# Colouring Square-Free Graphs without Long Induced Paths

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## Abstract

The Colouring problem is to decide if the vertices of a graph can be coloured with at most $k$ colours for a given integer $k$ such that no two adjacent vertices are coloured alike. The complexity of Colouring is fully understood for graph classes characterized by one forbidden induced subgraph $H$. Despite a huge body of existing work, there are still major complexity gaps if two induced subgraphs $H_1$ and $H_2$ are forbidden. We let $H_1$ be the $s$-vertex cycle $C_s$ and $H_2$ be the $t$-vertex path $P_t$. We show that Colouring is polynomial-time solvable for $s = 4$ and $t \leq 6$, which unifies several known results for Colouring on $(H_1, H_2)$-free graphs. Our algorithm is based on a novel decomposition theorem for $(C_4, P_6)$-free graphs without clique cutsets into homogeneous pairs of sets and a new framework for bounding the clique-width of a graph by the clique-width of its subgraphs induced by homogeneous pairs of sets. To apply this framework, we also need to use divide-and-conquer to bound the clique-width of subgraphs induced by homogeneous pairs of sets. To complement our positive result we also prove that Colouring is NP-complete for $s = 4$ and $t \geq 9$, which is the first hardness result on Colouring for $(C_4, P_t)$-free graphs.

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## 1 Introduction

Graph colouring has been a popular and extensively studied concept in computer science and mathematics since its introduction as a map colouring problem more than 150 years ago due to its many application areas crossing disciplinary boundaries and to its use as a benchmark problem in research into computational hardness. The corresponding decision problem, Colouring, is to decide, for a given graph $G$ and integer $k$, if $G$ admits a *$k$-colouring*, that is, a mapping $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$. Unless $\mathsf{P} = \mathsf{NP}$, it is not possible to solve Colouring in polynomial time for general graphs, not even if the number of colours is limited to 3 [37]. To get a better understanding of the borderline between tractable and intractable instances of Colouring, it is natural to restrict the input to some special graph class. Hereditary graph classes, which are classes of graphs closed under vertex deletion, provide a unified framework for a large collection of well-known graph classes. It is readily seen that a graph class is hereditary if and only if it can be characterized by a (unique) set $\mathcal{H}$ of minimal forbidden induced subgraphs. Graphs with no induced subgraph isomorphic to a graph in a set $\mathcal{H}$ are called *$\mathcal{H}$-free*.

Over the years, the study of Colouring for hereditary graph classes has evolved into a deep area of research in theoretical computer science and discrete mathematics (see, for example, [6, 22, 31, 44]). One of the best-known results is the classical result of Grötschel, Lovász, and Schrijver [24], who showed that Colouring is polynomial-time solvable for perfect graphs. Faster, even linear-time, algorithms are known for subclasses of perfect graphs, such as chordal graphs, bipartite graphs, interval graphs, and comparability graphs; see for example [22]. All these classes are characterized by infinitely many minimal forbidden induced subgraphs. Král', Kratochvíl, Tuza, and Woeginger [35] initiated a systematic study into the computational complexity of Colouring restricted to hereditary graph classes characterized by a *finite* number of minimal forbidden induced subgraphs. In particular they gave a complete classification of the complexity of Colouring for the case where $\mathcal{H}$ consists of a single graph $H$. Their dichotomy result led to two natural directions for further research:

1. Is it possible to obtain a dichotomy for Colouring on $H$-free graphs if the number of colours, $k$, is fixed (that is, $k$ no longer belongs to the input)?
2. Is it possible to obtain a dichotomy for Colouring on $\mathcal{H}$-free graphs if $\mathcal{H}$ has size 2?

We briefly discuss known results for both directions below and refer to [19] for a detailed survey. Let $C_s$ and $P_t$ denote the cycle on $s$ vertices and path on $t$ vertices, respectively. We start with the first question. If $k$ is fixed, then we denote the problem by $k$-Colouring. It is known that for every $k \geq 3$, the $k$-Colouring problem on $H$-free graphs is NP-complete whenever $H$ contains a cycle [16] or an induced claw [28, 36]. Therefore, only the case when $H$ is a disjoint union of paths remains. In particular, the situation where $H = P_t$ has been thoroughly studied. On the positive side, 3-Colouring $P_7$-free graphs and $k$-Colouring $P_5$-free graphs for any fixed $k \geq 1$ are shown to be polynomial-time solvable [3, 26]. On the negative side, Huang [29] proved NP-completeness for ($k = 5$, $t = 6$) and for ($k = 4$, $t = 7$). The cases ($k = 3$, $t \geq 8$) and ($k = 4$, $t = 6$) remain open, although some partial results are known [9, 10].

In this paper we focus on the second question, that is, we restrict the input of Colouring to $\mathcal{H}$-free graphs for $\mathcal{H} = \{H_1, H_2\}$. For two graphs $G$ and $H$, we use $G + H$ to denote the disjoint union of $G$ and $H$, and we write $rG$ to denote the disjoint union of $r$ copies of $G$. As a starting point, Král', Kratochvíl, Tuza, and Woeginger [35] identified the following three main sources of NP-completeness: (i) both $H_1$ and $H_2$ contain a claw; (ii) both $H_1$ and $H_2$ contain a cycle; and (iii) both $H_1$ and $H_2$ contain an induced subgraph from the set $\{4P_1, 2P_1 + P_2, 2P_2\}$. They also showed additional NP-completeness results by mixing the three types. Since then numerous papers [1, 7, 8, 13, 14, 25, 27, 29, 33, 35, 38, 41, 42, 43, 47] have been devoted to this problem, but despite all these efforts the complexity classification for Colouring on $(H_1, H_2)$-free graphs is still far from complete, and even dealing with specific pairs $(H_1, H_2)$ may require substantial work.

One of the "mixed" results obtained in [35] is that Colouring is NP-complete for $(C_s, H)$-free graphs when $s \geq 5$ and $H \in \{4P_1, 2P_1 + P_2, 2P_2\}$. This, together with the well-known result that Colouring can be solved in linear time for $P_4$-free graphs, implies the following dichotomy.

