Graph Isomorphism for (H_1, H_2)-Free Graphs: An Almost Complete Dichotomy*

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Abstract. We almost completely resolve the computational complexity of Graph Isomorphism for classes of graphs characterized by two forbidden induced subgraphs H_1 and H_2. Schweitzer settled the complexity of this problem restricted to (H_1, H_2)-free graphs for all but a finite number of pairs (H_1, H_2), but without explicitly giving the number of open cases. Grohe and Schweitzer proved that Graph Isomorphism is polynomial-time solvable on graph classes of bounded clique-width. By combining known results with a number of new results, we reduce the number of open cases to seven. By exploiting the strong relationship between Graph Isomorphism and clique-width, we simultaneously reduce the number of open cases for boundedness of clique-width for (H_1, H_2)-free graphs to five.

Keywords: Hereditary graph class · Induced subgraph · Clique-width · Graph isomorphism

1 Introduction

The Graph Isomorphism problem, which is that of deciding whether two given graphs are isomorphic, is a central problem in Computer Science. It is not known if this problem is polynomial-time solvable, but it is not NP-complete unless the polynomial hierarchy collapses [24]. Analogous to the use of the notion of NP-completeness, we can say that a problem is Graph Isomorphism-complete (abbreviated to GI-complete). Babai [1] proved that Graph Isomorphism can be solved in quasi-polynomial time.

In order to increase understanding of the computational complexity of Graph Isomorphism, it is natural to place restrictions on the input. This approach has yielded many graph classes on which Graph Isomorphism is polynomial-time solvable, and many other graph classes on which the problem remains GI-complete.

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We refer to [23] for a survey, but some recent examples include a polynomial-time algorithm for unit square graphs [20] and a complexity dichotomy for H-induced-minor-free graphs [2] for every graph H.

In this paper we consider the Graph Isomorphism problem for hereditary graph classes, which are the classes of graphs that are closed under vertex deletion. It is readily seen that a graph class $\mathcal{G}$ is hereditary if and only if there exists a family of graphs $\mathcal{F}_\mathcal{G}$, such that the following holds: a graph G belongs to $\mathcal{G}$ if and only if G does not contain any graph from $\mathcal{F}_\mathcal{G}$ as an induced subgraph. We implicitly assume that $\mathcal{F}_\mathcal{G}$ is a family of minimal forbidden induced subgraphs, in which case $\mathcal{F}_\mathcal{G}$ is unique. We note that $\mathcal{F}_\mathcal{G}$ may have infinite size. For instance, if $\mathcal{G}$ is the class of bipartite graphs, then $\mathcal{F}_\mathcal{G}$ consists of all odd cycles.

A natural direction for a *systematic* study of the computational complexity of Graph Isomorphism is to consider graph classes $\mathcal{G}$, for which $\mathcal{F}_\mathcal{G}$ is small, starting with the case where $\mathcal{F}_\mathcal{G}$ has size 1. A graph is *H-free* if it does not contain H as induced subgraph; conversely, we write $H \subseteq_i G$ to denote that H is an induced subgraph of G. The classification for H-free graphs [4] is due to an unpublished manuscript of Colbourn and Colbourn (see [16] for a proof).

Theorem 1 (see [4,16]). *Let H be a graph. Then* Graph Isomorphism *on H-free graphs is polynomial-time solvable if $H \subseteq_i P_4$ and* GI*-complete otherwise.*

Later, it was shown that Graph Isomorphism is polynomial-time solvable even for the class of permutation graphs [7], which form a superclass of the class of P_4-free graphs. Classifying the case where $\mathcal{F}_\mathcal{G}$ has size 2 is much more difficult than the size-1 case. Kratsch and Schweitzer [16] initiated this classification. Schweitzer [25] extended the results of [16] and proved that only a finite number of cases remain open. A graph is *(H_1, H_2)-free* if it has no induced subgraph isomorphic to H_1 or H_2. This leads to our research question:

> *Is it possible to determine the computational complexity of* Graph Isomorphism *for (H_1, H_2)-free graphs for all pairs H_1, H_2?*

We recall that the analogous research question for H-induced-minor-free graphs was fully answered by Belmonte, Otachi and Schweitzer [2], who also determined all graphs H for which the class of H-induced-minor-free graphs has bounded clique-width. Similar classifications for Graph Isomorphism [22] and boundedness of clique-width [12] are also known for H-free minor graphs.

