Computing Small Pivot-Minors*

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Abstract. A graph G contains a graph H as a pivot-minor if H can be obtained from G by applying a sequence of vertex deletions and edge pivots. Pivot-minors play an important role in the study of rank-width. However, so far, pivot-minors have only been studied from a structural perspective. We initiate a systematic study into their complexity aspects. We first prove that the Pivot-Minor problem, which asks if a given graph G contains a given graph H as a pivot-minor, is NP-complete. If H is not part of the input, we denote the problem by H-Pivot-Minor. We give a certifying polynomial-time algorithm for H-Pivot-Minor for every graph H with $|V(H)| \leq 4$ except when $H \in \{K_4, C_3 + P_1, 4P_1\}$, via a structural characterization of H-pivot-minor-free graphs in terms of a set $\mathcal{F}_H$ of minimal forbidden induced subgraphs.

1 Introduction

Computing whether a graph H appears as a "pattern" inside some other graph G is a well-studied problem in the area of structural and algorithmic graph theory.

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The definition of a pattern depends on the set of graph operations that we are allowed to use. For instance, if we can obtain H from G via a sequence of vertex deletions, edge deletions and edge contractions, then G contains H as a minor. The Minor problem is that of testing whether a given graph G contains a given graph H as a minor. This problem is known to be NP-complete even if G and H are trees of small diameter [19]. Hence, it is natural to fix the graph H and let the input consist of only G. This leads to the H-Minor problem, and a celebrated result of Robertson and Seymour [28] states that the H-Minor problem can be solved in cubic time for every graph H. If we only allow vertex deletions and edge contractions, then we obtain the H-Induced Minor problem. In contrast, this problem can be NP-complete (see [9] for an example of a "hard" graph H on 68 vertices). Other well-known containment relations include containing a graph H as a contraction, an induced subgraph, a subdivision, or an (induced) topological minor; see, e.g. [3,13,17,18,29] for some complexity results for these relations.

We focus on the pivot-minor containment relation, defined as follows. The *local complementation* at a vertex u in a graph G replaces every edge of the subgraph induced by the neighbours of u by a non-edge, and vice versa. We denote the resulting graph by $G * u$. An *edge pivot* is the operation that takes an edge uv, first applies a local complementation at u, then at v, and then at u again. We denote the resulting graph by $G \wedge uv = G * u * v * u$ and note that $G * u * v * u = G * v * u * v$, so $G \wedge uv = G \wedge vu$. Alternatively, we can define the edge pivot operation as follows. Consider the set S_u of neighbours of u that are non-adjacent to v, the set S_v of neighbours of v that are non-adjacent to u and the set S_{uv} of common neighbours of u and v. Replace every edge between any two vertices in distinct sets from $\{S_u, S_v, S_{uv}\}$ by a non-edge and vice versa. Then delete every edge between u and S_u and add every edge between u and S_v. Similarly, delete every edge between v and S_v and add every edge between v and S_u. See Fig. 1 for an example. A graph G contains a graph H as a *pivot-minor* if G can be modified into (an isomorphic copy of) H by a sequence of vertex deletions and edge pivots.

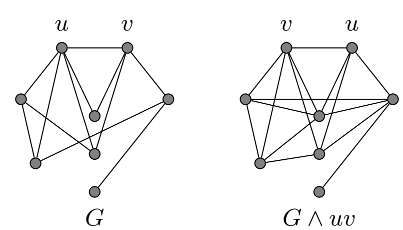

Fig. 1. An example of a graph before and after pivoting an edge.

Pivot-minors were called p-reductions by Bouchet [1] and have been studied from a structural perspective, as they form a very suitable tool for working with rank-width [22,26]. Rank-width is a well-known width parameter (see [25] for a survey) and pivot-minors play a similar role for rank-width as minors do for treewidth. Oum [23] showed that for every positive constant k the class of graphs of rank-width at most k is well-quasi-ordered under the pivot-minor relation. Kwon and Oum [16] proved that every graph of rank-width at most k is a pivot-minor of a graph of treewidth at most $2k$, and that a graph of linear rank-width at most k is a pivot-minor of a graph of path-width at most $k+1$.

Pivot-minors are closely related to so-called vertex-minors, introduced in the nineties as ℓ-reductions by Bouchet [1]. A graph G contains a graph H as a *vertex-minor* if G can be modified into (an isomorphic copy of) H by a sequence of vertex deletions and local complementations. Hence, if G contains H as a pivot-minor, then G contains H as a vertex-minor (but not necessarily vice versa). Bouchet [1] characterized circle graphs in terms of forbidden vertex-minors and by using this result, Geelen and Oum [12] were able to characterize circle graphs in terms of forbidden pivot-minors. Oum [24] conjectured that for each fixed bipartite circle graph H, every graph G of sufficiently large rank-width contains H as a pivot-minor. This conjecture is known to be true when G is a line graph, a bipartite graph or a circle graph (see [24]).