▶ **Theorem 1** ([35]). *Let $s \geq 5$ be a fixed positive integer. Then* Colouring *for $(C_s, P_t)$-free graphs is polynomial-time solvable when $t \leq 4$ and* NP*-complete when $t \geq 5$.*

Theorem 1 raises the natural question: what is the complexity of Colouring on $(C_s, P_t)$-free graphs when $s \in \{3, 4\}$? Huang, Johnson and Paulusma [30] proved that 4-Colouring, and thus Colouring, is NP-complete for $(C_3, P_{22})$-free graphs, while a result of Brandstädt, Klembt and Mahfud [5] implies that Colouring is polynomial-time solvable for $(C_3, P_6)$-free graphs. For $s = 4$, it is only known that Colouring is polynomial-time solvable for $(C_4, P_5)$-free graphs [41]. This is unless we fix the number of colours: for every $k \geq 1$ and $t \geq 1$, it is known that $k$-Colouring is polynomial-time solvable for $(C_4, P_t)$-free graphs [21].

**Our Results.** We first show, in section 3, that Colouring is polynomial-time solvable for $(C_4, P_6)$-free graphs. This case was explicitly mentioned as a natural case to consider in [19]. Our result unifies several previous results on colouring $(C_4, P_t)$-free graphs, namely: the polynomial-time solvability of Colouring for $(C_4, P_5)$-free graphs [41]; the polynomial-time solvability of $k$-Colouring for $(C_4, P_6)$-free graphs for every $k \geq 1$ [21]; and the recent 3/2-approximation algorithm for Colouring for $(C_4, P_6)$-free graphs [18]. It was not previously known if there exists an integer $t$ such that Colouring is NP-complete for $(C_4, P_t)$-free graphs. In section 4 we complement our positive result by giving an affirmative answer to this question: already the value $t = 9$ makes the problem NP-complete.

**Our Methodology.** The general research aim of our paper is to increase, in a systematic way, our insights in the computational hardness of Colouring and to narrow the complexity gaps between hard and easy cases. Clique-width is a well-known width parameter and having bounded clique-width is often the underlying reason for a large collection of NP-complete problems, including Colouring, to become tractable on a special graph class; this follows from results of [11, 17, 34, 45, 46]. However, the class of $(C_4, P_6)$-free graphs contains the class of split graphs, which may have arbitrarily large clique-width [40]. Hence, if we want to use clique-width to solve Colouring for $(C_4, P_6)$-free graphs, then we first need to preprocess the input graph. An *atom* is a graph with no clique cut set. In this paper we prove that $(C_4, P_6)$-free atoms *have* bounded clique-width. This implies a polynomial-time algorithm for Colouring on $(C_4, P_6)$-free graphs, as it is well known that Colouring is polynomial-time solvable on a hereditary graph class $\mathcal{G}$ if it is so on the atoms of $\mathcal{G}$ [49].

In order to prove that $(C_4, P_6)$-free atoms have bounded clique-width, we further develop the approach of [18] that was used to bound the chromatic number of $(C_4, P_6)$-free graphs as a linear function of their maximum clique size and to obtain a 3/2-approximation algorithm for Colouring for $(C_4, P_6)$-free graphs. The approach of [18] is based on a decomposition theorem for $(C_4, P_6)$-free atoms. For our purposes we derive a new variant of this decomposition theorem for so-called strong atoms, which are atoms that contain no universal vertices and no pairs of twin vertices. Another novel element in our approach is that we show how to bound the clique-width of a graph by the clique-width of its subgraphs induced by homogeneous pairs of sets, and this will be very useful for dealing with $(C_4, P_6)$-free strong atoms. To apply this method, we also need to use divide-and-conquer to bound the clique-width of subgraphs induced by homogeneous pairs of sets.

## 2 Preliminaries

For general graph theory notation we follow [2]. Let $G = (V, E)$ be a graph. The *neighbourhood* of a vertex $v$, denoted by $N_G(v)$, is the set of neighbours of $v$. For a set $X \subseteq V(G)$, let $N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X$. The *degree* of $v$, denoted by $d_G(v)$, is equal to $|N_G(v)|$. For $x \in V$ and $S \subseteq V$, we denote by $N_S(x)$ the set of neighbours of $x$ that are in $S$, i.e., $N_S(x) = N_G(x) \cap S$. For $X, Y \subseteq V$, we say that $X$ is *complete* (resp. *anti-complete*) to $Y$ if every vertex in $X$ is adjacent (resp. non-adjacent) to every vertex in $Y$. A vertex subset $K \subseteq V$ is a *clique cutset* if $G - K$ has more components than $G$ and $K$ induces a clique. A vertex is *universal* in $G$ if it is adjacent to all other vertices. For $S \subseteq V$, the subgraph *induced* by $S$, is denoted by $G[S]$.

A subset $D \subseteq V$ is a *dominating set* if every vertex not in $D$ has a neighbour in $D$. Let $u, v \in V$ be two distinct vertices. We say that a vertex $x \notin \{u, v\}$ *distinguishes* $u$ and $v$ if $x$ is adjacent to exactly one of $u$ and $v$. A set $H \subseteq V$ is a *homogeneous set* if no vertex in $V \setminus H$ can distinguish two vertices in $H$. A homogeneous set $H$ is *proper* if $1 < |H| < |V|$. A graph is *prime* if it contains no proper homogeneous set. We say that $u$ and $v$ are *twins* if $u$ and $v$ are adjacent and they have the same set of neighbours in $V \setminus \{u, v\}$. Note that the binary relation of being twins is an equivalence relation on $V$ and so $V$ can be partitioned into equivalence classes $T_1, \ldots, T_r$ of twins. The *skeleton* of $G$ is the subgraph induced by a set of $r$ vertices, one from each of $T_1, \ldots, T_r$. A *blow-up* of a graph $G$ is a graph $G'$ obtained by replacing each vertex $v$ of $G$ with a clique $K_v$ of size at least 1 such that $K_v$ and $K_u$ are complete in $G'$ if $u$ and $v$ are adjacent in $G$, and anti-complete otherwise. Since each equivalence class of twins is a clique and any two equivalence classes are either complete or anti-complete, every graph is a blow-up of its skeleton.

The *clique-width* of a graph $G$, denoted by $cw(G)$, is the minimum number of labels required to construct $G$ using the following four operations:

- $i(v)$: create a new graph consisting of a single vertex $v$ with label $i$;
- $G_1 \oplus G_2$: take the disjoint union of two labelled graphs $G_1$ and $G_2$;
- $\eta_{i,j}$: join each vertex with label $i$ to each vertex with label $j$ (for $i \neq j$);
- $\rho_{i \to j}$: rename label $i$ to $j$.

