

Three-Dimensional $\mathcal{N} = 4$ Gauge Theories in Omega Background

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ABSTRACT. We review the description of three-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry in the presence of an omega background as an $\mathcal{N} = 4$ supersymmetric quantum mechanics. We will focus throughout on a simple abelian example. The Hilbert space of supersymmetric ground states is populated by generalized vortex configurations, while half-BPS monopole operators act on the Hilbert space by creating and annihilating vortices, furnishing it with the structure of a Verma module for the quantized Coulomb branch chiral ring. Furthermore, by introducing two-dimensional $\mathcal{N} = (2, 2)$ boundary conditions, we find a finite version of the AGT correspondence between vortex partition functions and overlaps of Whittaker vectors for quantized Coulomb branch chiral rings.

1. Introduction

We review the description of three-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry in the presence of an omega background in the $x^{1,2}$ -plane as an $\mathcal{N} = 4$ supersymmetric quantum mechanics on the x^3 -axis, summarizing and illustrating results from the author's joint paper [7]. We focus exclusively on a simple abelian example, which is sufficient to illustrate the main points and will hopefully provide a foundation for the richer non-abelian examples treated in [7]. The setup is shown schematically in Figure 1.

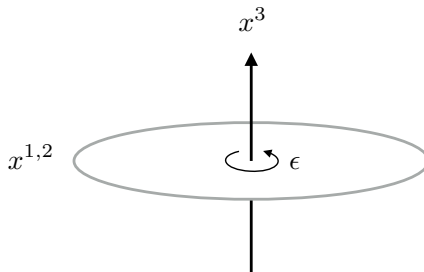


FIGURE 1. We will describe a three-dimensional gauge theory with $\mathcal{N} = 4$ supersymmetry with an omega background in the $x^{1,2}$ -plane as an $\mathcal{N} = 4$ supersymmetric quantum mechanics on the x^3 -axis.

We will provide an explicit description of the $\mathcal{N} = 4$ supersymmetric quantum mechanics on the x^3 -axis, which is summarized as follows:

- The supersymmetric ground states are vortex configurations localized at the origin of the $x^{1,2}$ -plane.
- Monopole operators on the x^3 -axis become half-BPS operators in the supersymmetric quantum mechanics that create and destroy vortices.

The monopole operators on the x^3 -axis generate a non-commutative algebra that quantizes the Coulomb branch in a given complex structure. The space of supersymmetric ground states transforms as a Verma module for this non-commutative algebra. Sending the omega background parameter $\epsilon \rightarrow 0$, we recover the exact Coulomb branch chiral ring. This provides a derivation of the proposed structure of 1-loop and non-perturbative quantum corrections to the Coulomb branch chiral ring developed in [5], and provides a complementary approach to the mathematical work of [4, 13].

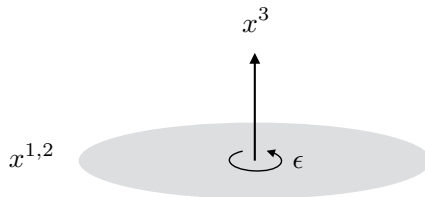


FIGURE 2. We will enrich the setup by introducing boundary conditions in the $x^{1,2}$ -plane preserving $\mathcal{N} = (2, 2)$ supersymmetry.

We will also enrich this setup by including boundary conditions that preserve a two-dimensional $\mathcal{N} = (2, 2)$ supersymmetry in the $x^{1,2}$ -plane [6]. Such boundary conditions define a boundary state in the Hilbert space of the $\mathcal{N} = 4$ supersymmetric quantum mechanics. We show that ‘Neumann’ boundary conditions lead to coherent states of vortices, or generalized Whittaker vectors. Furthermore, by evaluating partition functions on an interval with $\mathcal{N} = (2, 2)$ boundary conditions at each end, we provide a vast generalization and physical explanation for the ‘finite’ AGT correspondence introduced in [3].

2. Setup

2.1. 3d $\mathcal{N} = 4$ Supersymmetry. We work in flat euclidean $\mathbb{R}^3$ with coordinates x^1, x^2, x^3 and spinor indices α, β for the $SU(2)_E$ isometry group. The R-symmetry is $SU(2)_H \times SU(2)_C$ and we introduce indices A, B , and $\dot{A}, \dot{B}$ for the spinor representations of $SU(2)_H$ and $SU(2)_C$ respectively. We use uniform conventions for all $SU(2)$ indices: $(\sigma_i)^\alpha_\beta$ are the standard Pauli matrices, while spinor indices are raised and lowered as $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ with $\epsilon^{12} = \epsilon_{21} = 1$.

The supersymmetry generators are denoted by $Q_\alpha^{\dot{A}}$ with

$$(1) \quad \{Q_\alpha^{\dot{A}}, Q_\beta^{\dot{B}}\} = -2\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}P_{\alpha\beta} + 2\epsilon_{\alpha\beta}(\epsilon^{AB}Z^{\dot{A}\dot{A}} + \epsilon^{\dot{A}\dot{B}}Z^{AB})$$

where $P_{\alpha\beta}$ is the momentum generator and Z^{AB} , $Z^{\dot{A}\dot{B}}$ are central charges in the adjoint representation of $SU(2)_H$, $SU(2)_C$.

We are primarily concerned with supersymmetric gauge theories, in which

- scalars Q^A in hypermultiplets transform in the fundamental of $SU(2)_H$,
- scalars $\varphi^{\dot{A}\dot{B}}$ in vectormultiplets transform in the adjoint of $SU(2)_C$.

In particular, $SU(2)_H$ rotates the complex structure on the Higgs branch $\mathcal{M}_H$ while $SU(2)_C$ rotates the complex structure on the Coulomb branch $\mathcal{M}_C$. The central charges have the following form:

- Z^{AB} is a linear combination conserved charges for Coulomb branch flavor symmetries, with coefficients given by FI parameters t^{AB} .
- $Z^{\dot{A}\dot{B}}$ is a linear combination of conserved charges for gauge and Higgs branch flavor symmetries, with coefficients given by vectormultiplet scalars $\varphi^{\dot{A}\dot{B}}$ and mass parameters $m^{\dot{A}\dot{B}}$ respectively.

It is often convenient to decompose fields and parameters according their charges under a fixed maximal torus $U(1)_H \times U(1)_C$ of the R-symmetry. We will make the standard choice such that the supercharges $Q_\alpha^{1\dot{A}}, Q_\alpha^{2\dot{A}}$ have $U(1)_H$ charge $+\frac{1}{2}, -\frac{1}{2}$, while the supercharges $Q_\alpha^{A\dot{1}}, Q_\alpha^{A\dot{2}}$ have $U(1)_C$ charge $+\frac{1}{2}, -\frac{1}{2}$.

2.2. Example. Throughout this note, we will focus on a simple example: $G = U(1)$ with $N \geq 1$ hypermultiplets in the fundamental representation. We decompose the bosonic fields according to their charge under the $U(1)_H \times U(1)_C$:

- Hypermultiplet scalars Q_j^A with $j = 1, \dots, N$ decompose into complex components $(X_j, \bar{Y}_j)$ transforming with charge $+1$ under $G = U(1)$ and charge $(+\frac{1}{2}, -\frac{1}{2})$ under $U(1)_H$.
- Vectormultiplet scalars $\varphi^{\dot{A}\dot{B}}$ decompose into real and complex components $(\sigma, \varphi, \bar{\varphi})$ transforming with charge $(0, +1, -1)$ under $U(1)_C$.

There is a $G_H = PSU(N)$ flavor symmetry transforming the hypermultiplets. We can turn on mass parameters $m^{\dot{A}\dot{B}}$ corresponding to this symmetry by coupling to a background vectormultiplet and turning on a vacuum expectation value for the scalars in the Lie algebra of the maximal torus $T_H \subset G_H$. Here, we will turn on only complex masses $m^{11} \sim m_{\mathbb{C}} = (m_1, \dots, m_N)$ with $\sum_i m_i = 0$. This spontaneously breaks the $U(1)_C$ R-symmetry. The complex masses contribute to the central charges

$$(2) \quad Z^{i\dot{2}} \sim \sigma \quad Z^{\dot{1}i} \sim \varphi + m_{\mathbb{C}}.$$

To simplify notation we omit the symmetry generators: $\varphi + m_{\mathbb{C}}$ stands for an infinitesimal complex gauge transformation with parameter φ and T_H flavor transformation with parameter $m_{\mathbb{C}}$.