We study pivot-minors from an *algorithmic* perspective, that is, we consider the following research question:

Can we decide if a graph H is a pivot-minor of a graph G in polynomial time?

If both G and H are part of the input, then we obtain the following problem:

PIVOT-MINOR
Instance: A pair of graphs G and H.
Question: Does G have a pivot-minor isomorphic to H?

If H is not part of the input but fixed, then we obtain the H-PIVOT-MINOR problem. Question 7 in [25] asked for the complexity of H-PIVOT-MINOR, which has not been studied so far.

Our Results. We initiate a systematic study into the complexity of computing pivot minors. In Section 2 we prove that PIVOT-MINOR is NP-complete. Due to this, it is natural to study the computational complexity of H-PIVOT-MINOR, as proposed in [25]. To get a handle on this problem, we restrict ourselves to small graphs H. For every graph H on at most four vertices except for the complete graph K_4, the edgeless graph $4P_1$ and the triangle plus a vertex C_3+P_1, we give a certifying algorithm that solves H-PIVOT-MINOR in polynomial time.

To explain the idea behind our algorithms, we observe that *H-pivot-minor-free* graphs, that is, graphs that do not contain H as a pivot-minor, are closed under vertex deletion. It is well known and readily seen that a class of graphs is closed under vertex deletion if and only if it can be characterized by a (possibly infinite) set of minimal forbidden induced subgraphs. In Section 3, for every graph $H \notin \{K_4, C_3+P_1, 4P_1\}$ with $|V(H)| \leq 4$ we determine the set $\mathcal{F}_H$ of minimal forbidden induced subgraphs. We then test if the input graph G contains an induced subgraph $F \in \mathcal{F}_H$. If not, then G is H-pivot-minor-free. Otherwise, G

contains H as a pivot-minor. As the graph F found by our algorithm contains H as a pivot-minor, F is a certificate that can be used to verify H-pivot-minor containment in polynomial time: first delete all vertices of G not in F and then apply vertex deletions and edge pivots to obtain H from F. See [20] for a survey on certifying algorithms.

We discuss the graphs K_4, C_3+P_1 and $4P_1$ in Section 4. Computer experiments show that $\mathcal{F}_{4P_1}$ contains over 100,000 graphs, so it is likely that $\mathcal{F}_{4P_1}$ is not finite. We prove that $\mathcal{F}_{K_4}$ and $\mathcal{F}_{C_3+P_1}$ each contain infinitely many graphs. In the same section we discuss some further computer experiments and propose a general framework for future research.

2 When *H* Is Part of the Input

We prove that Pivot-Minor is NP-complete. We first introduce some terminology and basic results on matroids, which can be found in [27]. A *matroid* is a pair $M = (E, \mathcal{I})$ of a finite set E, called the *ground set*, and a set $\mathcal{I}$ of subsets of E satisfying the following three properties: (i) $\mathcal{I} \neq \emptyset$; (ii) if $Y \in \mathcal{I}$ and $X \subseteq Y$, then $X \in \mathcal{I}$; and (iii) if $X, Y \in \mathcal{I}$ with $|Y| = |X| + 1$, then there exists an element $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$. A set $X \subseteq E$ is *independent* in $M = (E, \mathcal{I})$ if $X \in \mathcal{I}$, otherwise X is *dependent*. The *rank* of a subset $X \subseteq E$ is the size of a largest independent subset of X. The *rank* of a matroid $M = (E, \mathcal{I})$ is the rank of E. A *base* of a matroid is a maximal independent set. A *circuit* of a matroid is a minimal dependent set. The *dual matroid* M^* of a matroid $M = (E, \mathcal{I})$ is a matroid on E such that X is a base of M^* if and only if $E \setminus X$ is a base in M. For a subset X of E, we define $M \setminus X$ to be the matroid $(E \setminus X, \mathcal{I}')$ such that $\mathcal{I}' = \{X' \subseteq E \setminus X \mid X' \in \mathcal{I}\}$. We define $M/X = (M^* \setminus X)^*$. A matroid N is a *minor* of a matroid M if $N = (M \setminus X)/Y$ for some disjoint sets X and Y. A matroid $M = (E, \mathcal{I})$ is *binary* if there is a matrix over the binary field whose columns are indexed by E such that X is independent in M if and only if the corresponding columns are linearly independent. It is known that the dual matroid of a binary matroid is also binary.

