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ON THE IMPLEMENTATION OF GRADIENT PLASTICITY WITH THE MATERIAL POINT METHOD

***T.J. Charlton, W.M. Coombs and C.E. Augarde**

Mechanics Research Group, School of Engineering and Computing Sciences,
Durham University, Durham, DH1 3LE

*t.j.charlton@durham.ac.uk

ABSTRACT

The Material Point Method (MPM) is a computational method which allows solid mechanics problems to be modelled using material points which move through a fixed background grid. State variables are stored at these material points and tracked throughout the simulation. The MPM is ideal for modelling geomechanics problems that require the ability to capture large deformations and non-linear material behaviour. A well documented grid crossing error exists in the MPM which occurs when material points move between grid elements. The Generalised Interpolation Material Point (GIMP) method was proposed to alleviate this problem [2]. An implicit implementation of the GIMP method is used in this work.

Conventional analysis techniques constructed in terms of stress and strain are able to handle large deformations well, however they are unable to deal with structural instabilities such as shear banding. Because there is no measure relating to the microstructure of the analysed material, the width of a shear band is highly mesh dependent. Gradient theories enrich these conventional theories with the addition of higher-order terms to include a length scale. Using gradient methods it is possible to model a shear band with a finite thickness without it being mesh dependent. Although there has been lots of work on gradient theories within the Finite Element Method (FEM) it is an area which has received less attention in the MPM.

In this work an existing gradient elasto-plasticity theory [4], used with the FEM, is applied to the GIMP method. The MPM and GIMP method are first introduced and the key equations that are required to include gradient elasto-plasticity are detailed. The effect of introducing a length scale is then demonstrated in a numerical example.

Key Words: *Material Point Method; Gradient Elasto-plasticity; GIMP*

1. Introduction to the Material Point Method

The Material Point Method (MPM), first developed by Sulsky *et al.* [6], is a meshfree method in which calculations take place on a fixed background grid while material properties are carried by a set of material points. Within each loadstep the mesh deforms as in the Finite Element Method (FEM), however variables are then mapped to the material points and the grid is reset so that the material points move through the background grid elements. This allows problems to be modelled where mesh distortions due to large deformations would otherwise not be possible without re-meshing. When particles cross from one background grid element to another, a grid crossing instability occurs which has been well documented in the literature [5]. To remedy this, the Generalised Interpolation Material Point (GIMP) method [2] was introduced.

1.1. Generalised Interpolation Material Point method

In the standard MPM, the material points represent a mass of material but do not have a physical size. Linear shape functions are used for constructing the strain-displacement matrix and for mapping variables to and from the background grid. The GIMP method modifies material points in the MPM to each have an associated influence domain, and the standard FEM shape functions N are replaced by weighting functions S_{vp} where the subscript v refers to vertices or grid nodes and p refers to particles or material points. These functions are constructed from the FEM shape functions and a particle characteristic function χ_p specifying the influence of the particle.

$$S_{vp} = \frac{1}{V_p}\int_{\Omega_p \cap \Omega} \chi_p(\xi) N_v(\xi) d\xi, \tag{1}$$

where V_p is the particle volume (or length in 1D), N_v are the standard linear shape functions, Ω is the physical domain and Ω_p is the influence domain of the particle. This results in an increased smoothness of shape functions between elements. To create GIMP shape functions in two or three dimensions, the tensor product of the one dimensional functions is taken. This change has been shown to significantly improve the stress response when compared to the MPM [2]. The GIMP method has been implemented by the author using a fully implicit approach using an updated Lagrangian framework. For brevity, only a short outline is given here.

A domain is discretised into a set of material points, these material points have influence domains which are initially defined in such a way that they cover the whole of the material with no gaps or overlaps. For both the MPM and GIMP method a regular mesh is used which covers not only the physical domain but also extends to where material is expected to move into during a simulation. It is possible to extend the grid during the simulation if the deformed configuration is not known.