In addition, there is a topological symmetry $G_C = U(1)_t$ and corresponding FI parameters t^{AB} . Here we only turn on a real FI parameter $t^{12} \sim t_{\mathbb{R}} < 0$. This preserves the $U(1)_H$ R-symmetry and contributes to the central charge

$$(3) \quad Z^{12} \sim t_{\mathbb{R}}.$$

This contribution vanishes on the elementary fields but acts non-trivially on monopole operators, which are charged under the $U(1)_t$ topological symmetry.

2.3. Supersymmetric Vacua. Supersymmetric vacua are determined classically by minimizing the potential and preserve all of the supercharges $Q_\alpha^{A\dot{A}}$. With

generic complex masses $m_{\mathbb{C}} = (m_1, \dots, m_N)$ and real FI parameter $t_{\mathbb{R}} < 0$, this requires that

$$(4) \quad \begin{aligned} \mu_{\mathbb{C}} &= 0 & \mu_{\mathbb{R}} + t_{\mathbb{R}} &= 0 \\ (\varphi + m_j)X_j &= 0 & \sigma X_j &= 0 \\ (-\varphi - m_j)Y_j &= 0 & -\sigma Y_j &= 0 \end{aligned}$$

modulo $U(1)$ gauge transformations, where

$$(5) \quad \mu_{\mathbb{C}} = \sum_{j=1}^N X_j Y_j \quad \mu_{\mathbb{R}} = \sum_{j=1}^N |X_j|^2 - |Y_j|^2$$

are the complex and real moment maps for the action of $G = U(1)$ on the hypermultiplets (X_j, Y_j) . In the language of $\mathcal{N} = 2$ supersymmetry, they arise from F-term and D-term contributions to the lagrangian respectively.

Setting the complex masses to vanish, $(m_1, \dots, m_N) = 0$, there is a moduli space of supersymmetric vacua known as the Higgs branch $\mathcal{M}_H$. This is protected from quantum corrections by supersymmetry and the classical description in terms of equations (4) is exact. In particular, $(\sigma, \varphi) = 0$ and $\mathcal{M}_H$ is the hyper-Kähler quotient,

$$(6) \quad \mathcal{M}_H = \left\{ \sum_{j=1}^N X_j Y_j = 0, \sum_{j=1}^N |X_j|^2 - |Y_j|^2 = -t_{\mathbb{R}} \right\} / U(1).$$

In the complex structure where X_j, Y_j are holomorphic, this is $T^*\mathbb{CP}^{N-1}$ with compact base parameterized by X_j and Kähler parameter $-t_{\mathbb{R}}$. A complex algebraic description is found by replacing the real moment map equation by the stability condition $X \neq 0$ and dividing by complex gauge transformations,

$$(7) \quad \mathcal{M}_H = \left\{ \sum_{j=1}^N X_j Y_j = 0, X_j \neq 0 \right\} / \mathbb{C}^*.$$

The coordinate ring of $\mathcal{M}_H$ in this complex structure is then generated by the hypermultiplet bilinears $X_i Y_j$ subject to the complex moment map constraint $\mu_{\mathbb{C}} = 0$. In physical language, this is the chiral ring generated by gauge invariant local operators annihilated by half of the supercharge Q_{α}^{1A} .

On the other hand, for generic complex masses $(m_1, \dots, m_N)$ but vanishing FI parameter $t_{\mathbb{R}} = 0$, there is a moduli space of supersymmetric vacua known as the Coulomb branch $\mathcal{M}_C$. Equations (4) require $X_j = Y_j = 0$ and the Coulomb branch is parametrized by the expectation values of (φ, σ) and the periodic dual photon $\gamma \sim \gamma + 2\pi$. However, the classical geometry $\mathbb{R}^3 \times S^1$ is modified by 1-loop quantum corrections to an N -centered Taub-NUT metric describing an S^1 fibration over $\mathbb{R}^3$ with singular fibers at $(\varphi, \sigma) = (-m_j, 0)$ for all $j = 1, \dots, N$. The $U(1)_t$ topological symmetry acts by rotating the S^1 fibers.

In the complex structure where φ is a holomorphic coordinate, the coordinate ring of the Coulomb branch coincides with the chiral ring generated by operators annihilated by another half of the supercharges $Q_{\alpha}^{A\dot{1}}$. The chiral operators are generated by φ and monopole operators $u^{\pm}$ of charge ± 1 under $G_C = U(1)_t$. The monopole operators can be defined classically by $u^{\pm} \sim e^{\pm(\sigma + i\gamma)}$. However, the

classical relations $u^+ u^- = 1$ are modified by 1-loop quantum corrections to

$$(8) \quad u^+ u^- = \prod_{j=1}^N (\varphi + m_j).$$

This identifies the Coulomb branch in a given complex structure with $\mathbb{C}^2/\mathbb{Z}_N$ with deformation parameters $(m_1, \dots, m_N)$. We will reproduce this quantum corrected chiral ring relation by localization to a supersymmetric quantum mechanics in the following sections.

Finally, turning on both generic complex masses $(m_1, \dots, m_N)$ and real FI parameter $t_{\mathbb{R}} < 0$, there are N isolated massive supersymmetric vacua,

$$(9) \quad \nu_i : \quad X_j = \sqrt{-t_{\mathbb{R}}} \delta_{ij} \quad Y_j = 0 \quad \varphi = -m_i \quad \sigma = 0.$$

The massive vacua can be identified with fixed points of the T_H -action on $\mathcal{M}_H$ generated by $(m_1, \dots, m_N)$: they are the coordinate hyperplanes in the $\mathbb{CP}^{N-1}$. Equivalently, since the topological symmetry rotates the S^1 fibers of $\mathcal{M}_C$, the massive vacua can be identified with fixed points of the G_C -action on $\mathcal{M}_C$. This illustrates an important theme: turning on mass parameters localizes the system to fixed points of the corresponding symmetry.

2.4. $\mathcal{N} = 4$ Quantum Mechanics. We will now identify a subalgebra of the 3d $\mathcal{N} = 4$ supersymmetry algebra (1) corresponding to an $\mathcal{N} = 4$ supersymmetric quantum mechanics on the x^3 -axis. It is then convenient to introduce a complex coordinate $z = x^1 + ix^2$ in the $x^{1,2}$ -plane. We will require that the system sits in a supersymmetric massive vacuum ν_i defined in (9) as $|z| \rightarrow \infty$.

First of all, let us denote by $U(1)_E \subset SU(2)_E$ the subgroup of rotations around the x^3 -axis under which the supercharges Q_1^{AA}, Q_2^{AA} have charge $-\frac{1}{2}, +\frac{1}{2}$ respectively. We now restrict attention to generators commuting with the diagonal subgroup

$$(10) \quad U(1)_\epsilon \subset U(1)_E \times U(1)_H.$$

The commuting supersymmetry generators are

$$(11) \quad Q^{\dot{A}} := Q_1^{1\dot{A}} \quad \tilde{Q}^{\dot{A}} := Q_2^{2\dot{A}}$$

with

$$(12) \quad \begin{aligned} \{Q^{\dot{A}}, Q^{\dot{B}}\} &= 0 \\ \{Q^{\dot{A}}, \tilde{Q}^{\dot{B}}\} &= 2\epsilon^{\dot{A}\dot{B}} H + 2Z^{\dot{A}\dot{B}} \\ \{\tilde{Q}^{\dot{A}}, \tilde{Q}^{\dot{B}}\} &= 0. \end{aligned}$$

where we define $H = P_3 + Z^{12}$. This is the supersymmetry algebra of an $\mathcal{N} = 4$ quantum mechanics on the x^3 -axis with R-symmetry $U(1)_H \times SU(2)_C$ and a distinguished flavor symmetry $U(1)_\epsilon$. This type of supersymmetry algebra can also be obtained by dimensional reduction of 2d $\mathcal{N} = (2, 2)$ supersymmetry.

It is straightforward to formally describe a 3d $\mathcal{N} = 4$ gauge theory as an infinite dimensional $\mathcal{N} = 4$ supersymmetric quantum mechanics on the x^3 -axis with supersymmetry algebra (12). Borrowing the supermultiplet terminology from 2d $\mathcal{N} = (2, 2)$ supersymmetry, we have

- Chiral multiplets with complex scalar components X_j, Y_j and $D_{\bar{z}}$.

- A vectormultiplet for the gauge group $\mathcal{G}$ of maps from the $x^{1,2}$ -plane into G that are constant at $|z| \rightarrow \infty$, with scalar components σ , φ and D_3 .
- A superpotential

$$(13) \quad W \sim \sum_j \int |dz|^2 X_j D_{\bar{z}} Y_j.$$

It also also important to note that the vectormultiplet fields can be organized into a twisted chiral multiplet with bottom component give by the complex scalar φ .