A major example of binary matroids arises from graphs. For a graph $G = (V, E)$, let $\mathcal{I}$ be the set of subsets X of E such that the subgraph (V, X) has no cycles. Then $M(G) = (E, \mathcal{I})$ is a matroid, called the *cycle matroid* of G and such matroids are binary. It is known that circuits of $M(G)$ are precisely the edge sets of cycles of G and if a graph H is a minor of G, then $M(H)$ is a minor of $M(G)$.

If G is connected and has n vertices and m edges, then $M(G)$ has rank $n-1$ because any spanning tree of G has $n-1$ edges, and $(M(G))^*$ has rank $m-n+1$.

For a binary matroid $M = (E, \mathcal{I})$, the *fundamental graph* of M with respect to a base B is the bipartite graph on E with the bipartition $(B, E \setminus B)$ such that $x \in B$, $y \in E \setminus B$ are adjacent if and only if $(B \setminus \{x\}) \cup \{y\}$ is a base of M. Conversely, for a bipartite graph G with a bipartition (A, B), we may define a binary matroid $\mathrm{Bin}(G, A, B)$ on $V(G)$ represented by the $A \times V(G)$ matrix

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ A & \begin{pmatrix} I_A & M_{A,B} \end{pmatrix} \end{array}$$

over the binary field where I_A is the $A \times A$ identity matrix and $M_{A,B}$ is the $A \times B$ submatrix of the adjacency matrix of G whose (x, y)-entry is 1 if and only if x and y are adjacent. We need the following lemma for our NP-hardness result.

Lemma 1 ([22, Corollary 3.6]). *The following statements hold:*

(i) Let N, M be binary matroids, and H, G be fundamental graphs of N and M respectively. If N is a minor of M, then H is a pivot-minor of G.
(ii) Let G be a bipartite graph with bipartition $A \cup B = V(G)$. If H is a pivot-minor of G, then there is a bipartition $A' \cup B' = V(H)$ such that $\mathrm{Bin}(H, A', B')$ is a minor of $\mathrm{Bin}(G, A, B)$.

Theorem 1. Pivot-Minor *is* NP*-complete.*

Proof. We reduce from the Hamilton Cycle problem, which asks if a graph has a Hamilton cycle. This problem is NP-complete even for 3-regular graphs [10]. Let $G = (V, E)$ be a 3-regular graph with n vertices and m edges. We may assume without loss of generality that $n \geq 5$ and that G is connected. As G is 3-regular, $2m = 3n$. Consequently, $(M(G))^*$ has rank $m - n + 1 = \frac{1}{2}n + 1$.

Let T be a spanning tree of G. Let G_T be the fundamental graph of $M(G)$ with respect to $E(T)$, which can be built in polynomial time. We claim G has a Hamilton cycle if and only if the n-vertex star $K_{1,n-1}$ is a pivot-minor of G_T.

For the forward direction, suppose G has a Hamilton cycle C. Then G contains C as a minor and thus $M(G)$ has $M(C)$ as a minor, and so G_T has every fundamental graph of $M(C)$ as a pivot-minor by Lemma 1(i). This proves the forward direction, because every fundamental graph of $M(C)$ is isomorphic to $K_{1,n-1}$.

For the reverse direction, suppose that $K_{1,n-1}$ is a pivot-minor of G_T. Then by Lemma 1(ii), $V(K_{1,n-1})$ has a bipartition (A', B') such that $\mathrm{Bin}(K_{1,n-1}, A', B')$ is a minor of $M(G) = \mathrm{Bin}(G_T, A, B)$ for some partition (A, B) of $V(G_T)$. As $K_{1,n-1}$ is connected, it admits only two possible bipartitions (that is, there is a unique way of partitioning the vertices of $K_{1,n-1}$ into two independent sets and there are two ways to order the sets). So $\mathrm{Bin}(K_{1,n-1}, A', B')$ is either $M(C)$ or its dual $(M(C))^*$, where C is the cycle on n vertices. Therefore $M(C)$ or $(M(C))^*$ is a minor of $M(G)$. Equivalently, $M(C)$ is a minor of $M(G)$ or $(M(G))^*$. Because the rank of $M(C)$ is $n-1$ and the rank of $(M(G))^*$ is $\frac{1}{2}n+1 < n-1$ (as $n \geq 5$) we find that $M(C)$ cannot be a minor of $(M(G))^*$. Thus, $M(C)$ is a minor of $M(G)$ and therefore $M(G)$ has a circuit of length at least n. This implies that G has a cycle of length n. □

3 When *H* Is Fixed

We give a certifying algorithm for recognizing H-pivot-minor-free graphs for every graph H on at most four vertices except for the cases where $H \in \{K_4, C_3 + P_1, 4P_1\}$ (see Section 4 for a further discussion on these three graphs). For each such graph H, we determine the minimal set $\mathcal{F}_H$ such that a graph G

contains H as a pivot-minor if and only if G contains an induced subgraph in $\mathcal{F}_H$. The cases where $H \in \{2P_1 + P_2, 2P_2\}$ are too involved to expect a combinatorial proof, so we rely on a computer-based proof for these cases. Of the remaining cases, the ones where H is not $3P_1$-free are more involved than the others. We therefore consider the $3P_1$-free cases in Section 3.1 and the remaining cases in Section 3.2.