Lokshtanov et al. [17] recently gave an FPT algorithm for Graph Isomorphism with parameter k on graph classes of treewidth at most k, and this has since been improved by Grohe et al. [13]. Whether an FPT algorithm exists when parameterized by clique-width is still open. Grohe and Schweitzer [14] proved membership of XP.

Theorem 2 ([14]). *For every c,* Graph Isomorphism *is polynomial-time solvable on graphs of clique-width at most c.*

Our Results. Combining known results [16,25] with Theorem 2, we narrow the list of open cases for Graph Isomorphism on (H_1, H_2)-free graphs to 14. Of these 14 cases, we prove that two are polynomial-time solvable (Section 3) and five others are GI-complete (Section 4). Thus we reduce the number of open cases to seven. In Section 5 we provide an explicit list of all known and open cases.

Besides Theorem 2, there is another reason why results for clique-width are of importance for Graph Isomorphism. Namely, Schweitzer [25] pointed out great similarities between proving unboundedness of clique-width of some graph class $\mathcal{G}$ and proving that Graph Isomorphism stays GI-complete for $\mathcal{G}$. We will illustrate these similarities by noting that our construction demonstrating that Graph Isomorphism is GI-complete for $(\text{gem}, P_1 + 2P_2)$-free graphs can also be used to show that this class has unbounded clique-width. This reduces the number of pairs (H_1, H_2) for which we do not know if the class of (H_1, H_2)-free graphs has bounded clique-width from six [11] to five. As such, our paper also continues a project [3,6,8,9,11,12] aiming to classify the boundedness of clique-width of (H_1, H_2)-free graphs for all pairs (H_1, H_2) (see [10] for a summary).

2 Preliminaries

We consider only finite, undirected graphs without multiple edges or self-loops. The *disjoint union* $(V(G) \cup V(H), E(G) \cup E(H))$ of two vertex-disjoint graphs G and H is denoted by $G + H$ and the disjoint union of r copies of a graph G is denoted by rG. For a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of G *induced* by S, which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. If $S = \{s_1, \ldots, s_r\}$, then we may write $G[s_1, \ldots, s_r]$ instead of $G[\{s_1, \ldots, s_r\}]$. Recall that for two graphs G and G' we write $G' \subseteq_i G$ to denote that G' is an induced subgraph of G. For a set of graphs $\{H_1, \ldots, H_p\}$, a graph G is $(H_1, \ldots, H_p)$*-free* if it has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; recall that if $p = 1$, we may write H_1-free instead of (H_1)-free. For a graph G, the set $N(u) = \{v \in V \mid uv \in E\}$ denotes the *(open) neighbourhood* of $u \in V(G)$ and $N[u] = N(u) \cup \{u\}$ denotes the *closed neighbourhood* of u. The *degree* $d_G(v)$ of a vertex v in a graph G is the number of vertices in G that are adjacent to v.

A *(connected) component* of a graph G is a maximal subset of vertices that induces a connected subgraph of G; it is *non-trivial* if it has at least two vertices, otherwise it is *trivial*. The *complement* $\overline{G}$ of a graph G has vertex set $V(\overline{G}) = V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

The graphs C_r, K_r, $K_{1,r-1}$ and P_r denote the cycle, complete graph, star and path on r vertices, respectively. Let $K_{1,n}^{+}$ and $K_{1,n}^{++}$ be the graphs obtained from $K_{1,n}$ by subdividing one edge once or twice, respectively. The graphs $K_{1,3}$, $\overline{2P_1 + P_2}$, $\overline{P_1 + P_3}$, $\overline{P_1 + P_4}$ and $\overline{2P_1 + P_3}$ are also called the claw, diamond, paw, gem and crossed house, respectively. We need the following result.

Lemma 1 ([25]). *For every fixed t,* Graph Isomorphism *is polynomial-time solvable on $(2K_{1,t}, K_t)$-free graphs.*

The graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, denotes the *subdivided claw*, that is, the tree that has only one vertex x of degree 3 and exactly three leaves, which are at distance h, i and j from x, respectively. Observe that $S_{1,1,1} = K_{1,3}$. A *subdivided star* is a graph obtained from a star by subdividing its edges an arbitrary number of times. A graph is a *path star forest* if all of its connected components are subdivided stars.