The complex masses $(m_1, \dots, m_N)$ are incorporated by coupling to a background vectormultiplet for the G_H symmetry and giving a vacuum expectation value to the bottom component of the twisted chiral multiplet in T_H . As above, this contributes a non-vanishing central charge Z^{11} proportional to $\varphi + m_{\mathbb{C}}$ breaking the $U(1)_C$ R-symmetry. Finally, the real FI parameter is incorporated by adding a twisted superpotential

$$(14) \quad \widetilde{W} \sim t_{\mathbb{R}} \int |dz|^2 \varphi.$$

which contributes a non-vanishing central charge Z^{12} proportional to $t_{\mathbb{R}}$, which preserves $U(1)_H$ R-symmetry.

2.5. $\mathcal{N} = 2$ Quantum Mechanics. For many purposes, it is convenient to describe the $\mathcal{N} = 4$ quantum mechanics in the language of $\mathcal{N} = 2$ quantum mechanics. There are two types of $\mathcal{N} = 2$ supersymmetric quantum mechanics that can be obtained from the dimensional reduction of $\mathcal{N} = (2, 0)$ supersymmetry and $\mathcal{N} = (1, 1)$ supersymmetry in two dimensions. Given a choice of complex structure on $\mathcal{M}_C$, we can define both an $\mathcal{N} = (2, 0)$ quantum mechanics and an S^1 family of $\mathcal{N} = (1, 1)$ quantum mechanics as follows.

First, we note that a complex structure on $\mathcal{M}_C$ is specified by a spinor $\xi_{\dot{A}} = (\xi_1, \xi_2)$ modulo complex rescalings, forming the homogeneous coordinates of a point on the twistor sphere $\mathbb{CP}^1$ over the Coulomb branch. It is convenient to fix the normalization $\xi_{\dot{A}} \xi^{\dagger \dot{A}} = |\xi_1|^2 + |\xi_2|^2 = 1$ where $\xi_{\dot{A}}^{\dagger} = (-\bar{\xi}_2, \bar{\xi}_1)$ and so describe the twistor sphere as a quotient of S^3 along the fibers of the Hopf fibration. It is important to note that $U(1)_C$ rotates the twistor sphere with fixed points $\xi_{\dot{A}} = (1, 0)$ and $(0, 1)$.

Now, given a choice of complex structure $\xi_{\dot{A}}$ on $\mathcal{M}_C$, let us define

$$(15) \quad \begin{aligned} Q_{\xi} &= \xi_{\dot{A}} Q^{\dot{A}} & \tilde{Q}_{\xi} &= \xi_{\dot{A}} \tilde{Q}^{\dot{A}} \\ Q_{\xi^{\dagger}} &= \xi_{\dot{A}}^{\dagger} Q^{\dot{A}} & \tilde{Q}_{\xi^{\dagger}} &= \xi_{\dot{A}}^{\dagger} \tilde{Q}^{\dot{A}} \end{aligned}$$

Then we have

- An $\mathcal{N} = (2, 0)$ quantum mechanics generated by the supercharges Q_{ξ} and $\tilde{Q}_{\xi^{\dagger}}$ with algebra

$$(16) \quad \{Q_{\xi}, \tilde{Q}_{\xi^{\dagger}}\} = 2(H + \xi_{\dot{A}} \xi_{\dot{B}}^{\dagger} Z^{\dot{A}\dot{B}}).$$

This supersymmetric quantum mechanics has R-symmetry $U(1)_H$ and inherits the $U(1)_{\epsilon}$ flavor symmetry.

- An S^1 family of $\mathcal{N} = (1, 1)$ supersymmetric quantum mechanics labelled by a phase ζ . This is generated by supercharges

$$(17) \quad \mathcal{Q}_{\zeta} = \zeta^{-1/2} Q_{\xi} + \zeta^{1/2} \tilde{Q}_{\xi} \quad \tilde{\mathcal{Q}}_{\zeta} = \zeta^{-1/2} Q_{\xi^{\dagger}} - \zeta^{1/2} \tilde{Q}_{\xi^{\dagger}}$$

with

$$(18) \quad \{Q_\zeta, \tilde{Q}_\zeta\} = -2H \quad \{Q_\zeta, Q_\zeta\} = 2Z_\zeta \quad \{\tilde{Q}_\zeta, \tilde{Q}_\zeta\} = 2Z_{\zeta^\dagger}$$

where $Z_\zeta = \xi_A \xi_{\bar{B}} Z^{\bar{A}, \bar{B}}$. This family of supersymmetric quantum mechanics inherits the $U(1)_\epsilon$ flavor symmetry but $U(1)_H$ transformations rotate the S^1 family by $\zeta \rightarrow \zeta e^{i\theta}$.

It is straightforward to decompose our $\mathcal{N} = 4$ supersymmetric quantum mechanics further in either of these two cases. For the purpose of this note, it will be convenient to phrase our computations in terms of the S^1 family of $\mathcal{N} = (1, 1)$ supersymmetric quantum mechanics. Furthermore, we will fix a complex structure $\xi_{\bar{A}} = (1, 0)$ on the Coulomb branch. This is left invariant by $U(1)_C$ and is therefore ‘adapted’ to this choice of maximal torus in $SU(2)_C$. In particular,

$$(19) \quad Q_\zeta^2 = Z^{\text{ii}} \sim \varphi + m_{\mathbb{C}},$$

showing that Q_ζ behaves as an equivariant differential for G gauge and T_H flavor transformations, a fact that will become important later.

With this choice, the supersymmetric quantum mechanics can be conveniently described in terms of $\mathcal{N} = (1, 1)$ real supermultiplets with bottom components σ , A_1 , A_2 and the real and imaginary parts of X_j , Y_j together with the real superpotential

$$(20) \quad h_\zeta = h + \text{Re}(W/\zeta)$$

where

$$(21) \quad h = \int |dz|^2 \sigma (\mu_{\mathbb{R}} + t_{\mathbb{R}} + 2iF_{z\bar{z}}) \quad W = \sum_j \int |dz|^2 X_j D_{\bar{z}} Y_j.$$

The supersymmetric ground states of the $\mathcal{N} = 4$ quantum mechanics are configurations solving the BPS equations for all of the supercharges $Q^{\bar{A}}$, $\tilde{Q}^{\bar{A}}$. This means they are supersymmetric ground states for every member of the S^1 family of $\mathcal{N} = (1, 1)$ quantum mechanics and therefore critical points of the real superpotential h_ζ for all $|\zeta| = 1$.

In Section 3, we will demonstrate that the critical points of the real superpotential h_ζ for all $|\zeta| = 1$ modulo gauge transformations are generalized vortices in the $x^{1,2}$ -plane. In the absence of the complex mass parameters, there is a moduli space of solutions $\mathcal{M}_n$ for each vortex number $n \in \mathbb{Z}_{\geq 0}$, which is a finite-dimensional non-compact Kähler manifold. Turning on complex mass parameters, the system is restricted to fixed points of the complex T_H transformation on $\mathcal{M}_n$ generated by $(m_1, \dots, m_N)$. The supercharge Q_ζ descends to the T_H -equivariant differential on $\mathcal{M}_n$ and supersymmetric ground states should be identified with the cohomology of Q_ζ .

This statement is quite subtle because the moduli spaces $\mathcal{M}_n$ and the fixed points of the complex T_H transformation generated by $m_{\mathbb{C}} = (m_1, \dots, m_N)$ are non-compact. Standard physical considerations from supersymmetric quantum mechanics suggest one should use L^2 harmonic forms on $\mathcal{M}_n$. However, as we explain in the next section, such subtleties can be avoided by turning on a mass parameter for the $U(1)_\epsilon$ flavor symmetry.

2.6. Omega Background. Recall that the supersymmetric quantum mechanics has flavor symmetry $U(1)_\epsilon \times T_H$ but so far we have only turned on complex mass parameters $(m_1, \dots, m_N)$ for T_H . From the point of view of supersymmetric quantum mechanics, there is no reason not to turn on a complex mass ϵ for the $U(1)_\epsilon$ flavor symmetry. From a three-dimensional perspective, this is known as an Ω -deformation in the $x^{1,2}$ -plane.