When implementing the GIMP method rather than MPM it is important to take into account the fact that material point influence domains can overlap multiple elements. This means that nodes can be affected not only by material points in the same element, but also those in adjacent elements, increasing the connectivity in the global stiffness matrix. At the start of each loadstep the location of each material point with respect to the background grid must be determined, and from this the weighting functions can be computed. A regular background grid is often chosen to make this process less expensive. Grid elements that do not contain material point domains are also determined so that they can be removed from the calculation for the current loadstep. External forces on the material points are then incremented and mapped to the grid nodes. Displacements are calculated by solving in the same way as the FEM, which then allows the calculation of stresses at the material points. At the end of each loadstep, material point positions and domains are updated while the background grid remains unchanged.

2. Gradient elasto-plasticity

Gradient theories extend the classical elasticity equations to account for microstructure of a material in a way that is not considered by conventional analyses. This is done by considering higher order derivatives, usually of displacement or strain. The approach used in this work and described below follows that of de Borst and Mühlhaus [4].

Unlike conventional methods in computational elasto-plasticity, in this approach the plastic consistency parameter $\dot{\gamma}$ is treated as an independent unknown which is solved for at nodes in addition to the nodal displacements. Beginning from the weak form of equilibrium and the Kuhn-Tucker-Karush consistency conditions for plasticity it is possible to develop two equations, (2) and (3), in terms of $\{\Delta\gamma\}$ and $\{\Delta\varepsilon\}$ which can be calculated from nodal values as shown in (4). For a more detailed explanation of this derivation the reader is referred to [3] or [4].

$$\int_v [B]^T[D^e][B]dv\{\Delta d\} - \int_v [B]^T[D^e]\{f_{,\sigma}\}\{h\}^T dv\{\Delta\Lambda\} = \int_S [N]^T dS - \int_v [B]^T\{\sigma_n\}dv \tag{2}$$

and

$$-\int_v \{h\}\{f_{,\sigma}\}^T[D^e][B]dv\{\Delta d\} + \int_v \{h\}\{f_{,\sigma}\}^T[D^e]\{f_{,\sigma}\}\{h\}^T dv\{\Delta\Lambda\} = \int_v \{h\}f(\sigma_n)dv. \tag{3}$$

where

$$\Delta\gamma = \{h\}^T\{\Delta\Lambda\} \quad \text{and} \quad \{\Delta\varepsilon\} = [B]\{\Delta d\} \tag{4}$$

where $\{h\}$ are Hermitian shape functions [1] and $[B]$ is the strain displacement matrix.

This can be re-written as a coupled system

$$\begin{bmatrix} [K_{aa}] & [K_{a\lambda}] \\ [K_{a\lambda}^T] & [K_{\lambda\lambda}] \end{bmatrix} \begin{bmatrix} \{\Delta d\} \\ \{\Delta\Lambda\} \end{bmatrix} = \begin{bmatrix} \int_s [N]^T\{t\}ds - \int_v [B]^T\{\sigma_n\}dv \\ \int_v \{h\}f(\sigma_n)dv \end{bmatrix}, \tag{5}$$

where

$$[K_{aa}] = \int_v [B]^T[D^e][B]dv, \quad [K_{a\lambda}] = -\int_v [B]^T[D^e]\{f_{,\sigma}\}\{h\}^T dv$$
$$\text{and} \quad [K_{\lambda\lambda}] = \int_v \{h\}\{f_{,\sigma}\}^T[D^e]\{f_{,\sigma}\}\{h\}^T dv. \tag{6}$$

With this formulation, where the plastic consistency parameter is solved for alongside displacements, it is possible to make the yield strength dependent not only on the plastic strain, but also its Laplacian. To do this, gradient terms must be introduced into the yield function. This means that the yield stress ρ_y used would be replaced with $\rho_y - c\frac{d^2\varepsilon^p}{dx^2}$, for the one dimensional case. $[K_{\lambda\lambda}]$ terms in $[K]$ are also required to be updated as

$$[K_{\lambda\lambda}] = \int_v \{h\}\{f_{,\sigma}\}^T[D^e]\{f_{,\sigma}\}\{h\}^T - c\{h\}\{p\}^T dv. \tag{7}$$

where $\{p\}$ are the laplacians of the Hermitian shape functions and c is a constant that can be related to the material length scale (see [4] for details).