3.1 When H Is $3P_1$-Free

The graph $\overline{G} = (V, \{uv \mid uv \notin E(G), u \neq v\}$ is the *complement* of a graph G. A *co-component* in a graph G is a maximal set of vertices in G that induces a connected subgraph in $\overline{G}$. The graph $G_1 + G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ is the *disjoint union* of two vertex-disjoint graphs G_1 and G_2. Recall that $K_{1,n-1}$ is the star on n vertices. The path and cycle on n vertices are denoted P_n and C_n, respectively; the *length* of a path or cycle is the number of edges it contains. The *paw*, *diamond*, *dart* and *claw* are the graphs $\overline{P_1 + P_3}$, $\overline{2P_1 + P_2}$, $\overline{P_1 + \text{paw}}$ and $K_{1,3}$, respectively (see also Fig. 2). A graph class is *pivot-minor-closed* if it is closed under vertex deletions and edge pivots. A graph G is $(H_1, \ldots, H_p)$*-free* for a set $\mathcal{H} = \{H_1, \ldots, H_p\}$ of graphs if G has no induced subgraph isomorphic to a graph in $\mathcal{H}$. Let $H \notin \{K_4, C_3 + P_1\}$ be a $3P_1$-free graph with $|V(H)| \leq 4$, so $H \in \{P_1, 2P_1, P_1 + P_2, P_2, 2P_2, P_3, P_4, C_3, C_4, \text{paw}, \text{diamond}\}$. The cases $H = P_1$, $H = P_2$ and $H = 2P_1$ are trivial. We now consider the other cases (we omit the proofs of Propositions 1–3).

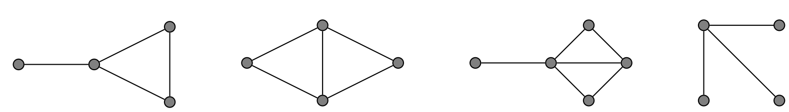

Fig. 2. Graphs referred to in Section 3.

Proposition 1. *For a graph G, P_3 is a pivot-minor of G if and only if P_3 is an induced subgraph of G.*

Proposition 2. *For a graph G, C_3 is a pivot-minor of G if and only if an odd cycle is an induced subgraph of G if and only if G is not bipartite.*

Proposition 3. *The following statements are equivalent for every graph G:*

(i) $P_1 + P_2$ is a pivot-minor of G.
(ii) $P_1 + P_2$, C_4 or the diamond is an induced subgraph of G.
(iii) G is neither a complete graph, an edgeless graph nor a star.

A graph is a *clique-star* if it consists of pairwise vertex-disjoint cliques K, $L_1, \ldots, L_p$ for some $p \geq 0$, such that every vertex of K is adjacent to every vertex of $L_1 \cup \cdots \cup L_p$ and there is no edge between any two distinct cliques L_i and L_j. Note that we may assume that $p \neq 1$, as if $p = 1$ then the clique-star is a complete graph, in which case we can set $p = 0$. We need the following lemma (we omit the proof).

Lemma 2. *The class of clique-stars is pivot-minor-closed.*

Proposition 4. *The following statements are equivalent for every graph G.*

(i) *P_4 is a pivot-minor of G.*
(ii) *C_4 is a pivot-minor of G.*
(iii) *P_4, C_4 or the dart is an induced subgraph of G.*
(iv) *G has a component that is not a clique-star.*

Proof. Both the P_4 and C_4 can be obtained from each other by pivoting one edge and so (i) and (ii) are equivalent. Pivoting an edge incident to a vertex of degree 2 and a vertex of degree 3 in the dart yields a bull (see Fig. 3), which contains P_4 as an induced subgraph. Therefore the dart contains P_4 as a pivot-minor, so (iii) implies (i) and (ii). As P_4, C_4 and the dart are not clique-stars, (iii) implies (iv). Lemma 2 implies that the class of graphs all of whose components are clique-stars is pivot-minor-closed, hence (i) and (ii) imply (iv).