The virtue of this deformation is that the combined $T_H \times U(1)_\epsilon$ action on $\mathcal{M}_n$ generated by $(m_1, \dots, m_N)$ then has only isolated fixed points on $\mathcal{M}_n$, so that the supersymmetric quantum mechanics has only isolated massive vacua. The mass parameter ϵ makes an additional contribution to the central charge Z^{ii} so that the supersymmetry algebra is modified to

$$(22) \quad Q_\zeta^2 \sim \varphi + m_\mathbb{C} + \epsilon.$$

The Hilbert space of the supersymmetric quantum mechanics is then identified with the the standard $T_H \times U(1)_\epsilon$ equivariant cohomology of $\mathcal{M}_n$, summed over all vortex numbers $n \geq 0$, with the equivariant differential given by $\mathcal{Q}_\zeta$.

We note that the Ω -background was introduced in the context of 4d $\mathcal{N} = 2$ supersymmetry on $\mathbb{R}^4$ with coordinates x^1, x^2, x^3, x^4 and deformation parameters ϵ and ϵ' corresponding to rotations in the $x^{1,2}$ -plane and $x^{3,4}$ -planes respectively [14]. Our construction can be obtained by sending $\epsilon' \rightarrow 0$, compactifying $x^4 \sim x^4 + 2\pi R$ and sending $R \rightarrow 0$.

3. Hilbert space

3.1. Half-BPS Equations. The supersymmetric ground states of the $\mathcal{N} = 4$ supersymmetric quantum mechanics are configurations preserving all of the supercharges $Q^A, \tilde{Q}^A$. Such configurations are supersymmetric ground states for every member of the S^1 family of $\mathcal{N} = (1, 1)$ quantum mechanics and are therefore critical points of the real superpotential h_ζ given in equation (21) for all $|\zeta| = 1$. This requires that

$$(23) \quad dh = 0 \quad dW = 0.$$

Expanding these equations and grouping them into real and complex equations, we find the following half-BPS equations in the three-dimensional gauge theory for the supercharges $Q^A, \tilde{Q}^A$,

$$(24) \quad -2iF_{z\bar{z}} = \mu_\mathbb{R} + t_\mathbb{R} \quad D_z \sigma = 0 \quad D_{\bar{z}} \sigma = 0$$

$$(25) \quad \mu_\mathbb{C} = 0 \quad D_{\bar{z}} X_j = 0 \quad D_{\bar{z}} Y_j = 0$$

$$(26) \quad \sigma X_j = 0 \quad -\sigma Y_j = 0.$$

Note that these equations are independent of the coordinate x^3 . We require that solutions tend to a supersymmetric vacuum ν_i from equation (9) and divide by gauge transformations in the $x^{1,2}$ -plane that tend to a constant value at $|z| \rightarrow \infty$.

In the absence of the complex masses $(m_1, \dots, m_N, \epsilon)$, the critical point equations are further supplemented by

$$(27) \quad \varphi X_j = 0 \quad -\varphi Y_j = 0.$$

As analyzed in more detail below, the solutions of these equations are labelled by a vortex number $\mathbf{n} \in \mathbb{Z}_{\geq 0}$, which is the flux through the $x^{1,2}$ -plane. For each $\mathbf{n} \in \mathbb{Z}_{\geq 0}$

there is a corresponding non-compact Kähler moduli space $\mathcal{M}_{\nu_i}^{\mathbf{n}}$ of solutions of complex dimension $\mathbf{n}N$.

Turning on complex masses $(m_1, \dots, m_N, \epsilon)$ for the $T_H \times U(1)_\epsilon$ flavor symmetry deforms the supplementary equations (27) to

$$(28) \quad (\varphi + m_j + \frac{\epsilon}{2} + \epsilon z D_z) X_j = 0 \quad (-\varphi - m_j + \frac{\epsilon}{2} + \epsilon z D_z) Y_j = 0.$$

This now requires that solutions are invariant under the combined complex gauge and flavor transformation generated by φ and $(m_1, \dots, m_N, \epsilon)$. This corresponds to the fixed points of the corresponding $T_H \times U(1)_\epsilon$ transformation on the moduli space $\mathcal{M}_\nu$. In our example there is a single fixed point for each vortex number $\mathbf{n} \in \mathbb{Z}_{\geq 0}$, which will contribute a single state $|\mathbf{n}\rangle$ to the Hilbert space of supersymmetric ground states.

Mathematically, the Hilbert space of supersymmetric ground states $\mathcal{H}_{\nu_i}$ with supersymmetric vacuum ν_i at infinity is identified with the $T_H \times U(1)_\epsilon$ -equivariant cohomology of the moduli space of generalized vortices,

$$(29) \quad \mathcal{H}_{\nu_i} = \bigoplus_{\mathbf{n} \geq 0} H_{T_H \times U(1)_\epsilon}^*(\mathcal{M}_{\nu_i}^{\mathbf{n}}, \mathbb{C}),$$

with the equivariant differential realized by any of the supercharges Q_ζ with $|\zeta| = 1$. In order to compute the equivariant cohomology, we will employ a complex algebraic description of the moduli spaces $\mathcal{M}_{\nu_i}^{\mathbf{n}}$. There is a natural basis $|\mathbf{n}\rangle$, $\mathbf{n} \in \mathbb{Z}_{\geq 0}$ for the equivariant cohomology in 1-1 correspondence with fixed points of $\mathcal{M}_{\nu_i}^{\mathbf{n}}$.

3.2. General structure. We begin by studying solutions to the half-BPS equations in the absence of complex mass parameters $(m_1, \dots, m_N, \epsilon)$. In this case, $\varphi = \sigma = 0$ everywhere. We then find a moduli space of solutions to the remaining equations,

$$(30) \quad \mathcal{M}_{\nu_i} = \left\{ \begin{array}{l} D_{\bar{z}} X_j = D_{\bar{z}} Y_j = 0 \\ \mu_{\mathbb{C}} = 0 \\ -2iF_{z\bar{z}} = \mu_{\mathbb{R}} + t_{\mathbb{R}} \end{array} \quad \middle| \quad X_j, Y_j \xrightarrow{|z| \rightarrow \infty} G \cdot \nu_i \right\} / \mathcal{G},$$

where $\mathcal{G}$ is the infinite-dimensional group of gauge transformations in the $x^{1,2}$ -plane that are constant at infinity and $G \cdot \nu_i$ denotes the $G = U(1)$ orbit of the supersymmetric vacuum ν_i on the hypermultiplet scalars.

The moduli space splits into disconnected components

$$(31) \quad \mathcal{M}_{\nu_i} = \bigcup_{\mathbf{n}} \mathcal{M}_{\nu_i}^{\mathbf{n}}$$

labelled by a vortex number $\mathbf{n} \in \pi_1(G) = \mathbb{Z}$ or flux through the $x^{1,2}$ -plane,

$$(32) \quad \mathbf{n} = \frac{1}{2\pi} \int_{\mathbb{R}^2} F.$$

With our choice $t_{\mathbb{R}} < 0$, only the components with $\mathbf{n} \geq 0$ are non-empty. They are Kähler manifolds of complex dimension $\mathbf{n}N$. In order to perform explicit computations in equivariant cohomology, it is convenient to introduce a complex algebraic description of the moduli spaces $\mathcal{M}_{\nu_i}^{\mathbf{n}}$.

3.3. Algebraic description. The complex algebraic description is obtained by dropping the real moment-map equation and instead dividing by complex gauge transformations $G_{\mathbb{C}} = \mathbb{C}^*$,

$$(33) \quad \mathcal{M}_{\nu_i} \simeq \left\{ \begin{array}{l} D_{\bar{z}} X_j = D_{\bar{z}} Y_j = 0 \\ \mu_{\mathbb{C}} = 0 \end{array} \mid X_j, Y_j \xrightarrow{|z| \rightarrow \infty} G_{\mathbb{C}} \cdot \nu_i \right\} / \mathcal{G}_{\mathbb{C}}.$$

Usually, a stability condition must be imposed in the algebraic quotient. However, any solution that tends to a supersymmetric vacuum ν_i at infinity is automatically stable, so no further conditions are necessary in the algebraic quotient (33). The equivalence between the descriptions (30) and (33) is a version of the Hitchin-Kobayashi correspondence for the generalized vortex equations.

From the complex algebraic point of view, the vortex number $\mathbf{n} \in \mathbb{Z}$ determines a complex $G_{\mathbb{C}}$ -bundle $\mathcal{O}(\mathbf{n})$ on the compactification $\mathbb{CP}^1$ of the $x^{1,2}$ -plane. A point in the moduli space $\mathcal{M}_{\nu_i}^{\mathbf{n}}$ is then specified by holomorphic sections X_j, Y_j of the associated bundle $\mathcal{O}(\mathbf{n})^N \oplus \mathcal{O}(-\mathbf{n})^N$ that satisfy the complex moment map constraint $\mu_{\mathbb{C}} = 0$ and lie in the complex orbit $G_{\mathbb{C}} \cdot \nu_i$ at infinity.