To apply this to the GIMP method it should be noted that the Hermitian functions used to map the plastic consistency parameter from the nodes to material points should remain the same as in the FEM. The derivatives of shape functions used for mapping the standard strains become the gradient weighting functions, which are derivatives of the GIMP weighting functions given in (1).

3. Demonstration in 1D

A 1D bar with a weakened section in the middle subjected to displacement at each end (as shown in Figure 1) was modelled to demonstrate the GIMP method with gradient plasticity. The bar has a Young's modulus of $E = 20000$, a yield strength of $\rho_y = 2$ (reduced by 10% in the weakened region) and has a softening(hardening) parameter of $H = -0.1E$. The bar has an initial length of 100 and is displaced by 0.01 which leads to elasto-plastic behaviour in the weakened section. The problem was modelled over 20 loadsteps using 160 elements, each with two material points. Using the gradient approach the length scale is introduced into the yield function which is defined in this case as

$$f = \rho_y + H\lambda - c\Delta\lambda \tag{8}$$

where $c = -l^2H$. The results of the simulation are shown in Figure 2 and Figure 3 where it can be seen that the plastic strains throughout the bar and the axial stress in the bar agree well with finite element simulations. In 1D the plastic strain ε^p is equivalent to λ.

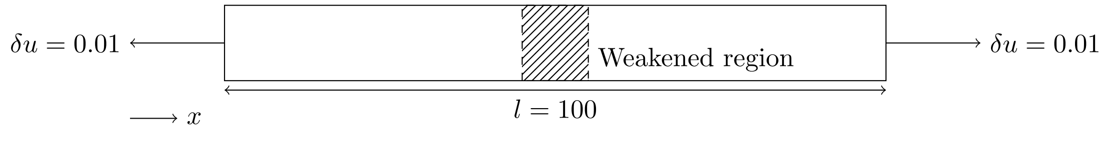

Figure 1: Bar with weakened central section subject to end displacements.

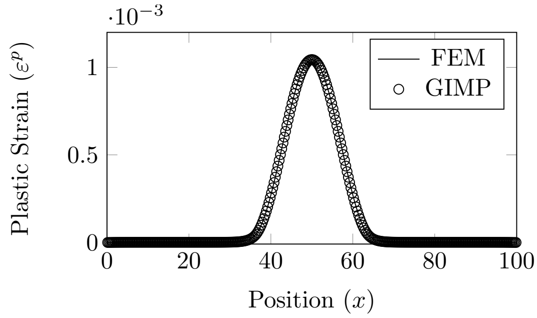

Figure 2: Plastic strain distribution throughout extended bar for both GIMP and FEM.

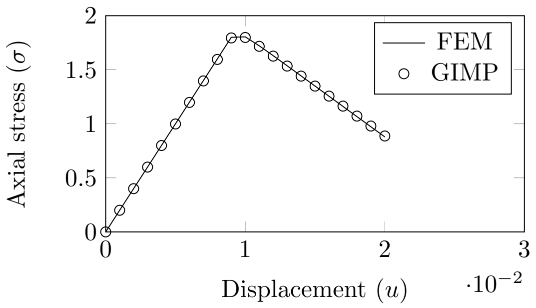

Figure 3: Displacement against axial strain. Gradient plasticity for GIMP and FEM.

4. Conclusions

This paper has outlined a framework to deal with problems containing strain localisations using gradient elasto-plasticity combined with the GIMP method. A fully implicit GIMP method has been used and the elasto-plastic gradient approach of de Borst [4] has been adopted as it provides a straightforward way to implement gradient effects in MPMs (as well as standard rate-independent plasticity). The plastic consistency parameter is treated as an additional nodal unknown that is mapped to the material points and the yield function at the material points becomes an additional error that must be minimised in the coupled non-linear algorithm. Hermitian shape functions are used for the mappings due to the requirement for higher-order continuity. The approach has been demonstrated in a 1D example in which the results using the GIMP method agree well with the FEM. Work is currently being undertaken to extend this to 2D, where the technique will be used to achieve a finite thickness in shear banding without mesh dependency.

References

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