Let G be a graph and let $X, Y \subseteq V(G)$ be disjoint sets. The edges between X and Y form a *perfect matching* if every vertex in X is adjacent to exactly one vertex in Y and vice versa. A vertex $x \in V(G) \setminus Y$ is *complete* (resp. *anti-complete*) to Y if it is adjacent (resp. non-adjacent) to every vertex in Y. Similarly, X is complete (resp. anti-complete) to Y if every vertex in X is complete (resp. anti-complete) to Y. A graph is *split* if its vertex set can be partitioned into a clique and an independent set. A graph is *complete multipartite* if its vertex set can be partitioned into independent sets $V_1, \ldots, V_k$ such that V_i is complete to V_j whenever $i \neq j$.

Lemma 2 ([21]). *Every connected ($\overline{P_1 + P_3}$)-free graph is either complete multipartite or K_3-free.*

Given two graphs G and H, an *isomorphism* from G to H is a bijection $f : V(G) \to V(H)$ such that $vw \in E(G)$ if and only if $f(v)f(w) \in E(H)$. For a function $f : X \to Y$, if $X' \subseteq X$, we define $f(X') := \{f(x) \in Y \mid x \in X'\}$. The Graph Isomorphism problem is defined as follows.

Graph Isomorphism *Instance:* Graphs G and H. *Question:* Is there an isomorphism from G to H?

The *clique-width* of a graph G, denoted by $\mathrm{cw}(G)$, is the minimum number of labels needed to construct G using the following four operations:

(i) create a new graph consisting of a single vertex v with label i;
(ii) take the disjoint union of two labelled graphs G_1 and G_2;
(iii) join each vertex with label i to each vertex with label j ($i \neq j$);
(iv) rename label i to j.

A class of graphs $\mathcal{G}$ has bounded clique-width if there is a constant c such that the clique-width of every graph in $\mathcal{G}$ is at most c; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

Let G be a graph. For an induced subgraph $G' \subseteq_i G$, the *subgraph complementation* operation (acting on G with respect to G') replaces every edge present in G' by a non-edge, and vice versa, that is, the resulting graph has vertex set $V(G)$ and edge set $(E(G) \setminus E(G')) \cup \{xy \mid x, y \in V(G'), x \neq y, xy \notin E(G')\}$. Similarly, for two disjoint vertex subsets S and T in G, the *bipartite complementation* operation with respect to S and T acts on G by replacing every edge with one end-vertex in S and the other in T by a non-edge and vice versa.

Let $k \geq 0$ be a constant and let γ be some graph operation. We say that a graph class $\mathcal{G}'$ is (k, γ)-*obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

(i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and
(ii) for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times.

We say that γ *preserves* boundedness of clique-width if for any finite constant k and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is (k, γ)-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [19].
Fact 2. Subgraph complementation preserves boundedness of clique-width [15].
Fact 3. Bipartite complementation preserves boundedness of clique-width [15].

We need the following lemmas on clique-width.

Lemma 3 ([5]). *The class of $\overline{2P_1 + P_3}$-free split graphs has bounded clique-width.*

Lemma 4 ([18]). *The class of $(P_2 + P_3)$-free bipartite graphs has bounded clique-width.*

We also need the special case of [12, Theorem 3] when $V_{0,i} = V_{i,0} = \emptyset$ for $i \in \{1, \ldots, n\}$.

Lemma 5 ([12]). *For $m \geq 1$ and $n > m + 1$ the clique-width of a graph G is at least $\lfloor \frac{n-1}{m+1} \rfloor + 1$ if $V(G)$ has a partition into sets $V_{i,j}$ $(i, j \in \{1, \ldots, n\})$ with the following properties:*

1. *$|V_{i,j}| \geq 1$ for all $i, j \geq 1$.*
2. *$G[\cup_{j=1}^{n} V_{i,j}]$ is connected for all $i \geq 1$.*
3. *$G[\cup_{i=1}^{n} V_{i,j}]$ is connected for all $j \geq 1$.*
4. *For $i, j, k, \ell \geq 1$, if a vertex of $V_{i,j}$ is adjacent to a vertex of $V_{k,\ell}$, then $|k - i| \leq m$ and $|\ell - j| \leq m$.*

3 New Polynomial-Time Results

In this section we prove Theorem 3, which states that Graph Isomorphism is polynomial-time solvable on $(\overline{2P_1 + P_3}, P_2 + P_3)$-free graphs (see also Fig. 1). The complexity of Graph Isomorphism on $(\overline{2P_1 + P_3}, 2P_2)$-free graphs was previously unknown, but since this class is contained in the class of $(\overline{2P_1 + P_3}, P_2 + P_3)$-free graphs, Theorem 3 implies that Graph Isomorphism is also polynomial-time solvable on this class. Before proving Theorem 3, we first prove a useful lemma.

