setminus \{u, v\}$ (note that no multiple edges or self-loops are created in this way). A graph G contains a graph H as a *minor* if H can be obtained from G using operations from $S = \{\mathsf{vd}, \mathsf{ed}, \mathsf{ec}\}$. For $S = \{\mathsf{vd}, \mathsf{ec}\}$ we say that G contains H as an *induced minor*, and for $S = \{\mathsf{ec}\}$ we say that G contains H as a *contraction*. For a fixed graph H (that is, H is not part of the input), the corresponding three decision problems are denoted by H-Minor, H-Induced Minor and H-Contractibility, respectively.

A celebrated result by Robertson and Seymour [15] states that the H-Minor problem can be solved in cubic time for every fixed pattern graph H. The problems H-Induced Minor and H-Contractibility are harder. Fellows et al. [5] gave an example of a graph H on 68 vertices for which H-Induced Minor is NP-complete, whereas Brouwer and Veldman [4] proved that H-Contractibility is NP-complete even when $H = P_4$ or $H = C_4$ (the graphs C_k and P_k denote the cycle and path on k vertices, respectively). Both complexity classifications are still not settled, as there are many graphs H for which the complexity is unknown (see also [12]).

We observe that P_k-Induced Minor and C_k-Induced Minor are polynomial-time solvable for all k; it suffices to check if G contains P_k as an induced subgraph, that is, if G is not *P_k-free*, or if G contains an induced cycle of length at least k. In order to obtain similar results to those for minors and induced minors, we need to restrict the input of the P_k-Contractibility and C_k-Contractibility problems to some special graph class.

Of particular relevance is the closely related problem of determining the *cyclicity* [9] of a graph, that is, the length of a longest cycle to which a given graph can be contracted. Cyclicity was introduced by Blum [3] under the name *co-circularity,* due to a close relationship with a concept in topology called circularity (see also [1]). Later Hammack [9] coined the current name for the concept and gave both structural results and polynomial-time algorithms

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for a number of special graph classes. He also proved that the problem of determining the cyclicity is NP-complete for general graphs [10].

Van 't Hof, Paulusma and Woeginger [13] proved that P_4-Contractibility is NP-complete for P_6-free graphs, but polynomial-time solvable for P_5-free graphs. Their results can be extended in a straightforward way to obtain a complexity dichotomy for P_k-Contractibility restricted to P_ℓ-free graphs except for one missing case, namely when $k = 5$ and $\ell = 6$. Fiala, Kamiński and Paulusma [6] proved that P_k-Contractibility is NP-complete on line graphs (and thus for claw-free graphs) for $k \geq 7$ and polynomial-time solvable on claw-free graphs for $k \leq 4$. The problems of determining the computational complexity for the missing cases $k = 5$ and $k = 6$ were left open. The same authors also proved that C_6-Contractibility is NP-complete for claw-free graphs [10], which implies that determining the cyclicity of a claw-free graph is NP-hard.

Hammack [9] proved that C_k-Contractibility is polynomial-time solvable on planar graphs for every $k \geq 3$. Later, Kamiński, Paulusma and Thilikos [14] proved that H-Contractibility is polynomial-time solvable on planar graphs for every graph H. Golovach, Kratsch and Paulusma [8] proved that the H-Contractibility problem is polynomial-time solvable on AT-free graphs for every triangle-free graph H. Hence, as C_3-Contractibility is readily seen to be polynomial-time solvable for general graphs, C_k-Contractibility and P_k-Contractibility are polynomial-time solvable on AT-free graphs for every integer $k \geq 3$. Heggernes et al. [11] proved that P_k-Contractibility is polynomial-time solvable on chordal graphs for every $k \geq 1$. Later, Belmonte et al. [2] proved that H-Contractibility is polynomial-time solvable on chordal graphs for every graph H. Heggernes et al. [11] also proved that P_6-Contractibility is NP-complete even for the class of bipartite graphs.