A *clique-width expression* for $G$ is an algebraic expression that describes how $G$ can be recursively constructed using these operations. A *$k$-expression* for $G$ is a clique-width expression using at most $k$ distinct labels. For instance, this is a 3-expression for the induced path on four vertices $a, b, c, d$:

$$\eta_{3,2}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))).$$

Clique-width is of fundamental importance in computer science since all problems expressible in monadic second-order logic using quantifiers over vertex subsets but not over edge subsets become polynomial-time solvable for graphs of bounded clique-width [11]. Although this meta-theorem does not directly apply to Colouring, a result of Kobler and Rotics [34], combined with the approximation algorithm of Oum and Seymour [45] for finding a $p$-expression, showed that Colouring can be added to the list of such problems.

▶ **Theorem 2** ([34]). Colouring *can be solved in polynomial time for graphs of bounded clique-width.*

## 3 The Polynomial-Time Result

In this section, we shall prove that the chromatic number of any $(C_4, P_6)$-free graph can be found in polynomial time.

▶ **Theorem 3.** Colouring *is polynomial-time solvable on the class of $(C_4, P_6)$-free graphs.*

A graph is called an *atom* if it contains no clique cutset. The main ingredient for proving Theorem 3 is a new structural property of $(C_4, P_6)$-free atoms below which asserts that $(C_4, P_6)$-free atoms have bounded clique-width.

▶ **Theorem 4.** *Every $(C_4, P_6)$-free atom has bounded clique-width.*

The proof of Theorem 4 is deferred to subsection 3.3.

*Proof of Theorem 3 (assuming Theorem 4).* Let $G$ be a $(C_4, P_6)$-free graph. We find the clique decomposition of Tarjan [49] in $O(mn)$ time and this gives a binary decomposition tree $T$ where the root of $T$ is $G$ and the leaves are induced subgraphs of $G$ without clique cutsets. Tarjan [49] showed that there are at most $O(n)$ leaves and that the chromatic number of any node in $T$ is the maximum of the chromatic numbers of its children. Therefore, determining $\chi(G)$ reduces to determining the chromatic number of atoms. Now it follows from Theorem 4 that each atom has bounded clique-width and thus the chromatic number can be found in polynomial time by Theorem 2. ◀

The remainder of the section is organized as follows. In subsection 3.1, we present the key tools on clique-width that play an important role in the proof of Theorem 4. In subsection 3.2, we list structural properties around a 5-cycle in a $(C_4, P_6)$-free graph that are frequently used in later proofs. We then present our main proof, the proof of Theorem 4, in subsection 3.3.

## 3.1 Clique-width

Let $G = (V, E)$ be a graph and $H$ be a proper homogeneous set in $G$. Then $V \setminus H$ is partitioned into two subsets $N$ and $M$ where $N$ is complete to $H$ and $M$ is anti-complete to $H$. Let $h \in H$ be an arbitrary vertex and $G_h = G - (H \setminus \{h\})$. We say that $H$ and $G_h$ are *factors* of $G$ with respect to $H$. Suppose that $\tau$ is a $k_1$-expression for $G_h$ using labels $1, \ldots, k_1$ and $\sigma$ is a $k_2$-expression for $H$ using labels $1, \ldots, k_2$. Then substituting $i(h)$ in $\tau$ with $\rho_{1\to i} \ldots \rho_{k_2 \to i}\sigma$ results in a $k$-expression for $G$ where $k = \max\{k_1, k_2\}$.

▶ **Lemma 1** ([12])**.** *The clique-width of any graph $G$ is the maximum clique-width of any prime induced subgraph of $G$.*

A bipartite graph is a *chain* graph if it is $2P_2$-free. A *co-bipartite chain graph* is the complement of a bipartite chain graph. Let $G$ be a (not necessarily bipartite) graph such that $V(G)$ is partitioned into two subsets $A$ and $B$. We say that a $k$-expression for $G$ is *nice* if all vertices in $A$ end up with the same label $i$ and all vertices in $B$ end up with the same label $j$ with $i \neq j$. It is well-known that any co-bipartite chain graph whose vertex set is partitioned into two cliques has a nice 4-expression.

▶ **Lemma 2** (Folklore)**.** *There is a nice 4-expression for any co-bipartite chain graph.*

We now use divide-and conquer to show that a special graph class has clique-width at most 4. This plays a crucial role in our proof of the main theorem (Theorem 4).

▶ **Lemma 3.** *Let $G$ be a $C_4$-free graph such that $V(G)$ is partitioned into two subsets $A$ and $B$ that satisfy the following conditions: (i) $A$ is a clique; (ii) $B$ is $P_4$-free; (iii) no vertex in $A$ has two non-adjacent neighbours in $B$; (iv) there is no induced $P_4$ in $G$ that starts with a vertex in $A$ followed by three vertices in $B$. Then there is a nice 4-expression for $G$.*

**Proof.** We use induction on $|B|$. If $B$ contains at most one vertex, then $G$ is a co-bipartite chain graph and the lemma follows from Lemma 2. So, we assume that $B$ contains at least two vertices. Since $B$ is $P_4$-free, either $B$ or $\overline{B}$ is disconnected [48]. Suppose first that $B$ is disconnected. Then $B$ can be partitioned into two nonempty subsets $B_1$ and $B_2$ that are anti-complete to each other. Let $A_1 = N(B_1) \cap A$ and $A_2 = A \setminus A_1$. Clearly, $G[A_i \cup B_i]$ with the partition $(A_i, B_i)$ satisfies all the conditions of the lemma for each $1 \leq i \leq 2$. Note also that, by (iii), $A_1$ is anti-complete to $B_2$ and $A_2$ is anti-complete to $B_1$. By the inductive hypothesis there is a nice 4-expression $\tau_i$ for $G[A_i \cup B_i]$ in which all vertices in $A_i$ and $B_i$ have labels 2 and 4, respectively. Now $\rho_{1\to 2}(\eta_{1,2}(\tau_1 \oplus \rho_{2\to 1}\tau_2))$ is a nice 4-expression for $G$.