Using a complex gauge transformation, we can pass to a holomorphic frame in which the sections are described concretely as polynomials $X_j(z), Y_j(z)$ of degree at most $\mathbf{n}, -\mathbf{n}$ in the affine coordinate z . We then have the following description of the moduli space $\mathcal{M}_{\nu_i}^{\mathbf{n}}$:

- If $\mathbf{n} > 0$ then only the $X_j(z)$ are nonzero. Hitting the supersymmetric vacuum ν_i at infinity requires the leading coefficient of $X_j(z)$ with $j \neq i$ to vanish while the leading coefficient of $X_i(z)$ is nonvanishing. A constant complex gauge transformation sets the leading coefficient of $X_i(z)$ to 1, such that

$$(34) \quad X_j(z) = \delta_{ij} z^{\mathbf{n}} + \sum_{l=0}^{\mathbf{n}-1} x_{j,l} z^l.$$

The coefficients $x_{i,l}$ are unconstrained and parameterize $\mathcal{M}_{\nu_i}^{\mathbf{n}} \simeq \mathbb{C}^{N\mathbf{n}}$.

- If $\mathbf{n} = 0$, both X_j and Y_j are nonzero constants. However, the requirement that they hit the vacuum ν_i at infinity sets them equal to their vacuum values. Thus $\mathcal{M}_{\nu_i}^0$ is a point.
- If $\mathbf{n} < 0$ then only the $Y_j(z)$ can be nonzero. This is incompatible with the vacuum ν_i , so $\mathcal{M}_{\nu_i}^{\mathbf{n}}$ is empty.

The complex algebraic description of the moduli space is familiar in the physics literature from the work of Morrison and Plesser [12] and the moduli matrix construction of vortices [10, 9]. Mathematically, we are describing based holomorphic maps $\mathbb{CP}^1 \rightarrow [\mathcal{M}_H]$ into the Higgs branch stack $[\mathcal{M}_H] = [\mu_{\mathbb{C}}^{-1}(0)/G_{\mathbb{C}}]$, sending the point at infinity to the complex orbit $G_{\mathbb{C}} \cdot \nu_i$.

3.4. Fixed points and the Hilbert space. Turning on complex masses $(m_1, \dots, m_N, \epsilon)$ makes the supersymmetric quantum mechanics completely massive. Equations (28) force the system to the $T_H \times U(1)_{\epsilon}$ fixed points on $\mathcal{M}_{\nu_i}$. The Hilbert space of supersymmetric ground states is identified with the $T_H \times U(1)_{\epsilon}$ equivariant cohomology of $\mathcal{M}_{\nu_i}$ with a natural basis labelled by the equivariant fixed points of $\mathcal{M}_{\nu_i}$.

The equivariant fixed points are straightforward to identify using the algebraic description of the moduli spaces $\mathcal{M}_{\nu_i}^{\mathbf{n}}$. Let us consider an infinitesimal combined

gauge and $T_H \times U(1)_\epsilon$ flavor transformation generated by $(\varphi, m_1, \dots, m_N, \epsilon)$. This sends

$$(35) \quad X_j(z) \mapsto (\varphi + m_j + \frac{\epsilon}{2} + \epsilon z \partial_z) X_j(z).$$

For $\mathbf{n} \geq 0$, there is a unique fixed point $X_j(z) = \delta_{ij} z^{\mathbf{n}}$ with

$$(36) \quad \varphi = -m_i - (\mathbf{n} + \frac{1}{2})\epsilon.$$

which is simply origin of $\mathcal{M}_{\nu_j}^{\mathbf{n}} = \mathbb{C}^{N\mathbf{n}}$. Denoting the corresponding state in the quantum mechanics as $|\mathbf{n}\rangle$, we find that

$$(37) \quad \mathcal{H} = \bigoplus_{\mathbf{n} \geq 0} \mathbb{C} |\mathbf{n}\rangle.$$

The Hilbert space has a natural inner product from the supersymmetric quantum mechanics: $\langle \mathbf{n}' | \mathbf{n} \rangle$ is given by computing the path integral of the supersymmetric quantum mechanics with $|\mathbf{n}\rangle$ at $x^3 \rightarrow -\infty$ and $\langle \mathbf{n}' |$ at $x^3 \rightarrow \infty$. The path integral is zero unless $\mathbf{n} = \mathbf{n}'$, in which case it is given by the equivariant integral $\int_{\mathcal{M}_{\nu}^{\mathbf{n}}}$ of the product of equivariant cohomology classes representing $\langle \mathbf{n}' |$ and $|\mathbf{n}\rangle$.

At this stage, there is a slight ambiguity in the normalization of the states $|\mathbf{n}\rangle$. A natural choice is that $|\mathbf{n}\rangle$ is the Poincaré dual of the fundamental class of the fixed point of $\mathcal{M}_{\nu_i}^{\mathbf{n}} = \mathbb{C}^{N\mathbf{n}}$, in other words an equivariant δ -function supported at the origin. In this case, we would find

$$(38) \quad \langle \mathbf{n}' | \mathbf{n} \rangle = \delta_{\mathbf{n}', \mathbf{n}} \omega_{\mathbf{n}}$$

where $\omega_{\mathbf{n}}$ is the equivariant weight of the tangent space to $\mathcal{M}_{\nu_i}^{\mathbf{n}} = \mathbb{C}^{N\mathbf{n}}$ at the origin. Alternatively, we could normalize $|\mathbf{n}\rangle$ by the equivariant weight of the tangent space at the fixed point, so that

$$(39) \quad \langle \mathbf{n}' | \mathbf{n} \rangle = \frac{\delta_{\mathbf{n}', \mathbf{n}}}{\omega_{\mathbf{n}}}$$

From a physical perspective, neither normalization is especially preferred. Here we choose the latter normalization (39).

The only remaining task is to compute the equivariant weight of the tangent space to the origin in $\mathcal{M}_{\nu_i}^{\mathbf{n}} = \mathbb{C}^{N\mathbf{n}}$. This is parameterized by the subleading coefficients $x_{j,l}$ in the expansion (34), which transform as

$$(40) \quad \begin{aligned} x_{j,l} &\mapsto (\varphi + m_j + (l + \frac{1}{2})\epsilon) x_{j,l} \\ &= (m_j - m_i + (l - \mathbf{n})\epsilon) x_{j,l} \end{aligned}$$

where we evaluate $\varphi = -m_i - (\mathbf{n} + \frac{1}{2})\epsilon$ at the origin in $\mathcal{M}_{\nu_i}^{\mathbf{n}} = \mathbb{C}^{N\mathbf{n}}$. Therefore, the inner product on the Hilbert space is given by

$$(41) \quad \langle \mathbf{n}' | \mathbf{n} \rangle = \delta_{\mathbf{n}', \mathbf{n}} \prod_{j=1}^N \prod_{l=0}^{\mathbf{n}-1} \frac{1}{m_j - m_i + (l - \mathbf{n})\epsilon}.$$

It is often convenient to introduce a characteristic polynomial for the Higgs branch flavor symmetry, $P(x) = \prod_{j=1}^N (x + m_j)$, and write the inner product as

$$(42) \quad \langle \mathbf{n}' | \mathbf{n} \rangle = \delta_{\mathbf{n}', \mathbf{n}} \prod_{l=0}^{\mathbf{n}-1} \frac{1}{P(-m_i + (l - \mathbf{n})\epsilon)}.$$

4. Monopole operators

4.1. Monopole Operators. We now consider half-BPS operators in the $\mathcal{N} = 4$ supersymmetric quantum mechanics preserving the supercharges $Q^{\mathbf{i}}, \tilde{Q}^{\mathbf{i}}$. Such operators arise from Coulomb branch chiral ring operators in the original three-dimensional theory annihilated by Q_{α}^{1A} .