2 Research Question

We consider the class of bipartite graphs, for which we still have a limited understanding of the Contractibility problem. In contrast to a number of other graph classes, as discussed above, bipartite graphs are not closed under edge contraction, which means that getting a handle on the H-Contractibility problem is more difficult. We therefore focus on the $H = P_k$ and $H = C_k$ cases of the following underlying research question for H-Contractibility restricted to bipartite graphs:

Do the computational complexities of H-Contractibility for general graphs *and bipartite graphs coincide for every graph H?*

This question belongs to a more general framework where we aim to research whether for graph classes not closed under edge contraction, one is still able to obtain "tractable" graphs H, for which the H-Contractibility problem is NP-complete in general. For instance, claw-free graphs are not closed under edge contraction. However, there does exist a graph H, namely $H = P_4$, such that H-Contractibility is polynomial-time solvable on claw-free graphs and NP-complete for general graphs. Hence, being claw-free at the start is a sufficiently strong property for P_4-Contractibility to be polynomial-time solvable, even though applying contractions might take us out of the class. It is not known whether being bipartite at the start is also sufficiently strong.

3 Our Contribution

We recall that the H-Contractibility problem is already NP-hard if $H = C_4$ or $H = P_4$. Hence, with respect to our research question we will need to consider small graphs H. While we do not manage to give a conclusive answer, we do improve upon the aforementioned result from Heggernes et al. [11] on bipartite graphs by showing that even P_5-Contractibility is NP-complete for bipartite graphs.

Theorem 3.1 *P_5-Contractibility is NP-complete for bipartite graphs.*

We prove Theorem 3.1 via a reduction from Hypergraph 2-Colourability, which is defined as follows. Let $(Q, \mathcal{S})$ be a hypergraph, where $\mathcal{S}$ is a collection of subsets of Q. A 2-*colouring* of $(Q, \mathcal{S})$ is a partition (Q_1, Q_2) of Q with $Q_1 \cap S \neq \emptyset$ and $Q_2 \cap S \neq \emptyset$ for every $S \in \mathcal{S}$. The Hypergraph 2-Colourability problem is that of deciding whether a given hypergraph $(Q, \mathcal{S})$ admits a 2-colouring. This problem is well known to be NP-complete (see [7]).

We also prove the following result via a reduction from Hypergraph 2-Colourability.

Theorem 3.2 *The C_6-Contractibility problem is NP-complete for bipartite graphs.*

As an immediate consequence, we obtain the following result.

Corollary 3.3 *The problem of determining whether the cyclicity of a bipartite graph is at least 6 is NP-complete.*

4 Future Work

We proved that P_5-Contractibility is NP-complete for the class of bipartite graphs, which strengthens a result in [11], where this was shown for P_6-Contractibility restricted to bipartite graphs. As P_3-Contractibility is readily seen to be polynomial-time solvable for general graphs, this leaves us with one stubborn case, namely when $k = 4$.

Problem 4.1 *Determine the complexity of P_4-Contractibility for bipartite graphs.*

One approach for settling Problem 4.1 would be to first consider *chordal bipartite* graphs, which are bipartite graphs in which every induced cycle has length 4. We believe this is an interesting question on its own.

Problem 4.2 *Determine the complexity of P_4-Contractibility for chordal bipartite graphs.*

We also proved that the C_6-Contractibility problem is NP-complete for bipartite graphs, which implied that determining the cyclicity of a bipartite graph is NP-hard. As mentioned, C_3-Contractibility is polynomial-time solvable for general graphs. This leaves us with the following two open cases.

Problem 4.3 *Determine the complexity of C_k-Contractibility for bipartite graphs when $4 \leq k \leq 5$.*

The 2-Disjoint Connected Subgraphs problem takes as input a graph G and two disjoint subsets Z_1 and Z_2 of $V(G)$. It asks whether $V(G)$ can be partitioned into sets A_1 and A_2, such that $Z_1 \subseteq A_1$, $Z_2 \subseteq A_2$ and both A_1 and A_2 induce connected subgraphs of G. Telle and Villanger [16] gave an $O^*(1.7804^n)$-time algorithm for solving this problem, which is known to be NP-complete even if $|Z_1| = 2$ [13]. Here, the O^* notation suppresses factors of polynomial order. By using their algorithm as a subroutine we can prove the following result.

Proposition 4.4 *There exists an $O^*(1.7804^n)$-time algorithm for solving P_4-Contractibility on n-vertex graphs.*

The proof of the aforementioned NP-completeness result for 2-Disjoint Connected Subgraphs in [13] can be easily modified to hold for bipartite graphs (by subdividing each edge in the hardness construction). This brings us to our final open problem.

Problem 4.5 *Does there exist an exact algorithm for P_4-*Contractibility *for bipartite graphs that is faster than $O^*(1.7804^n)$ time?*

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