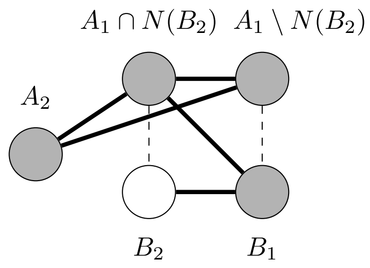


**Figure 1** The case $\overline{B}$ is disconnected. Shaded circles represent cliques. A thick line between two sets represents that the two sets are complete, and a dotted line represents that the edges between the two sets can be arbitrary. Two sets are anti-complete if there is no line between them.

Suppose now that $\overline{B}$ is disconnected. This means that $B$ can be partitioned into two subsets $B_1$ and $B_2$ that are complete to each other. Since $G$ is $C_4$-free, either $B_1$ or $B_2$ is a clique. Without loss generality, we may assume that $B_1$ is a clique. Moreover, we choose the partition $(B_1, B_2)$ such that $B_1$ is maximal. Then every vertex in $B_2$ is not adjacent to some vertex in $B_2$ for otherwise we could have moved such a vertex to $B_1$. If $B_2 = \emptyset$ then $G$ is a co-bipartite chain graph and so the lemma follows from Lemma 2. Therefore, we assume in the following that $B_1, B_2 \neq \emptyset$. Let $A_1 = N(B_1) \cap A$ and $A_2 = A \setminus A_1$. Note that $A_2$ is anti-complete to $B_1$.

We claim that $N(B_2) \cap A$ is complete to $B_1$. Suppose, by contradiction, that $a \in N(B_2) \cap A$ and $b_1 \in B_1$ are not adjacent. By definition, $a$ has a neighbour $b \in B_2$. Recall that $b$ is not adjacent to some vertex $b' \in B_2$. Now $a, b, b_1, b'$ induces either a $P_4$ or a $C_4$, depending on whether $a$ and $b'$ are adjacent. This contradicts (iv) or the $C_4$-freeness of $G$. This proves the claim. Therefore, $A_2$ is anti-complete to $B_2$ and $N(B_2) \cap A = N(B_2) \cap A_1$ (see Figure 1). Consequently, $G[(A_1 \cap N(B_2)) \cup B_2]$ with the partition $(A_1 \cap N(B_2), B_2)$ satisfies all the conditions of the lemma. By the inductive hypothesis there is a nice 4-expression $\tau$ for $G[(A_1 \cap N(B_2)) \cup B_2]$ in which all vertices in $A \cap N(B_2) = A_1 \cap N(B_2)$ and $B_2$ have labels 2 and 4, respectively. In addition, note that $(A_1 \setminus N(B_2), B_1)$ is a co-bipartite chain graph. It then follows from Lemma 2 that there is a nice 4-expression $\epsilon$ for it in which all vertices in $A_1 \setminus N(B_2)$ and $B_1$ have labels 1 and 3, respectively. Then

$$\sigma = \rho_{3\to4}(\rho_{1\to2}(\eta_{3,4}(\eta_{2,3}(\eta_{1,2}(\epsilon \oplus \tau)))))$$

is a nice 4-expression for $G - A_2$. Let $\delta$ be a 2-expression for $A_2$ in which all vertices in $A_2$ have label 1. Then $\rho_{1\to2}(\eta_{1,2}(\delta \oplus \sigma))$ is a nice 4-expression for $G$. This completes the proof. ◀

Let $G = (V, E)$ be a graph and $X$ and $Y$ two disjoint subsets of $V(G)$. We say that $(X, Y)$ is a *homogeneous pair of sets* in $G$ if no vertex in $V \setminus (X \cup Y)$ distinguishes two vertices in $X$ or in $Y$. If both $X$ and $Y$ are cliques then $(X, Y)$ is a *homogeneous pair of cliques*. Note that homogeneous sets are special cases of homogeneous pair of sets where one of $X$ and $Y$ is empty. We establish a novel framework via existing results on clique-width for bounding the clique-width of a graph by the clique-width of its subgraphs induced by homogeneous pairs of sets.

▶ **Lemma 4.** *Let $G$ be a graph such that $V(G)$ can be partitioned into a subset $V_0$ of vertices of constant size, a constant number of pairs of sets $(S_i, T_i)$ for $1 \leq i \leq r$ and a subset $V'$ of vertices such that (i) for each $1 \leq i \leq r$, $(S_i, T_i)$ is a homogeneous pair of sets in*

*$G - (V_0 \cup \bigcup_{j=1}^{i-1}(S_j \cup T_j)$; (ii) for each $1 \leq i \leq r$, $G[S_i \cup T_i]$ has bounded clique-width; and (iii) $G[V']$ has bounded clique-width. Then $G$ has bounded clique-width.*

**Proof.** Let $G_1 = G - V_0$ and $G_{i+1} = G_i - (S_i \cup T_i)$ for $1 \leq i \leq r$. Note that $G_{r+1} = G[V']$. First of all, it follows from [39] that $G$ has bounded clique-width if and only if $G_1$ has bounded clique-width. In addition, (i) says that $(S_i, T_i)$ is a homogeneous pair of sets in $G_i$. Let $N_i$ and $M_i$ be sets of vertices in $G_i$ that are complete to $S_i$ and $T_i$, respectively. For each $i$ we do in $G_i$ two *bipartite complementations* on the pairs $(S_i, V(G_i) \setminus N_i)$ and $(T_i, V(G_i) \setminus M_i)$, which means that we interchange edges and non-edges between the pairs. This results in a graph $G'$ on the same vertex set as $G_1$ that is the disjoint union of $G[S_i \cup T_i]$ and $G[V']$. It follows from [32] that $G_1$ has bounded clique-width if and only if each $G[S_i \cup T_i]$ and $G[V']$ have bounded clique-width. Now the lemma follows from our assumptions (ii) and (iii). ◀

## 3.2 Structure around a 5-Cycle

Let $G = (V, E)$ be a graph and $H$ be an induced subgraph of $G$. We partition $V \setminus V(H)$ into subsets with respect to $H$ as follows: for any $X \subseteq V(H)$, we denote by $S(X)$ the set of vertices in $V \setminus V(H)$ that have $X$ as their neighbourhood among $V(H)$, i.e.,

$$S(X) = \{v \in V \setminus V(H) : N_{V(H)}(v) = X\}.$$

For $0 \leq j \leq |V(H)|$, we denote by $S_j$ the set of vertices in $V \setminus V(H)$ that have exactly $j$ neighbours among $V(H)$. Note that $S_j = \bigcup_{X \subseteq V(H):|X|=j} S(X)$. We say that a vertex in $S_j$ is a *$j$-vertex*. Let $G$ be a $(C_4, P_6)$-free graph and $C = 1, 2, 3, 4, 5$ be an induced $C_5$ in $G$. We partition $V \setminus C$ with respect to $C$ as above. All indices below are modulo 5. Since $G$ is $C_4$-free, there is no vertex in $V \setminus C$ that is adjacent to vertices $i$ and $i + 2$ but not to vertex $i + 1$. In particular, $S(1, 3)$, $S_4$, etc. are empty. The following properties of $S(X)$ were proved in [25] using the fact that $G$ is $(C_4, P_6)$-free.