In an abelian theory, one such operator is the complex scalar φ which acts on the vortex state $|\mathbf{n}\rangle$ by evaluation at the corresponding fixed point

$$(43) \quad \varphi|\mathbf{n}\rangle = (-m_i - (\mathbf{n} + \tfrac{1}{2})\epsilon)|\mathbf{n}\rangle.$$

However, there are also monopole operators v_A labelled by an integer magnetic charge A , which are defined by removing a small S_p^2 around a point p and imposing singular boundary conditions in the path integral,

$$(44) \quad F = A \sin \theta d\theta \wedge d\phi + \dots \quad \sigma = -\frac{A}{2r} + \dots$$

where (r, θ, ϕ) are spherical coordinates around the point p [2]. In this section, we explain how these monopole operators can be understood from our $\mathcal{N} = 4$ supersymmetric quantum mechanics and how they act on the Hilbert space of supersymmetric ground states of the quantum mechanics.

As a preliminary observation, we note that the monopole operator v_A creates A units of flux on a small sphere S_p^2 surrounding the point p where it is inserted,

$$(45) \quad \frac{1}{2\pi} \int_{S^2} F = A \in \mathbb{Z}.$$

Therefore, by topological considerations alone, we must have

$$(46) \quad v_A|\mathbf{n}\rangle = \begin{cases} c_{A,\mathbf{n}}|\mathbf{n} + A\rangle & \text{if } \mathbf{n} + A \geq 0 \\ 0 & \text{if } \mathbf{n} + A < 0 \end{cases}.$$

In other words, monopole operators create and annihilate vortices. Our task is therefore reduced to computing explicitly the coefficients $c_{A,\mathbf{n}}$. Since the vortex states $|\mathbf{n}\rangle$ are orthogonal, this is equivalent to computing the non-zero correlation functions $\langle \mathbf{n} + A | v_A | \mathbf{n} \rangle$.

4.2. Quarter BPS Equations. As noted above, the monopole operators preserve the supercharges $Q^{\mathbf{i}}, \tilde{Q}^{\mathbf{i}}$ of the $\mathcal{N} = 4$ supersymmetric quantum mechanics. This is equivalent to preserving the supercharge Q_{ζ} for all phases ζ . They should therefore correspond to singular solutions of the instanton equations for every member of the S^1 family of $\mathcal{N} = (1, 1)$ quantum mechanics.

The instanton equations for the supercharge Q_{ζ} are gradient flow equations for the real superpotential h_{ζ} ,

$$(47) \quad D_3\Phi = -\frac{\delta h_{\zeta}}{\delta \Phi}$$

Imposing the instanton equations for all $|\zeta| = 1$ we find

$$(48) \quad \frac{\delta W}{\delta \Phi} = 0 \quad D_3\sigma = -\frac{\delta h}{\delta \Phi}$$

where Φ stands for the scalar components of real $\mathcal{N} = (1, 1)$ supermultiplets, namely σ, A_1, A_2 and the real and imaginary parts of X_j, Y_j . Expanding and grouping

into real and complex equations, we find the following quarter-BPS equations in the three-dimensional gauge theory for the supercharges $Q^{\dot{1}}, \tilde{Q}^{\dot{1}}$,

$$(49) \quad -2iF_{z\bar{z}} - D_3\sigma = \mu_{\mathbb{R}} + t_{\mathbb{R}} \quad F_{3z} - iD_z\sigma = 0 \quad F_{3\bar{z}} + iD_{\bar{z}}\sigma = 0$$

$$(50) \quad \mu_{\mathbb{C}} = 0 \quad D_{\bar{z}}X_j = 0 \quad D_{\bar{z}}Y_j = 0$$

$$(51) \quad (D_3 + \sigma)X = 0 \quad (D_3 - \sigma)Y = 0.$$

These equations are again supplemented by

$$(52) \quad (\varphi + m_j + \frac{\epsilon}{2} + \epsilon z D_z)X_j = 0 \quad (-\varphi - m_j + \frac{\epsilon}{2} + \epsilon z D_z)Y_j = 0.$$

Note that x^3 -independent solutions of these equations reduce in axial gauge $A_3 = 0$ to supersymmetric ground states preserving all of the supercharges of the $\mathcal{N} = 4$ quantum mechanics.

In principle, a correlation function $\langle \mathbf{n} + A | v_A | \mathbf{n} \rangle$ in the supersymmetric quantum mechanics localizes to an equivariant integral over the moduli space of solutions to (49)-(51) that tend to vortex solutions as $x^3 \rightarrow \pm\infty$ with $\mathbf{n}$ and $\mathbf{n} + A$ units of flux in the $x^{1,2}$ -plane, with a monopole singularity generating A units of flux at some point on the x^3 -axis. Instead we will compute the action of monopole operators $v_A | \mathbf{n} \rangle$ directly using a complex algebraic description of the solutions to equations (49)-(51).

4.3. Algebraic Approach. Let us recall from section the complex algebraic description of the moduli space $\mathcal{M}_{\nu_i}^{\mathbf{n}}$ of solutions to the x^3 -independent half-BPS equations for $Q^{\dot{A}}, \tilde{Q}^{\dot{A}}$ with vacuum ν_i at $|z| \rightarrow \infty$. A point in $\mathcal{M}_{\nu_i}$ is specified by the following ‘holomorphic data’:

- A complex line bundle $E \cong \mathcal{O}(\mathbf{n})$.
- Holomorphic sections (X_j, Y_j) of the associated bundle $\mathcal{O}(\mathbf{n})^N \oplus \mathcal{O}(-\mathbf{n})^N$ obeying $\sum_j X_j Y_j = 0$ and lying in the complex orbit $G_{\mathbb{C}} \cdot \nu_i$ at $|z| \rightarrow \infty$.

This description was sufficient to build an explicit description of the Hilbert space of supersymmetric ground states $\mathcal{H}_{\nu_i}$ as the $U(1)_\epsilon \times T_H$ -equivariant cohomology of $\mathcal{M}_{\nu_i}$ with equivariant parameters $(m_1, \dots, m_N, \epsilon)$.

Let us now assume that at some point $x^3 = s_0$ we have solution specified by a point in $\mathcal{M}_{\nu_i}^{\mathbf{n}}$. We will now ask how the holomorphic data evolves as a function of $x^3 \geq s_0$ by solving the quarter-BPS equations for $Q^{\dot{1}}, \tilde{Q}^{\dot{1}}$. Choosing axial gauge $A_3 = 0$, it follows from the quarter-BPS equations (49)-(51) that

$$(53) \quad \partial_3 A_{\bar{z}} = -iD_{\bar{z}}\sigma \quad \partial_3 X = -\sigma X \quad \partial_3 Y = -\sigma Y.$$

This shows that evolution in the x^3 -direction is a complex gauge transformation with parameter $i\sigma$. Therefore provided σ is smooth, the holomorphic type of the bundle $E \simeq \mathcal{O}(\mathbf{n})$ cannot change. Together with $D_{\bar{z}}X = 0$, $D_{\bar{z}}Y = 0$ and $\mu_{\mathbb{C}} = 0$, this ensures that the holomorphic data are constant in the x^3 -direction. More precisely, the holomorphic data at nearby s and s' are related by a globally invertible, holomorphic gauge transformation $g(z; s, s')$.

However, at a collection of points $\{s_i\}$ the holomorphic data can jump due to the presence of a monopole operator on the x^3 -axis with a singularity for σ . The holomorphic data at $x^3 < s_i$ and $x^3 > s_i$ are then related by a ‘singular’ complex

gauge transformation $g(z)$ that is only invertible in the complement of the origin $z = 0$. In our example, we consider the singular gauge transformations,

$$(54) \quad g(z) \sim z^A \quad A \in \mathbb{Z},$$

corresponding to the insertion of a monopole operator v_A at $x^3 = s_i$ and $z = 0$. This is known as a ‘Hecke modification’ of the holomorphic data. Such modifications were analyzed by Kapustin and Witten [11] in a four-dimensional lift of our current setup.

4.4. Action on Hilbert Space. The action of the singular gauge transformation $g(z) = z^A$ on the holomorphic data is summarized as follows:

- If $A \geq 0$, the gauge transformation sends $X_i(z) \mapsto z^A X_i(z)$. This creates A vortices at the origin of the z -plane.
- If $A < 0$, the transformation sends $X_i(z) \mapsto z^{-|A|} X_i(z)$. Regularity of this modification requires that $X_i(z)$ have a zero of order A at $z = 0$. In other words, there must exist A vortices at the origin of the z -plane to be destroyed by the monopole operator.

To determine the coefficients $c_{A,n}$ in equation (46), we examine the action of the singular gauge transformation in a neighborhood of the fixed points of $\mathcal{M}_{\nu_i}^n$ and $\mathcal{M}_{\nu_i}^{n+A}$. Note that if $A > 0$, the singular gauge transformation z^A is a composition of A singular gauge transformations z . In terms of monopole operators, we therefore write $v_A = (v_+)^A$. Similarly, if $A < 0$ we write $v_A = (v_-)^{|A|}$. Thus it suffices to determine the action of v_+ and v_- .