It remains to prove that (iv) implies (iii). Suppose that G has a component D that is not a clique-star. Also assume that G is (P_4, C_4)-free. It is well known that the complement of a connected P_4-free graph on at least two vertices is disconnected [4]. Hence we can partition $V(D)$ into two sets A and B, such that every vertex of A is adjacent to every vertex of B. Moreover, as D is not a complete graph, we may assume that B is not a clique. If A is not a clique either, then two non-adjacent vertices of A, together with two non-adjacent vertices of B, form an induced C_4, a contradiction. Hence A is a clique. We may assume that A is chosen to be maximal subject to the condition that every vertex of A is adjacent to every vertex of B and B contains two non-adjacent vertices.

Suppose B induces a connected subgraph. Then, since $G[B]$ is P_4-free, connected, and contains at least two vertices, we can partition B into two non-empty sets B_1 and B_2 such that every vertex of B_1 is adjacent to every vertex of B_2. As B is not a clique, this means that at least one of B_1 and B_2, say B_2, is not a clique. Then, by the same argument as before, B_1 must be a clique. This implies that every vertex of B_1 is adjacent to every other vertex of B_1 and to every vertex of B_2. However, this contradicts the maximality of A, as we could have chosen $A \cup B_1$ instead. Hence B does not induce a connected subgraph of D.

Let $J_1, \ldots, J_r$ be the components of $D[B]$ for some $r \geq 2$. Since D is not a clique-star, one of $J_1, \ldots, J_r$, say J_1, is not complete. If follows that J_1 contains an induced P_3, say on vertices u, v, w. Then u, v, w, together with a vertex of A and a vertex of J_2, induce a dart. □

Proposition 5. *The following statements are equivalent for every graph G.*

(i) The paw is a pivot-minor of G.
(ii) The diamond is a pivot-minor of G.
(iii) The paw, the diamond or an odd cycle of length at least 5 is an induced subgraph of G.
(iv) G has a component that is neither bipartite nor complete.

Proof. By pivoting one edge, the diamond can be obtained from the paw and so (i) and (ii) are equivalent. Since every odd cycle on at least five vertices contains the paw as a pivot-minor, (iii) implies (i) and (ii). As the classes of complete graphs and bipartite graphs are pivot-minor-closed, (i) and (ii) imply (iv).

To prove that (iv) implies (iii), suppose (iii) is false. Let D be a component of G. We claim that D is bipartite or complete. If not, C_3 is a proper induced subgraph of D. Let K be a maximal clique of D containing the vertices of a C_3. As D is not complete and K is maximal, there is a vertex $u \in V(D) \setminus K$ that has both a neighbour and a non-neighbour in K. Since K is a clique of size at least 3, D contains the paw or diamond as an induced subgraph, a contradiction. □

We proved the next proposition by computer (see [6] for source code).

Proposition 6 (proved by computer). *The set $\mathcal{F}_{2P_2}$ has size 9.*

A sequence S of vertex deletions and edge pivots is an *H-pivot-minor-sequence* of a graph G if H can be obtained from G after applying the operations of S.

Theorem 2. *For $H \in \{P_1, 2P_1, P_2, P_1 + P_2, P_3, C_3, 2P_2, P_4, C_4, paw, diamond\}$, there is a polynomial-time algorithm for H-*Pivot-Minor* that gives an H-pivot-minor-sequence (if one exists).*

Proof. The cases when $H \in \{P_1, 2P_1, P_2\}$ are trivial. By Propositions 1, 3, 4 and 6, the set $\mathcal{F}_H$ of minimal obstructions is finite if $H \in \{P_3, P_1 + P_2, P_4, C_4, 2P_2\}$. If $H = C_3$, by Proposition 2, we need to find an odd cycle F, which we do in polynomial time by testing bipartiteness. If $H \in \{\text{paw}, \text{diamond}\}$, then we use condition (iv) in Proposition 5 to decide if a graph has H as a pivot-minor; this allows us to find a forbidden induced subgraph F efficiently by using the argument in its proof. Then the theorem follows, as in polynomial time we can find the vertex deletions and edge pivots that modify F into H. □

3.2 When H Is Not $3P_1$-Free

We now consider the cases where $H \in \{3P_1, 2P_1 + P_2, P_1 + P_3, \text{claw}\}$. The *bull* is the graph obtained from P_5 by adding an edge between the second vertex and the fourth vertex. The graph W_4 is obtained from C_4 by adding one vertex adjacent to all vertices of C_4. The graph BW_3 is the bipartite graph on seven vertices obtained from C_6 by adding one vertex adjacent to three pairwise non-adjacent vertices of the cycle. We will work with the complement of BW_3, denoted by $\overline{BW_3}$. See Fig. 3 for pictures of the bull, W_4 and $\overline{BW_3}$.