- **(P1)** $S_5 \cup S(i - 1, i, i + 1)$ is a clique.
- **(P2)** $S(i)$ is complete to $S(i + 2)$ and anti-complete to $S(i + 1)$. Moreover, if neither $S(i)$ nor $S(i + 2)$ are empty then both sets are cliques.
- **(P3)** $S(i, i + 1)$ is complete to $S(i + 1, i + 2)$ and anti-complete to $S(i + 2, i + 3)$. Moreover, if neither $S(i, i + 1)$ nor $S(i + 1, i + 2)$ are empty then both sets are cliques.
- **(P4)** $S(i - 1, i, i + 1)$ is anti-complete to $S(i + 1, i + 2, i + 3)$.
- **(P5)** $S(i)$ is anti-complete to $S(j, j + 1)$ if $j \neq i + 2$. Moreover, if a vertex in $S(i + 2, i + 3)$ is not anti-complete to $S(i)$ then it is universal in $S(i + 2, i + 3)$.
- **(P6)** $S(i)$ is anti-complete to $S(i + 1, i + 2, i + 3)$.
- **(P7)** $S(i - 2, i + 2)$ is anti-complete to $S(i - 1, i, i + 1)$.
- **(P8)** Either $S(i)$ or $S(i+1, i+2)$ is empty. By symmetry, either $S(i)$ or $S(i-1, i-2)$ is empty.
- **(P9)** At least one of $S(i - 1, i)$, $S(i, i + 1)$ and $S(i + 2, i - 2)$ is empty.

## 3.3 Proof of Theorem 4

In this section, we give a proof of Theorem 4. A graph is *chordal* if it does not contain any induced cycle of length at least 4. The following structure of $(C_4, P_6)$-free graphs discovered by Brandstädt and Hoàng [4] is of particular importance in our proofs below.

▶ **Theorem 5** ([4]). *Let $G$ be a $(C_4, P_6)$-free atom. Then the following statements hold: (i) every induced $C_5$ is dominating; (ii) if $G$ contains an induced $C_6$ which is not dominating, then $G$ is the join of a blow-up of the Petersen graph (Figure 2) and a (possibly empty) clique.*

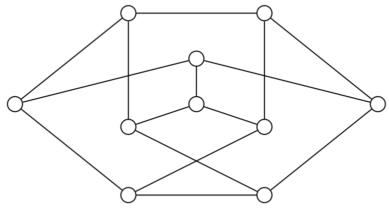

**Figure 2** The Petersen graph.

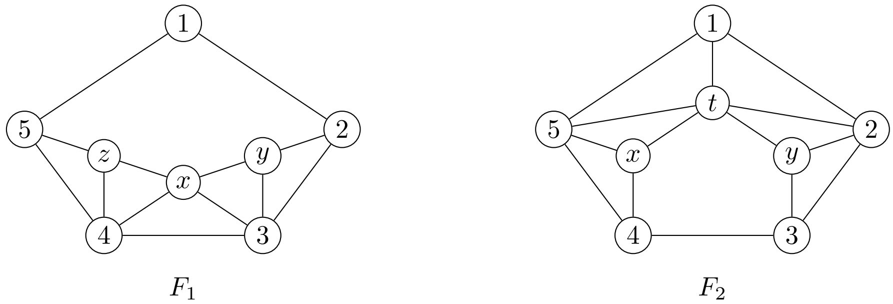


**Figure 3** Two special graphs $F_1$ and $F_2$.

We say that an atom is *strong* if it has no pair of twin vertices or universal vertices. Note that a pair of twin vertices and a universal vertex in a graph give rise to two special kinds of proper homogeneous sets such that one of the factors decomposed by these homogeneous sets is a clique. Therefore, removing twin vertices and universal vertices does not change the clique-width of the graph by Lemma 1. So, to prove Theorem 4 it suffices to prove the theorem for strong atoms. We follow the approach in [18]. In [18], the first and second authors showed how to derive a useful decomposition theorem for $(C_4, P_6)$-free atoms by eliminating a sequence $F_1$, $C_6$, $F_2$ and $C_5$ (see Figure 3 for the graphs $F_1$ and $F_2$) of induced subgraphs and then employing Dirac's classical theorem [15] on chordal graphs. Here we adopt the same strategy and show in Lemma 5–Lemma 8 below that if a $(C_4, P_6)$-free strong atom $G$ contains an induced $C_5$ or $C_6$, then it has bounded clique-width. The remaining case is therefore that $G$ is chordal and so $G$ is a clique by Dirac's theorem [15]. Since cliques have clique-width 2, Theorem 4 follows. It turns out that we can easily prove Lemma 5 and Lemma 6 via the framework formulated in Lemma 4 using the structure of the graphs discovered in [18]. The difficulty is, however, that we have to extend the structural analysis in [18] extensively for Lemma 7 and Lemma 8 and provide new insights on bounding the clique-width of certain special graphs using divide-and-conquer (see Lemma 3).

▶ **Lemma 5.** *If a $(C_4, P_6)$-free strong atom $G$ contains an induced $F_1$, then $G$ has bounded clique-width.*

▶ **Lemma 6.** *If a $(C_4, F_1, P_6)$-free strong atom $G$ contains an induced $C_6$, then $G$ has bounded clique-width.*

▶ **Lemma 7.** *If a $(C_4, C_6, F_1, P_6)$-free strong atom $G$ contains an induced $F_2$, then $G$ has bounded clique-width.*

▶ **Lemma 8.** *If a $(C_4, C_6, F_1, F_2, P_6)$-free strong atom $G$ contains an induced $C_5$, then $G$ has bounded clique-width.*

We illustrate our techniques by giving a proof of Lemma 7 below and omit the proofs of the other lemmas.