Lemma 6. *Let G be a $\overline{2P_1 + P_3}$-free graph containing an induced K_5 with vertex set K^G. Then $V(G)$ can be partitioned into sets $A_1^G, \ldots, A_p^G, N_1^G, \ldots, N_p^G, B^G$ for some $p \geq 5$ such that:*

(i) $K^G \subseteq \bigcup A_i^G$;

(ii) $G[\bigcup A_i^G]$ *is a complete multipartite graph, with partition* $A_1^G, \ldots, A_p^G$*;*
(iii) *For every* $i \in \{1, \ldots, p\}$*, every vertex of* N_i^G *has a neighbour in* A_i^G*, but is anti-complete to* A_j^G *for every* $j \in \{1, \ldots, p\} \setminus \{i\}$*; and*
(iv) B^G *is anti-complete to* $\bigcup A_i^G$*.*

Furthermore, given K^G*, this partition is unique (up to permuting the indices on the* A_i^G*s and corresponding* N_i^G*s) and can be found in polynomial time.*

Proof. Let G be a $\overline{2P_1 + P_3}$-free graph containing an induced K_5 with vertex set K^G. If a vertex $v \in V(G) \setminus K^G$ has two neighbours $x, x' \in K^G$ and two non-neighbours $y, y' \in K^G$, then $G[x, x', y, v, y']$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore every vertex in $V(G) \setminus K^G$ has either at most one non-neighbour in K^G or at most one neighbour in K^G. Let L^G denote the set of vertices that are either in K^G or have at most one non-neighbour in K^G and note that L^G is uniquely defined by the choice of K^G.

We claim that $G[L^G]$ is a complete multipartite graph. Suppose, for contradiction, that $G[L^G]$ is not complete multipartite. Then $G[L^G]$ contains an induced $P_1 + P_2 = \overline{P_3}$, say on vertices v, v', v'' (note that some of these vertices may be in K^G). Now each of v, v', v'' has at most one non-neighbour in K^G and if a vertex $w \in \{v, v', v''\}$ is in K^G, then it is adjacent to every vertex in $K^G \setminus \{w\}$. Therefore, since $|K^G| = 5$, there must be vertices $u, u' \in K^G \setminus \{v, v', v''\}$ that are complete to $\{v, v', v''\}$. Now $G[u, u', v', v, v'']$ is a $\overline{2P_1 + P_3}$. This contradiction completes the proof that $G[L^G]$ is complete multipartite.

We let $A_1^G, \ldots, A_p^G$ be the partition classes of the complete multipartite graph $G[L^G]$. Note that $p \geq 5$, since each A_i^G contains at most one vertex of K^G. We claim that each vertex not in L^G has neighbours in at most one set A_i^G. Suppose, for contradiction, that there is a vertex $v \in V(G) \setminus L^G$ with neighbours in two distinct sets A_i^G, say v is adjacent to $u \in A_1^G$ and $u' \in A_2^G$. Since $v \notin L^G$, the vertex v has at most one neighbour in K^G. Since $|K^G| = 5$, there must be two vertices $y, y' \in K^G \setminus (A_1^G \cup A_2^G)$ that are non-adjacent to v. Now $G[u, u', y, v, y']$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore every vertex not in L^G has neighbours in at most one set A_i^G. Let N_i^G be the set of vertices in $V(G) \setminus L^G$ that have neighbours in A_i^G and let B^G be the set of vertices in $V(G) \setminus L^G$ that are anti-complete to L^G. Finally, note that the partition of $V(G)$ into sets $A_1^G, \ldots, A_p^G, N_1^G, \ldots, N_p^G, B^G$ can be found in polynomial time and is unique (up to permuting the indices on the A_i^Gs and corresponding N_i^Gs). □

Theorem 3. Graph Isomorphism *is polynomial-time solvable on* $(\overline{2P_1 + P_3}, P_2 + P_3)$*-free graphs.*