We write G/v to denote $(G \wedge zv) - v$ if a vertex v has a neighbour z and $G - v$ if v is isolated. Two graphs are *pivot-equivalent* if they can be obtained from

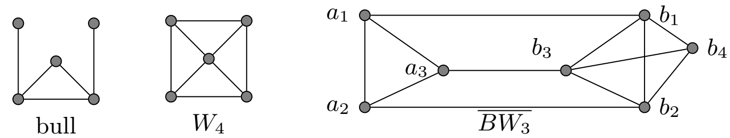

Fig. 3. Forbidden graphs from Section 3.2.

each other by a sequence of edge pivots. For two distinct neighbours x, y of v, because $(G \wedge xv) - v = (G \wedge yv \wedge xy) - v = (G \wedge yv - v) \wedge xy$, we find that $(G \wedge xv) - v$ is pivot-equivalent to $(G \wedge yv) - v$ and thus the choice of neighbour of v does not change the pivot-equivalence of graphs G/v. We need two lemmas (we omit the proofs). Lemma 4 holds in the context of binary delta-matroids or matrix pivots (see [2,24]) and its proof is inspired by the analogous proof for vertex-minors in [12].

Lemma 3. *Let v, x, y be distinct vertices of a graph G. If $xy \in E(G)$, then $(G \wedge xy) - v$ is pivot-equivalent to $G - v$ and $(G \wedge xy)/v$ is pivot-equivalent to G/v.*

Lemma 4. *If a graph H is a pivot-minor of a graph G and $v \in V(G) \setminus V(H)$, then H is a pivot-minor of $G - v$ or $(G \wedge vw) - v$ for some neighbour w of v in G.*

The proofs for the cases where $H \in \{P_1 + P_3, \text{claw}\}$ rely on the proof for the $H = 3P_1$ case. Our proof for the $H = 3P_1$ case focuses on showing that if a graph G contains $3P_1$ as a pivot-minor, then G contains a graph from $\{3P_1, W_4, \overline{BW_3}\}$ as an induced subgraph. We will do this by induction on $|V(G)|$. Since $3P_1$ is edgeless, we cannot pivot any edge in it. Therefore, the above claim holds if $|V(G)| \leq 3$, and so we may assume that $|V(G)| \geq 4$. If G has a pivot-minor isomorphic to $3P_1$, then by Lemma 4, there is a vertex $v \in V(G)$ such that $G - v$ or G/v contains a pivot-minor isomorphic to $3P_1$ for some neighbour w of v. Clearly, if $G - v$ contains $3P_1$, W_4 or $\overline{BW_3}$ as an induced subgraph then G also contains this graph as an induced subgraph. Therefore, by the induction hypothesis, we may assume that G/v contains $3P_1$ as a pivot-minor. Lemmas 5, 6 and 7, we show that if G/v contains an induced subgraph isomorphic to $3P_1$, W_4 or $\overline{BW_3}$, then G contains an induced subgraph in $\{3P_1, W_4, \overline{BW_3}\}$; these lemmas (we omit the proofs) will form the main steps in our induction.

Lemma 5. *Let vw be an edge of a graph G. If $(G \wedge vw) - v$ contains $3P_1$ as an induced subgraph, then G contains $3P_1$ or W_4 as an induced subgraph.*

Lemma 6. *Let vw be an edge of a graph G. If $G \wedge vw$ contains W_4 as an induced subgraph, then G contains $3P_1$, W_4 or $\overline{BW_3}$ as an induced subgraph.*

Lemma 7. *Let G be a graph containing an edge vw. If $G \wedge vw$ contains $\overline{BW_3}$ as an induced subgraph, then G contains $3P_1$, W_4 or $\overline{BW_3}$ as an induced subgraph.*

Proposition 7. *A graph G contains $3P_1$ as a pivot-minor if and only if G contains a graph from $\{3P_1, W_4, \overline{BW_3}\}$ as an induced subgraph.*

Proof. We first prove the "if" part. Suppose G contains a graph $H \in \{3P_1, W_4, \overline{BW_3}\}$ as an induced subgraph. If $H = W_4$, then by pivoting an edge incident to the vertex of degree 4 we obtain a graph which contains $3P_1$ as an induced subgraph. If $H = \overline{BW_3}$, then let $U_1 = \{a_1, a_2, a_3\}$ and $U_2 = \{b_1, b_2, b_3, b_4\}$ be the two cliques of H and $a_ib_i \in E(H)$ for $i = 1, 2, 3$. By pivoting an edge a_1b_1, we obtain a subgraph induced by $\{a_2, a_3, b_2, b_3, b_4\}$ that is isomorphic to W_4.