**Proof of Lemma 7.** Let $G$ be a $(C_4, C_6, F_1, P_6)$-free strong atom that contains an induced subgraph $H$ that is isomorphic to $F_2$ with $V(H) = \{1, 2, 3, 4, 5, t, x, y\}$ such that $1, 2, 3, 4, 5, 1$ induces the *underlying* 5-cycle $C$, and $t$ is adjacent to 5, 1 and 2, $x$ is adjacent to 4, 5 and $y$ is adjacent to 2 and 3. Moreover, $t$ is adjacent to both $x$ and $y$, see Figure 3. We partition $V(G)$ with respect to $C$. We choose $H$ such that $C$ has $|S_2|$ maximized. Note that $x \in S(4, 5)$, $y \in S(2, 3)$ and $t \in S(5, 1, 2)$.

The overall strategy is to first decompose $G$ into a subset $V_0$ of constant size, constant number of homogeneous pairs of sets, and a subset $V'$, and then finish off the proof via Lemma 4 by showing that each homogeneous pair of sets and $G[V']$ have bounded clique-width where Lemma 3 is employed.

We start with the decomposition. Since $S(2, 3)$ and $S(4, 5)$ are not empty, it follows from **(P8)** that $S_1 = S(2) \cup S(5)$. If both $S(2)$ and $S(5)$ are not empty, say $u \in S(2)$ and $v \in S(5)$, then $u, 2, 3, 4, 5, v$ induces either a $P_6$ or a $C_6$, depending on whether $u$ and $v$ are adjacent. This shows that $S_1 = S(i)$ for some $i \in \{2, 5\}$. Now we argue that $S_2 = S(2, 3) \cup S(4, 5)$. If $S(3, 4)$ contains a vertex $z$, then $z$ is adjacent to $x$ and $y$ by **(P3)** but not adjacent to $t$ by **(P7)**. This implies that $t, x, z, y$ induces a $C_4$, So, $S(3, 4) = \emptyset$. If $S(1, 2)$ contains a vertex $z$, then $z$ is adjacent to $y$ by **(P3)** and so $1, z, y, 3, 4, 5, 1$ induces a $C_6$, a contradiction. This shows that $S(1, 2) = \emptyset$. By symmetry, $S(5, 1) = \emptyset$. Therefore, $S_2 = S(2, 3) \cup S(4, 5)$. The following properties among subsets of $G$ were proved in [18].

**(a)** Each vertex in $S(5, 1, 2)$ is either complete or anti-complete to $S_2$.
**(b)** $S(2, 3)$ and $S(4, 5)$ are cliques.
**(c)** Each vertex in $S(3, 4, 5) \cup S(4, 5, 1)$ is either complete or anti-complete to $S(4, 5)$. By symmetry, each vertex in $S(1, 2, 3) \cup S(2, 3, 4)$ is either complete or anti-complete to $S(2, 3)$.
**(d)** $S(4, 5)$ is anti-complete to $S(2, 3, 4)$. By symmetry, $S(2, 3)$ is anti-complete to $S(3, 4, 5)$.
**(e)** $S(1, 2, 3)$ is complete to $S(5, 1, 2)$. By symmetry, $S(5, 1, 2)$ is complete to $S(4, 5, 1)$.
**(f)** $S(4, 5)$ is complete to $S(4, 5, 1)$. By symmetry, $S(2, 3)$ is complete to $S(1, 2, 3)$.
**(g)** $S(1, 2, 3)$ is complete to $S(2, 3, 4)$. By symmetry, $S(3, 4, 5)$ is complete to $S(4, 5, 1)$.
**(h)** $S_5$ is complete to $S_2$.

Recall that $S_1 = S(i)$ for some $i \in \{2, 5\}$. By symmetry, we may assume that $S_1 = S(5)$. Note that $S(5)$ is complete to $S(4, 5, 1)$ by Theorem 5 and anti-complete to $S(1, 2, 3)$ by **(P6)**. It follows from **(P1)**, **(P4)**, **(P7)**, **(e)**, **(f)** and **(g)** that $S(i-1, i, i+1) \cup \{i\}$ is a homogeneous clique in $G$ and therefore $S(i-1, i, i+1) = \emptyset$ for $i = 2, 5$. Similarly, $S(4, 5)$ is a homogeneous clique in $G$ by **(P7)**, **(a)-(d)**, **(f)** and **(h)** and so $S(4, 5) = \{x\}$. Let $T = \{t \in S(5, 1, 2) : t \text{ is complete to } S_2\}$.

**(1)** *$S(5)$ is anti-complete to $S(5, 1, 2) \setminus T$.*
Let $u \in S(5)$ and $t' \in S(5, 1, 2) \setminus T$. If $u$ and $t'$ are adjacent, then $u, t', 2, 3, 4, x$ induces either a $P_6$ or a $C_6$, depending on whether $u$ and $x$ are adjacent. ∎

By **(1)** and **(d)**, $(S(5, 1, 2) \setminus T) \cup \{1\}$ is a homogeneous set in $G$ and so $S(5, 1, 2) \setminus T = \emptyset$. In other words, $S(5, 1, 2)$ is complete to $S_2$. We now partition $S(5)$ into $X = \{v \in S(5) : v \text{ has a neighbour in } S(2, 3)\}$ and $Y = S(5) \setminus X$.

**(2)** *$X$ is anti-complete to $S(3, 4, 5)$.*
Let $v \in X$ and $s \in S(3, 4, 5)$ be adjacent. By the definition of $X$, $v$ has a neighbour $y' \in S(2, 3)$. By **(d)**, $y'$ is not adjacent to $s$ and so $v, y', 3, s$ induces a $C_4$. ∎

**(3)** *$X$ is complete to $S(5,1,2)$.*
Assume, by contradiction, that $v \in X$ and $t' \in T$ are not adjacent. By the definition of $X$, $v$ has a neighbour $y' \in S(2,3)$. Since $t'$ is adjacent to $y'$, $v, 5, t', y'$ induces a $C_4$. ∎

**(4)** *$X$ is anti-complete to $Y$.*
Suppose that $u \in X$ and $v \in Y$ are adjacent. Let $y' \in S(2,3)$ be a neighbour of $u$. Note that $x$ is adjacent to neither $u$ nor $v$ by **(P5)**. But now $x, 4, 3, y', u, v$ induces a $P_6$. ∎

**(5)** *$X$ is complete to $S_5$.*
Suppose that $v \in X$ and $u \in S_5$ are not adjacent. Let $y' \in S(2,3)$ be a neighbour of $v$. By **(h)**, $y'$ and $u$ are adjacent. Then $u, 5, v, y'$ induces a $C_4$. ∎

It follows from **(P1)**-**(P7)**, **(a)**-**(d)**, **(f)**, **(h)** and **(2)**-**(5)** that $(X, S(2,3))$ is a homogeneous pair of sets in $G$.