Proof. As Graph Isomorphism can be solved component-wise, we need only consider connected graphs. Therefore, as Graph Isomorphism is polynomial-time solvable on $(K_5, P_2 + P_3)$-free graphs by Lemma 1, and we can test whether a graph is K_5-free in polynomial time, it only remains to consider the class of connected $(\overline{2P_1 + P_3}, P_2 + P_3)$-free graphs G that contain an induced K_5. Let K^G

$\overline{2P_1 + P_3}$ $\qquad$ $P_2 + P_3$

Fig. 1. Forbidden induced subgraphs from Theorem 3.

be the vertices of such a K_5 in G (note that such a set K^G can be found in polynomial time, but it is not necessarily unique). Let $A_1^G, \ldots, A_p^G, N_1^G, \ldots, N_p^G, B^G$ be defined as in Lemma 6 and let $L^G = \bigcup A_i^G$ and $D^G = V(G) \setminus L^G$.

Suppose G and H are connected $(\overline{2P_1 + P_3}, P_2 + P_3)$-free graphs that each contain an induced K_5. If G and H have bounded clique-width (which happens in Case 1 below), then by Theorem 2 we are done. Otherwise, note that if K^G and K^H are vertex sets that induce a K_5 in G and H, respectively, then Lemma 6 implies that L^G, D^G, L^H and D^H are uniquely defined. Therefore, we fix one choice of K^G and, for each choice of K^H, test whether there is an isomorphism $f : G \to H$ such that $f(L^G) = L^H$ (we use this approach in Cases 2 and 3 below). Clearly, we may assume that the vertex partitions given by Lemma 6 for G and H have the same value of p and that $|A_i^G| = |A_i^H|$ and $|N_i^G| = |N_i^H|$ for all $i \in \{1, \ldots, p\}$ and $|B^G| = |B^H|$. Furthermore, for any claims we prove about G and its vertex sets, we may assume that the same claims hold for H (otherwise such an isomorphism f does not exist). We start by proving the following four claims.

Claim 1. *$G[D^G]$ is P_3-free*
Indeed, suppose, for contradiction, that $G[D^G]$ contains an induced P_3, say on vertices u, u', u''. Since $|K^G| = 5$ and each vertex in D^G has at most one neighbour in K^G, there must be vertices $v, v' \in K^G$ that are anti-complete to $\{u, u', u''\}$. Then $G[v, v', u, u', u'']$ is a $P_2 + P_3$, a contradiction. $\diamond$

Claim 2. *If $v \in N_j^G$ for some $j \in \{1, \ldots, p\}$ and there are two adjacent vertices $u, u' \in D^G \setminus N_j^G$, then v is complete to $\{u, u'\}$.*
Since $G[D^G]$ is P_3-free by Claim 1, the vertex v must be either complete or anti-complete to $\{u, u'\}$. Suppose, for contradiction, that v is anti-complete to $\{u, u'\}$. Since $v \in N_j^G$, v has a neighbour $v' \in A_j^G$. Since $|K^G \setminus A_j^G| \geq 4$ and each vertex in D^G has at most one neighbour in K^G, there is a vertex $v'' \in K^G \setminus A_j^G$ that is non-adjacent to both u and u'. Since $v'' \notin A_j^G$, v'' is also non-adjacent to v, but is adjacent to v'. Now $G[u, u', v, v', v'']$ is a $P_2 + P_3$, a contradiction. $\diamond$

Claim 3. *If $G[D^G]$ has at least two components and one of these components C has at least three vertices, then there is an $i \in \{1, \ldots, p\}$ such that $D^G \setminus C \subset N_i^G \cup B^G$ and all but at most one vertex of C belongs to N_i^G.*
By Claim 1, $G[D^G]$ is a disjoint union of cliques. As G is connected, $D^G \setminus C$

cannot be a subset of B^G. Hence, for some $i \in \{1,\ldots,p\}$, there must be a vertex $x \in N_i^G \setminus C$. Therefore, by Claim 2, at most one vertex of C can lie outside of N_i^G. As $|C| \geq 3$, it follows that $C \cap N_i^G$ contains at least two vertices. As the vertices in C are pairwise adjacent, by Claim 2 it follows that $D^G \setminus C \subset N_i^G \cup B^G$. $\diamond$