Next, we prove the "only if" part. Suppose G contains $3P_1$ as a pivot-minor. We use induction on $|V(G)| = n$ to prove that G contains a graph from $\{3P_1, W_4, \overline{BW_3}\}$ as an induced subgraph. We may assume that $n \geq 4$.

As $n \geq 4 > |V(3P_1)|$, Lemma 4 implies that there is a vertex $v \in V(G)$ such that $G - v$ or $(G \wedge vw) - v$, for some neighbour w of v, contains $3P_1$ as a pivot-minor. If $G - v$ contains $3P_1$ as a pivot-minor, then by the induction hypothesis, $G - v$ contains an induced subgraph in $\{3P_1, W_4, \overline{BW_3}\}$, hence so does G. Now we assume that $(G \wedge vw) - v$, for some neighbour w of v, contains $3P_1$ as a pivot-minor. By the induction hypothesis, $(G \wedge vw) - v$ contains $3P_1$, W_4 or $\overline{BW_3}$ as an induced subgraph. Applying Lemmas 5, 6 and 7, respectively, we find that G contains an induced graph in $\{3P_1, W_4, \overline{BW_3}\}$. □

Proposition 8. *The following statements are equivalent for every graph G.*

(i) $P_1 + P_3$ is a pivot-minor of G.
(ii) $P_1 + P_3$, $K_{2,3}$, W_4 or $\overline{BW_3}$ is an induced subgraph of G.
(iii) G contains $3P_1$ as a pivot minor and G is not a clique-star.

Proof. It is easy to verify that $K_{2,3}$, W_4 and $\overline{BW_3}$ contain $P_1 + P_3$ as a pivot-minor. Therefore (ii) implies (i). To prove that (i) implies (iii), suppose that G contains $P_1 + P_3$ as a pivot-minor. Since $3P_1$ is a pivot-minor of $P_1 + P_3$, it follows that G contains $3P_1$ as a pivot-minor. It is easy to verify that all clique-stars are $(P_1 + P_3)$-free. Since the class of clique-stars is pivot-minor-closed by Lemma 2, it follows that all clique-stars are $(P_1 + P_3)$-pivot-minor-free. Hence G is not a clique-star. Therefore (i) implies (iii).

It remains to show that (iii) implies (ii). Suppose (ii) does not hold, that is, G is $(P_1 + P_3, K_{2,3}, W_4, \overline{BW_3})$-free. A graph is $\overline{P_1 + P_3}$-free if and only if every component of it is either complete multipartite or C_3-free [21]. Hence, as G is $(P_1 + P_3)$-free, every co-component of G is either a disjoint union of cliques or $3P_1$-free. If every co-component of G is $3P_1$-free, then since co-components are complete to each other, it follows that G is $3P_1$-free. Then G is $(3P_1, W_4, \overline{BW_3})$-free. Then, by Proposition 7, G is $3P_1$-pivot-minor-free. Assume that G has a co-component D that contains an induced $3P_1$. Then D is a disjoint union of (at least three) cliques. As G is $K_{2,3}$-free, every other co-component of G is $2P_1$-free, in which case it consists of a single vertex. Therefore the vertices in all the other co-components of G form a dominating clique. Hence G is a clique-star. □

For $H =$ claw, we need a lemma (we omit the proof) that allows us to focus on *connected* graphs.

Lemma 8. *A graph G is* (bull, claw, P_5)*-free if and only if every component of G is $3P_1$-free.*

Combining Lemma 8 with Proposition 7, it is easy to prove the following (we omit the proof).

Proposition 9. *A graph G contains the claw as a pivot-minor if and only if G contains a graph from* $\{\text{claw}, P_5, \text{bull}, W_4, \overline{BW_3}\}$ *as an induced subgraph.*

We proved the next proposition by computer (see [6] for source code).

Proposition 10 (proved by computer). *The set* $\mathcal{F}_{2P_1+P_2}$ *has size* 19.

In the same way as for Theorem 2 we use Propositions 7–10 to prove:

Theorem 3. *For* $H \in \{3P_1, 2P_1+P_2, P_1+P_3, \mathit{claw}\}$*, there is a polynomial-time algorithm for H-*Pivot-Minor *that gives an H-pivot-minor-sequence (if one exists).*

4 Future Work

We aim to continue determining the complexity of H-Pivot-Minor. We do not know yet if there is a graph H for which H-Pivot-Minor is NP-complete. Our current technique for proving polynomial-time solvability is to find the set $\mathcal{F}_H$ of minimal forbidden induced subgraphs or a structural characterization verifiable in polynomial time. Our research led to the following framework for future work.