**(6)** *For each connected component $A$ of $Y$, each vertex in $S(5,1,2) \cup S(3,4,5)$ is either complete or anti-complete to $A$.*
Let $A$ be an arbitrary component of $Y$. Suppose that $s \in S(5,1,2) \cup S(3,4,5)$ distinguishes an edge $aa'$ in $A$, say $s$ is adjacent to $a$ but not adjacent to $a'$. we may assume by symmetry that $s \in S(5,1,2)$. Then $a', a, s, 2, 3, 4$ induces a $P_6$, a contradiction. ∎

**(7)** *Each component of $Y$ has a neighbour in both $S(5,1,2)$ and $S(3,4,5)$.*
Suppose that a component $A$ of $Y$ does not have a neighbour in one of $S(5,1,2)$ and $S(3,4,5)$, say $S(5,1,2)$. Then $S_5 \cup S(3,4,5) \cup \{5\}$ is a clique cutset of $G$ by **(4)**. ∎

**(8)** *Each component of $Y$ is a clique.*
Let $A$ be an arbitrary component of $Y$. By **(7)**, $A$ has a neighbour $s \in S(5,1,2)$ and $r \in S(3,4,5)$. Note that $s$ and $r$ are not adjacent. Moreover, $\{s,r\}$ is complete to $A$ by **(6)**. Now **(8)** follows from the fact that $G$ is $C_4$-free. ∎

**(9)** *$Y$ is complete to $S_5$.*
Suppose, by contradiction, that $v \in Y$ and $u \in S_5$ are not adjacent. By **(7)**, $v$ has a neighbour $s \in S(5,1,2)$ and $r \in S(3,4,5)$. Then $v, s, u, r$ induces a $C_4$. ∎

It follows from **(P1)**, **(h)**, **(5)** and **(9)** that each vertex in $S_5$ is a universal vertex in $G$ and so $S_5 = \emptyset$. Let $S'(3,4,5) = \{s \in S(3,4,5) : s \text{ has a neighbour in } Y\}$ and $S''(3,4,5) = S(3,4,5) \setminus S'(3,4,5)$. Note that $S''(3,4,5)$ is anti-complete to $Y$. We now show further properties of $Y$ and $S'(3,4,5)$.

**(10)** *$S'(3,4,5)$ is complete to $S(2,3,4)$.*
Suppose, by contradiction, that $r' \in S'(3,4,5)$ is not adjacent to $s \in S(2,3,4)$. By the definition of $S'(3,4,5)$, $r$ has a neighbour $v \in Y$. Then $v, r, 4, s, 2, 1$ induces a $P_6$. ∎

**(11)** *Each vertex in $S(5,1,2)$ is either complete or anti-complete to $Y$.*
Let $t' \in S(5,1,2)$ be an arbitrary vertex. Suppose that $t'$ has a neighbour $u \in Y$. Let $A$ be the component of $Y$ containing $u$. Then $t'$ is complete to $A$ by **(6)**. It remains to show that $t'$ is adjacent to each vertex $u' \in Y \setminus A$. By **(7)**, $u$ has a neighbour $s \in S(3,4,5)$. Note that $C' = u, t,' y, 3, s$ induces a $C_5$. Moreover, $x$ and $s$ are not adjacent for otherwise $x, s, u, t'$ induces a $C_4$. This implies that $x$ is adjacent only to $t'$ on $C'$. On the other hand, $u'$ is not adjacent to any of $u$, 3 and $y$. This implies that $u'$ is adjacent to either $s$

or $t'$ by Theorem 5. If $u'$ is not adjacent $t'$, then $u'$ is adjacent to $s$. This implies that $u', s, 3, y, t', x$ induces a $P_6$ or $C_6$, depending on whether $u'$ and $x$ are adjacent. Therefore, $u'$ is adjacent to $t'$. Since $u'$ is an arbitrary vertex in $Y \setminus A$, this proves **(11)**. ∎

**(12)** $S'(3,4,5)$ *is anti-complete to* $x$.
Suppose not. Let $s \in S'(3,4,5)$ be adjacent to $x$. By definition, $s$ has a neighbour $y' \in Y$. Note that $x$ and $y'$ are not adjacent by **(P5)**. By **(6)** and **(7)**, $y$ has a neighbour $t \in T = S(5,1,2)$. So, $t$ is adjacent to $x$. But now $s, y', t, x$ induces a $C_4$. ∎

It follows from **(P1)**-**(P7)**, **(d)**, **(2)**, **(4)**, **(10)**, **(11)** and **(12)** that $(Y, S'(3,4,5))$ is a homogeneous pair of sets in $G$. Let $S'(5,1,2) = \{s \in S(5,1,2) : s \text{ is complete to } Y\}$. Then $S(5,1,2) \setminus S'(5,1,2)$ is anti-complete to $Y$ by **(11)**. It follows from **(3)** that both $S'(5,1,2)$ and $S(5,1,2) \setminus S'(5,1,2)$ are homogeneous cliques in $G$. So, $|S(5,1,2)| \le 2$. *Now* $V(G)$ *is partitioned into a subset* $V_0 = C \cup S(5,1,2) \cup \{x\}$ *of vertices of size at most* 8*, two homogeneous pairs of sets* $(X, S(2,3))$ *and* $(Y, S'(3,4,5))$*, and a subset* $V' = S''(3,4,5) \cup S(2,3,4)$.