Claim 4. *Let $i \in \{1,\ldots,p\}$. If $G[D^G]$ contains at least two non-trivial components and there is a vertex v in A_i^G with two non-neighbours in the same component of $G[D^G]$, then v is anti-complete to D^G. Furthermore, there is at most one vertex in A_i^G with this property.*
Suppose $v \in A_i^G$ has two non-neighbours x, x' in some component C of $G[D^G]$. By Claim 1, $G[D^G]$ is a disjoint union of cliques, so x must be adjacent to x'. We claim that v is anti-complete to $D^G \setminus C$. Suppose, for contradiction, that v has a neighbour $y \in D^G \setminus C$. Since every vertex of D^G has at most one neighbour in K^G, there must be a vertex $z \in K^G \setminus A_i^G$ that is non-adjacent to x, x' and y and so $G[x, x', y, v, z]$ is a $P_2 + P_3$. This contradiction implies that v is indeed anti-complete to $D^G \setminus C$. Now $G[D^G \setminus C]$ contains another non-trivial component C' and we have shown that v is anti-complete to C'. Repeating the same argument with C' taking the place of C, we find that v is anti-complete to $D^G \setminus C'$, and therefore v is anti-complete to D^G. Finally, suppose, for contradiction, that there are two vertices $v, v' \in A_i^G$ that are both anti-complete to D^G. Let x, x' be adjacent vertices in D^G and let $z \in K^G \setminus A_i^G$ be a vertex non-adjacent to x and x'. Then $G[x, x', v, z, v']$ is a $P_2 + P_3$, a contradiction. $\diamond$

We now start a case distinction and first consider the following case.

Case 1. *$G[D^G]$ contains at most one non-trivial component.*
In this case we will show that G has bounded clique-width, and so we will be done by Theorem 2. By Claim 1, every component of $G[D^G]$ is a clique. Since $G[D^G]$ contains at most one non-trivial component, we may partition D^G into a clique C and an independent set I (note that C or I may be empty). If $|C| \geq 3$ and $|I| \geq 1$, then by Claim 3 there is an $i \in \{1,\ldots,p\}$ such that at most one vertex of $C \cup I$ is outside N_i^G; if such a vertex exists, then by Fact 1 we may delete it. Now if $|C| \leq 3$, then by Fact 1 we may delete the vertices of C. Thus we may assume that either $C = \emptyset$ or $|C| \geq 4$ and furthermore, if $|C| \geq 4$ and $|I| \geq 1$, then $C \cup I \subseteq N_i^G$ for some $i \in \{1,\ldots,p\}$. Note that $I \cap B^G = \emptyset$ since G is connected, so $B^G \subset C$. Hence $G[B^G]$ is a complete graph, so it has clique-width at most 2. Applying a bipartite complementation between B^G and $C \setminus B^G$ removes all edges between B^G and $V(G) \setminus B^G$. By Fact 3, we may thus assume that $B^G = \emptyset$.

Let M be the set of vertices in L^G that have neighbours in I. We claim that M is complete to all but at most one vertex of C. We may assume that $|C| \geq 4$ and $|I| \geq 1$, otherwise the claim follows trivially. Therefore, as noted above, $C \cup I \subseteq N_i^G$ for some $i \in \{1,\ldots,p\}$. Suppose $u \in M$ has a neighbour $u' \in I$ and note that this implies $u \in A_i^G$, $u' \in N_i^G$. Suppose, for contradiction, that u has two non-neighbours $v, v' \in C$ and let $w \in K^G \setminus A_i^G$. Then $G[v, v', u', u, w]$ is a $P_2 + P_3$, a contradiction. Therefore if $u \in M$, then u has at most one non-neighbour in C. Now suppose that there are two vertices $u, u' \in M$. It follows that $u, u' \in A_i^G$, so these vertices must be non-adjacent. Furthermore, each of these vertices has at most one non-neighbour in C. If u and u' have different neighbourhoods in C, then without loss of generality we may assume that there are vertices $x, y, y' \in C$ such that u is adjacent to x, y and y' and u' is adjacent to y and y', but not to x. Now $G[y, y', u, u', x]$ is a $\overline{2P_1 + P_3}$, a contradiction. Hence every vertex in M has the same neighbourhood in C, which consists of all but at most one vertex of C and the claim holds. If the vertices of M are not complete to C, then we delete one vertex of C (we may do so by Fact 1), after which M will be complete to C. Hence we may assume that M is complete to C.