Let us therefore consider the action of the monopole operator v_+ on the state $|\mathbf{n} - 1\rangle$. A vortex configuration in a neighborhood of the origin of $\mathcal{M}_{\nu_i}^{n-1}$ has the general form

$$(55) \quad X_j(z) = z^{n-1} \delta_{ij} + \sum_{l=0}^{n-2} x_{j,l+1} z^l.$$

This is mapped by the singular gauge transformation $g(z) = z$ to

$$(56) \quad g(z) X_j(z) = z^n \delta_{ij} + \sum_{l=1}^{n-1} x_{j,l} z^l.$$

Thus the image of $g(z)$ is the subspace of $\mathcal{M}_{\nu_i}^n \cong \mathbb{C}^{nN}$ with $x_{j,0} = 0$ for all $j = 1, \dots, N$. This means that $|\mathbf{n} - 1\rangle$ is mapped to $|\mathbf{n}\rangle$, times an equivariant δ -function imposing the constraints $x_{j,0} = 0$. We therefore multiply by the equivariant weights of the coordinates $x_{j,0}$ for $j = 1, \dots, N$. The result is

$$(57) \quad v_+ |\mathbf{n} - 1\rangle = P(-m_i - n\epsilon) |\mathbf{n}\rangle,$$

where $P(u) = \prod_{j=1}^N (u + m_j)$.

On the other hand, to compute the action of the monopole operator v_- on the vortex state $|\mathbf{n}\rangle$ we consider the singular gauge transformation $g(z) = z^{-1}$. This time the subspace of $\mathcal{M}_{\nu_i}^n$ defined by $x_{j,0} = 0$ maps isomorphically onto $\mathcal{M}_{\nu_i}^{n-1}$. We therefore have $v_- |\mathbf{n}\rangle = |\mathbf{n} - 1\rangle$ for $\mathbf{n} > 0$, and $v_- |0\rangle = 0$. An alternative perspective on this computation in terms of correspondences can be found in [7].

We can therefore summarize the action of Coulomb branch chiral ring operators on the Hilbert space of supersymmetric ground states $\mathcal{H}_{\nu_i}$ by

$$\begin{aligned} \varphi|\mathbf{n}\rangle &= (-m_i - (\mathbf{n} + \tfrac{1}{2})\epsilon)|\mathbf{n}\rangle \\ v_+|\mathbf{n}\rangle &= P(\varphi + \tfrac{1}{2}\epsilon)|\mathbf{n} + 1\rangle \\ v_-|\mathbf{n}\rangle &= |\mathbf{n} - 1\rangle. \end{aligned} \quad (58)$$

A short computation shows that the monopole operators obey the algebra

$$\begin{aligned} v_+v_- &= P(\varphi + \tfrac{1}{2}\epsilon) & v_-v_+ &= P(\varphi - \tfrac{1}{2}\epsilon) \\ [\varphi, v_{\pm}] &= \mp \epsilon v_{\pm}. \end{aligned} \quad (59)$$

This is a non-commutative deformation of the Coulomb branch chiral ring (8). It is a deformation quantization of the Coulomb branch with holomorphic symplectic form $d\varphi \wedge d \log u^+$. The complex masses $(m_1, \dots, m_N)$ are the period of the quantization.

4.5. Some Representation Theory. The deformation quantization (59) is a spherical rational Cherednik algebra in the mathematical literature. It is graded by the topological symmetry $G_C \simeq U(1)_t$ under which φ, u^+, u^- have charge 0, $-1, +1$. For generic complex masses $(m_1, \dots, m_N)$, every vortex state $|\mathbf{n}\rangle \in \mathcal{H}_{\nu_i}$ can be obtained by acting on $|0\rangle$ with the monopole operators v_+ of negative grading. The Hilbert spaces of supersymmetric ground states $\mathcal{H}_{\nu_i}$ transform as in equation (58) as highest-weight Verma modules of the spherical rational Cherednik algebra with respect to this grading.

In the special case $N = 2$, the deformation quantization is isomorphic to a central quotient of the universal enveloping algebra $U(\mathfrak{sl}_2)$, with the quadratic Casimir element fixed by the complex masses $(m_1, \dots, m_N)$. In particular, defining the generators

$$h = 2\varphi \quad e = -v_- \quad f = v_+ \quad (60)$$

we find

$$[h, e] = 2\epsilon e, \quad [h, f] = -2\epsilon f, \quad [e, f] = \epsilon h, \quad (61)$$

and

$$C_2 = \frac{1}{2}h^2 + ef + fe = \frac{1}{2}((m_1 - m_2)^2 - \epsilon^2). \quad (62)$$

The enveloping algebra $U(\mathfrak{sl}_2)$ at a generic value of the central charge admits two irreducible Verma modules, which can be identified with the Hilbert spaces of supersymmetric ground states $\mathcal{H}_{\nu_1}, \mathcal{H}_{\nu_2}$ associated to the two isolated massive vacua.

5. Boundary conditions and overlaps

In this section, we enrich the setup considered previously by adding boundary conditions $\mathcal{B}$ that preserve a 2d $\mathcal{N} = (2, 2)$ supersymmetry algebra in the $x^{1,2}$ -plane with vector R-symmetry $U(1)_H$. Large families of boundary conditions of this type were introduced in [6]. Such boundary conditions preserve the supercharges $Q^i, \tilde{Q}^i$ of the $\mathcal{N} = 4$ supersymmetric quantum mechanics. Correlation functions involving such boundary conditions can then be performed by localization to the appropriate solutions of the quarter-BPS equations (49)-(51).

Boundary condition of this type that are compatible with a real FI parameter $t_{\mathbb{R}} < 0$ and generic complex masses $(m_1, \dots, m_N)$ will define a state in the Hilbert space $\mathcal{H}_{\nu_i}$ of supersymmetric ground states. In section 5.1, we will construct this

boundary state for a class of Neumann boundary conditions in our abelian example. In this case, the boundary state is a coherent state of vortices, or equivalently a generalized eigenvector of the monopole operators $u^\pm$. Mathematically, it defines a generalized Whittaker vector in $\mathcal{H}_\nu$.

Compactifying the three-dimensional theory on an interval with such boundary conditions at either end leads to a 2d $\mathcal{N} = (2, 2)$ gauge theory. In section 5.2, we show that the vortex partition function of this 2d $\mathcal{N} = (2, 2)$ theory in Ω -background is an inner product of the corresponding boundary states in $\mathcal{H}_{\nu_i}$. This can be viewed as a finite version of the AGT correspondence, vastly extending and providing the correct physical setup for the beautiful mathematical work [3].

5.1. Neumann Boundary Conditions. We will focus here on boundary conditions that involve Neumann boundary conditions for the gauge field and therefore preserve the gauge symmetry at the boundary. The boundary conditions for a $G = U(1)$ vectormultiplet are [6]

$$(63) \quad F_{3j}| = 0 \quad \partial_3 \varphi| = 0 \quad \sigma + i\gamma| = \tau_{2d}$$

where γ is the dual photon and $\tau_{2d} = t_{2d} + i\theta_{2d}$ is a combination of a boundary FI parameter and theta angle. The exponential $\xi = e^{\tau_{2d}}$ transforms as the bottom component of a 2d $\mathcal{N} = (2, 2)$ twisted chiral multiplet.

The remaining boundary conditions for the N hypermultiplets are labelled by a sign vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ with

$$(64) \quad \begin{array}{ll} \varepsilon_j = + & : \quad D_s X_j| = 0 \quad Y_j| = 0 \\ \varepsilon_j = - & : \quad D_s Y_j| = 0 \quad X_j| = 0. \end{array}$$

We therefore label Neumann boundary conditions by $\mathcal{N}_{\varepsilon, \xi}$.