We now apply Lemma 4 to finish off the proof by showing that each of $G[X \cup S(2,3)]$, $G[Y \cup S'(3,4,5)]$, and $G[V']$ has bounded clique-width. First of all, $G[V']$ has clique-width 4 by Lemma 2. Secondly, note that no vertex in $S(1,2)$ can have two non-adjacent neighbours in $X$ since $G$ is $C_4$-free. If there is an induced $P_4 = y', x_1, x_2, x_3$ such that $y' \in S(2,3)$ and $x_i \in X$, then $x_3, x_2, x_1, y', 3, 4$ induces a $P_6$ in $G$. Now if $P = x_1, x_2, x_3, x_4$ is an induced $P_4$ in $G[X]$, any neighbour $y_1$ of $x_1$ is not adjacent to $x_3$ and $x_4$. But then $P \cup \{y_1\}$ contains such a labelled $P_4$ in $G[X \cup S(2,3)]$. Therefore, $G[X \cup S(2,3)]$ with the partition $(X, S(2,3))$ satisfies all the conditions of Lemma 3 and so has clique-width at most 4. Finally, note that each vertex in $S(3,4,5)$ can have neighbours in at most one component of $Y$ due to **(7)**, **(11)** and the fact that $G$ is $C_4$-free. It then follows from **(6)**-**(8)** that $G[Y \cup S'(3,4,5)]$ with the partition $(Y, S'(3,4,5))$ satisfies all the condition in Lemma 3 (where $A = S'(3,4,5)$ and $B = Y$) and so has clique-width at most 4. This completes our proof. ◀

We are now ready to prove our main theorem.

**Proof of Theorem 4.** Let $G$ be a $(C_4, P_6)$-free atom. Let $G'$ be the graph obtained from $G$ by removing twin vertices and universal vertices. It follows from Lemma 5–Lemma 8 that if $G'$ contains an induced $C_5$ or $C_6$, then $G'$ has bounded clique-width. Therefore, we can assume that $G'$ is also $(C_5, C_6)$-free and therefore $G'$ is chordal. It then follows from a well-known result of Dirac [15] that $G'$ is a clique whose clique-width is 2. Finally, $cw(G) = cw(G')$ by Lemma 1 and this completes the proof. ◀

## 4 The Hardness Result

A graph is a *split graph* if its vertex set can be partitioned into two disjoint sets $C$ and $I$ such that $C$ is a clique and $I$ is an independent set. The pair $(C, I)$ is called a *split partition* of $G$. A split graph is *complete* if it has a *complete* split partition, that is, a partition $(C, I)$ such that $C$ and $I$ are complete to each other. A *list assignment* of a graph $G = (V, E)$ is a function $L$ that prescribes, for each $u \in V$, a finite list $L(u) \subseteq \{1, 2, \dots\}$ of colours for $u$. The *size* of a list assignment $L$ is the maximum list size $|L(u)|$ over all vertices $u \in V$. A colouring $c$ *respects* $L$ if $c(u) \in L(u)$ for all $u \in V$. The List Colouring problem is to decide whether a given graph $G$ has a colouring $c$ that respects a given list assignment $L$. We sketch a proof of our hardness result, in which we construct a graph $G'$ that is neither $(sP_2 + P_8)$-free nor $(sP_2 + P_4 + P_5)$-free for any $s \ge 0$. Hence, a different construction is needed for tightening our hardness result (if possible).

▶ **Theorem 6.** Colouring *is* NP*-complete for* $(C_4, 3P_3, P_3 + P_6, 2P_5, P_9)$*-free graphs.*

*Proof Sketch.* We reduce from List Colouring on complete split graphs with a list assignment of size at most 3. It is known that List Colouring is NP-complete for this graph class [20].

Let $G$ be a complete split graph with a list assignment $L$ of size at most 3. From $(G, L)$ we construct an instance $(G', k)$ of Colouring as follows. Let $k \leq 3|V(G)|$ be the size of the union of all lists $L(u)$. Let $(C, I)$ be a complete split partition of $V(G)$. Let $G'$ be the graph of size $O(|V(G)|k)$ obtained from $G$ as follows. Take a clique $X$ on $k$ vertices $x_1, \ldots, x_k$. For each $u \in V(G)$, introduce a clique $Y_u$ of size $k - |L(u)|$ such that every vertex of $Y_u$ is adjacent to $u$ and to every $x_i$ with $i \in L(u)$ (so, each vertex in every $Y_u$ is adjacent to exactly one vertex of $V(G)$, namely vertex $u$). By construction, $G$ has a colouring that respects $L$ if and only if $G'$ has a $k$-colouring. Moreover, it can be readily checked that $G'$ is $(C_4, 3P_3, P_3 + P_6, 2P_5, P_9)$-free. ◇

## 5 Conclusions

We gave an almost complete dichotomy for Colouring restricted to $(C_4, P_t)$-free graphs and leave open only the cases when $7 \leq t \leq 8$. We believe the techniques developed in this paper could be useful for solving open questions regarding Colouring on other hereditary graph classes. The natural candidate class for a polynomial-time result of Colouring is the class of $(C_4, P_7)$-free graphs. However, this may require significant efforts for the following reason. Lozin and Malyshev [38] determined the complexity of Colouring for $\mathcal{H}$-free graphs for every finite set of graphs $\mathcal{H}$ consisting only of 4-vertex graphs except when $\mathcal{H}$ is $\{K_{1,3}, 4P_1\}$, $\{K_{1,3}, 2P_1 + P_2\}$, $\{K_{1,3}, 2P_1 + P_2, 4P_1\}$ or $\{C_4, 4P_1\}$. Solving any of these open cases would be considered as a major advancement in the area. Since $(C_4, 4P_1)$-free graphs are $(C_4, P_7)$-free, polynomial-time solvability of Colouring on $(C_4, P_7)$-free graphs implies polynomial-time solvability for Colouring on $(C_4, 4P_1)$-free graphs. As a first step, we aim to apply the techniques of this paper to $(C_4, 4P_1)$-free graphs.

We recall that the complexity of Colouring on $(C_s, P_t)$-free graphs is known for all $s \geq 5$ and $t \geq 1$ (Theorem 1) and that the complexity of Colouring on $(C_3, P_t)$-free graphs is also known due to the results of [5] and [30] except if $7 \leq t \leq 21$. The class of $(C_3, P_7)$-free graphs is also a natural class to consider. Interestingly, every $(C_3, P_7)$-free graph is 5-colourable. This follows from a result of Gravier, Hoàng and Maffray [23] who proved that for any two integers $r, t \geq 1$, every $(K_r, P_t)$-free graph can be coloured with at most $(t-2)^{r-2}$ colours. On the other hand, 3-Colouring is polynomial-time solvable for $P_7$-free graphs [3]. Hence, in order to solve Colouring for $(C_3, P_7)$-free graphs we may instead consider 4-Colouring for $(C_3, P_7)$-free graphs. This problem seems also highly non-trivial.

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