1. For a graph H, determine if $\mathcal{F}_H$ is finite (or has a polynomial characterization). We have some preliminary results for the remaining graphs H on at most four vertices, namely K_4, $C_3 + P_1$ and $4P_1$. Using a computer, we found that $\mathcal{F}_{4P_1}$ contains over 100,000 graphs even if we only list graphs on at most twelve vertices. As such, it is likely that $\mathcal{F}_{4P_1}$ is not finite. If $H = K_4$ and $H = C_3 + P_1$, then the set $\mathcal{F}_H$ has infinite size. We also started to extend our computer approach to graphs H on more than four vertices, which yielded large finite sets $\mathcal{F}_H$ for certain graphs H. The largest finite set we have found is $\mathcal{F}_{P_2+C_4} = \mathcal{F}_{P_2+P_4}$, which contains 7932 graphs. In addition to $\mathcal{F}_{4P_1}$, we found that $\mathcal{F}_{3P_2}$ also contains over 100,000 graphs, but it is not yet feasible for us to test if the set of minimal forbidden graphs found so far is complete. Besides some further tests by computer, we also need to answer the question of whether $\mathcal{F}_H$ is infinite whenever H contains an induced subgraph H' for which $\mathcal{F}_{H'}$ is infinite.

2. For a graph H, determine if H-pivot-minor-free graphs have bounded rankwidth. If for a fixed graph H, the class of H-pivot-minor-free graphs has rank-width at most k for some constant k, then we can decide in polynomial time if a given graph G contains H as a pivot-minor. We first check in polynomial time [26] if the rank-width $\text{rw}(G)$ of G is at least $k+1$ or at most $3k+1$. If $\text{rw}(G) \geq k+1$, then G has H as a pivot-minor. If $\text{rw}(G) \leq 3k+1$, then we can decide in cubic time if G has H as a pivot-minor by adapting the approach for vertex-minor testing on graphs of bounded rank-width from [5], namely via expression in monadic second order logic with modulo-2 counting (we refer to a future paper for the details).

3. For a graph H, follow a hybrid approach by combining approaches 1 and 2.
In fact, for a graph H, it suffices to determine a sufficiently precise set $\mathcal{F}' \supseteq \mathcal{F}_H$, after which we can try to prove boundedness of rank-width of the superclass of $\mathcal{F}'$-free graphs using techniques for hereditary graph classes (see e.g. [8,14,15]).

4. For a graph H, determine whether the class of H-pivot-minor-free graphs is well-quasi-ordered by the induced subgraph relation.
For every graph H, the set $\mathcal{F}_H$ is an antichain with respect to the induced subgraph relation. Suppose that the class of H-pivot-minor-free graphs is a subclass of a hereditary class $\mathcal{H}$ that is defined by a finite collection of forbidden induced subgraphs such that $\mathcal{H}$ is well-quasi-ordered by the induced subgraph relation. Then all graphs in $\mathcal{F}_H$ are either one of the finitely-many minimal forbidden induced subgraphs for $\mathcal{H}$, or belong to $\mathcal{H}$. Since $\mathcal{H}$ is well-quasi-ordered by the induced subgraph relation and the graphs in $\mathcal{F}_H$ form an antichain, it follows that $\mathcal{F}_H$ is finite. For example, since the graph W_4 contains $3P_1$ as a pivot-minor and the class of $(3P_1, W_4)$-free graphs is well-quasi-ordered by the induced subgraph relation [7], it follows that $\mathcal{F}_{3P_1}$ is finite. Thus, even without finding the precise graphs in $\mathcal{F}_H$, it may be possible to establish that the class of H-pivot-minor-free graphs is well-quasi-ordered by the induced subgraph relation, and so conclude that the H-Pivot Minor problem is polynomial-time solvable by finiteness of $\mathcal{F}_H$.

We note that approaches 2 and 3 do not yield certifying algorithms, while approach 4 only gives a non-constructive proof that such an algorithm exists. Besides the above, a proof for the Minor Recognition conjecture [11] for binary matroids would also yield a technique to obtain complexity results for pivot-minors. In particular, if this conjecture is true, then for every graph H the H-Pivot-Minor problem is polynomial-time solvable for bipartite graphs. This follows from Lemma 1, which implies that a connected bipartite graph H is a pivot-minor of a bipartite graph G if and only if for binary matroids M and N that have G and H as fundamental graphs, respectively, N or the dual of N is a minor of M (if H is not connected, then we try all possible ways of making duals per component of H).

Finally, it would be interesting to perform a similar complexity study with respect to vertex-minors, starting by taking both G and H as part of the input.

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