Recall that monopole operators are given semi-classically by

$$(65) \quad v_\pm \sim e^{\pm(\sigma + i\gamma)}$$

and one might therefore expect that $v_\pm| \sim \xi^\pm$ for a monopole operator brought to the Neumann boundary condition. However, 1-loop quantum corrections modify this relation so that the boundary Ward identity for the action of bulk monopole operators $v_\pm$ on the boundary state is given by [6]

$$(66) \quad \begin{aligned} v_+|\mathcal{N}_\varepsilon\rangle &= \xi \prod_{i \text{ s.t. } \varepsilon_i = +} (\varphi + m_i + \tfrac{\varepsilon}{2}) |\mathcal{N}_\varepsilon\rangle, \\ v_-|\mathcal{N}_\varepsilon\rangle &= \xi^{-1} \prod_{i \text{ s.t. } \varepsilon_i = -} (-\varphi - m_i + \tfrac{\varepsilon}{2}) |\mathcal{N}_\varepsilon\rangle. \end{aligned}$$

Note that the factors appearing on the right are the equivariant weights of the chiral fields with Neumann boundary conditions: X_j if $\varepsilon_j = +$ and Y_j if $\varepsilon_j = -$. It is easy to check that this is compatible with the algebra (58). The states $|\mathcal{N}_{\varepsilon, \zeta}\rangle$ are known as generalized Whittaker vectors.

One way to derive equation (66) directly would be to compute the overlaps $\langle n|\mathcal{N}_{\varepsilon, \xi}\rangle$ from the path integral with Neumann boundary condition at $x^3 \rightarrow -\infty$ and the vortex configuration corresponding to the fixed point $X_j(z) = \delta_{ij} z^n$ at $x^3 \rightarrow +\infty$. This would reduce to an equivariant integral over solutions to the quarter BPS equations for $\tilde{Q}^1, Q^1$ with these boundary conditions.

Let us examine in more detail the Neumann boundary condition with $\varepsilon = (+, \dots, +)$ where X_j all have Neumann boundary conditions and Y_j all have Dirichlet boundary conditions. In this case, the boundary state obeys

$$(67) \quad \begin{aligned} v_+ |\mathcal{N}_{\varepsilon, \xi}\rangle &= \xi P(\varphi + \frac{\epsilon}{2}) |\mathcal{N}_{\varepsilon, \xi}\rangle \\ v_- |\mathcal{N}_{\varepsilon, \xi}\rangle &= \xi^{-1} |\mathcal{N}_{\varepsilon, \xi}\rangle. \end{aligned}$$

with solution

$$(68) \quad |\mathcal{N}_{(+, \dots, +), \xi}\rangle = \xi^{\varphi/\epsilon} \sum_{\mathbf{n} \geq 0} |\mathbf{n}\rangle.$$

This state can be characterized as a coherent state of vortices: it is an eigenvector of the annihilation operator v_- and has non-vanishing overlap with all vortex states $|\mathbf{n}\rangle$, $\mathbf{n} \geq 0$. This is to be expected as this Neumann boundary condition is compatible with all vortex configurations for $t_{\mathbb{R}} < 0$.

In the opposite case, $\varepsilon = (-, \dots, -)$, there is no non-trivial solution of the boundary Ward identities (66) in $\mathcal{H}_{\nu_j}$ and therefore $|\mathcal{N}_{\varepsilon, \xi}\rangle = 0$. This is compatible with the observation that this Neumann boundary condition is incompatible with solutions of the vortex equations for $t_{\mathbb{R}} < 0$ so that $\langle \mathbf{n} | \mathcal{N}_{\varepsilon, \xi} \rangle = 0$ for all $\mathbf{n} \geq 0$. The intermediate cases are discussed in [6].

5.2. Overlaps. With the above results, we can now compute the partition function of our theory on an interval with Neumann boundary condition $\mathcal{N}_{\varepsilon, \xi}$ and $\mathcal{N}_{\varepsilon', \xi'}$ at either end - see figure. Let us denote the partition function of this system by $Z_{\nu_i}(q)$ where we define $q = \xi/\xi'$. This partition function can be computed in two ways:

- 1) In the $\mathcal{N} = 4$ supersymmetric quantum mechanics, the Neumann boundary conditions defines states $|\mathcal{N}_{\varepsilon, \xi}\rangle$ and $|\mathcal{N}_{\varepsilon', \xi'}\rangle$ in $\mathcal{H}_{\nu_i}$ and the partition function Z_{ν_i} is the overlap $\langle \mathcal{N}_{\varepsilon', \xi'} | \mathcal{N}_{\varepsilon, \xi} \rangle$.
- 2) Since the partition function is independent of the length of the interval, we can send this length to zero to obtain a 2d $\mathcal{N} = (2, 2)$ theory $\mathcal{T}_{2d}$. The partition function $Z_{\nu_i}(q)$ is then identified with the vortex partition function of $\mathcal{T}_{2d}$.

The equivalence of these computations can be viewed as a finite analogue of the AGT correspondence, providing a vast generalization and the correct physical setup of the beautiful mathematical work [3].

We consider the case of Neumann boundary conditions with $\varepsilon = \varepsilon' = (+, \dots, +)$. The overlap of boundary states (68) is

$$(69) \quad Z_{\nu_i}(q) = \langle \mathcal{N}_{\varepsilon, \xi} | \mathcal{N}_{\varepsilon, \xi'} \rangle = \sum_{\mathbf{n} \geq 0} \frac{q^{\frac{m_j}{\epsilon} - \frac{1}{2} - \mathbf{n}}}{\prod_{l=0}^{\mathbf{n}-1} P(-m_i + (l - \mathbf{n})\epsilon)}$$

This is exactly vortex partition function of the 2d $\mathcal{N} = (2, 2)$ theory $\mathcal{T}_{2d}$ with gauge group $U(1)$ and N chiral multiplets X_j of charge $+1$ and an exponentiated complexified FI parameter $q = \xi/\xi'$. [1, 8]. Mathematically, it is the equivariant J-function of $\mathbb{CP}^{N-1}$.

5.3. Differential equations. The vortex partition functions (69) are generalized hypergeometric functions, which satisfy an N -th order differential equation in the parameter q . This differential equation can be explicitly derived from the

relation $\mathcal{Z}_{\nu_i}(q) = \langle \mathcal{N}_{\varepsilon', \zeta'} | \mathcal{N}_{\varepsilon, \zeta} \rangle$ and the defining properties of the boundary states. As above, we focus in the case $\varepsilon = \varepsilon' = (+, \dots, +)$.

Our starting point is the differential equation

$$(70) \quad \epsilon \xi \frac{d}{d\xi} |\mathcal{N}_{\varepsilon, \xi}\rangle = \varphi |\mathcal{N}_{\varepsilon, \xi}\rangle,$$

which follows immediately from equation (68). Recalling that $v_+ v_- = P(\varphi + \frac{\epsilon}{2})$ we now have

$$(71) \quad \begin{aligned} P\left(\epsilon q \frac{\partial}{\partial q} + \frac{\epsilon}{2}\right) \mathcal{Z}_{\nu_i}(q) &= \langle \mathcal{N}_{\varepsilon, \xi'} | P(\varphi + \frac{\epsilon}{2}) | \mathcal{N}_{\varepsilon, \xi} \rangle \\ &= \langle \mathcal{N}_{\varepsilon, \xi'} | v_+ v_- | \mathcal{N}_{\varepsilon, \xi} \rangle \\ &= q^{-1} \mathcal{Z}_{\nu_i}(q), \end{aligned}$$

which is the N -th order generalized hypergeometric equation satisfied by (69). Note that the derivation did not depend on the choice of vacuum ν_i : the N different choices of vacuum produce a basis linearly independent solutions.

6. Vortex quantum mechanics

In sections 2 and 3, we argued that a 3d $\mathcal{N} = 4$ gauge theory in an Ω -background in the $x^{1,2}$ -plane localizes to an $\mathcal{N} = 4$ supersymmetric quantum mechanics on the x^3 -axis. The space of supersymmetric vacua decomposed as a direct sum

$$(72) \quad \mathcal{H}_\nu = \bigoplus_{\mathbf{n} \geq 0} \mathcal{H}_\nu^{\mathbf{n}},$$

where each summand $\mathcal{H}_{\nu_i}^{\mathbf{n}}$ is given by the equivariant cohomology of a moduli space of vortices $\mathcal{M}_\nu^{\mathbf{n}}$ with vortex number $\mathbf{n}$.

An alternative approach is to describe each summand in isolation as a massive gauged supersymmetric quantum mechanics $Q(\nu, \mathbf{n})$, whose Higgs branch is the moduli space of vortices $\mathcal{M}_{\nu_i}^{\mathbf{n}}$ and whose space of supersymmetric vacua is $\mathcal{H}_{\nu_i}$. The matter content of each quantum mechanics is known from the brane construction of $\mathcal{M}_\nu^{\mathbf{n}}$. The monopole operators v_A are realized as a family of half-BPS interfaces between quantum mechanics $Q(\nu, \mathbf{n})$ and $Q(\nu, \mathbf{n} + A)$. This approach is explored in section 6 of reference [7] and shown to reproduce the results that we have presented here.

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