Now note that for all $i \in \{1, \ldots, p\}$, the graph $G_i = G[(A_i^G \setminus M) \cup (N_i^G \cap C)]$ is a $\overline{2P_1 + P_3}$-free split graph, so it has bounded clique-width by Lemma 3. Furthermore $G_i' = G[(A_i^G \cap M) \cup (N_i^G \cap I)]$ is a $(P_2 + P_3)$-free bipartite graph, so it has bounded clique-width by Lemma 4. Let G_i'' be the graph obtained from the disjoint union $G_i + G_i'$ by complementing A_i^G and $(N_i^G \cap C)$. By Fact 2, G_i'' also has bounded clique-width. Therefore the disjoint union G^* of all the G_i''s has bounded clique-width. Now G can be constructed from G^* by complementing L^G, complementing C and applying a bipartite complementation between C and M. Hence, by Facts 2 and 3, G has bounded clique-width. This completes Case 1.

We may now assume that Case 1 does not apply, that is, $G[D^G]$ has at least two non-trivial components. This leads us to our second and third cases.

Case 2. *$G[D^G]$ contains at least two non-trivial components, but is K_4-free.* Recall that $G[D^G]$ is P_3-free by Claim 1, so every component of $G[D^G]$ is a clique. Let C be a non-trivial component of $G[D^G]$ and let $x, y \in C$. Then x is adjacent to y and $x, y \in N_i^G \cup N_j^G \cup B^G$ for some (not necessarily distinct) $i, j \in \{1, \ldots, p\}$. By Claim 2, every vertex z in a component of $G[D^G]$ other than C must also be in $N_i^G \cup N_j^G \cup B^G$. As $G[D^G]$ contains at least two non-trivial components, repeating this argument with another non-trivial component implies that every vertex of D^G lies in $N_i^G \cup N_j^G \cup B^G$. Without loss of generality, we may therefore assume that $N_k^G = \emptyset$ for $k \geq 3$.

Since $G[D^G]$ is K_4-free, for each $i \in \{1, \ldots, p\}$ the graph $G[D^G \cup A_i^G]$ is K_5-free. This means that every K_5 in G is entirely contained in L^G. By Claim 4, for $i \geq 3$, $|A_i^G| = 1$ and so $L^G \setminus (A_1^G \cup A_2^G)$ must be a clique. The vertices of $L^G \setminus (A_1^G \cup A_2^G)$ have no neighbours outside L^G and are adjacent to every other vertex of L^G, so these vertices are in some sense interchangeable. Indeed, $N[v] = L^G$ for every $v \in L^G \setminus (A_1^G \cup A_2^G)$, and so every bijection that permutes the vertices of $L^G \setminus (A_1^G \cup A_2^G)$ and leaves the other vertices of G unchanged is an isomorphism from G to itself. Let G' be the graph obtained from G by deleting all vertices in A_i^G for $i \geq 6$ (if any such vertices are present). Now G' is K_6-free and thus $(K_6, P_2 + P_3)$-free. We can test isomorphism of such graphs G' in polynomial time by Lemma 1. If there is an isomorphism between two such graphs G' and H', then, as the vertices of $L^G \setminus (A_1^G \cup A_2^G)$ are interchangeable, we can extend it to a full isomorphism of G and H by mapping the remaining vertices of $L^G \setminus (A_1^G \cup A_2^G)$ to $L^H \setminus (A_1^H \cup A_2^H)$ arbitrarily. This completes Case 2.

Case 3. *$G[D^G]$ contains at least two non-trivial components and contains an induced K_4.*

Recall that $G[D^G]$ is P_3-free by Claim 1, so every component of $G[D^G]$ is a clique. We claim that $D^G \subseteq N_i^G \cup B^G$ for some $i \in \{1,\ldots,p\}$. Let C be a component of $G[D^G]$ that contains at least four vertices, and let C' be a component of $G[D^G]$ other than C, and note that such components exist by assumption. By Claim 3, there is an $i \in \{1,\ldots,p\}$ such that $D^G \setminus C \subset N_i^G \cup B^G$ and all but at most one vertex of C belongs to N_i^G. In particular, this implies that $C' \subset N_i^G \cup B^G$. By Claim 2, it follows that C cannot have a vertex in N_j^G for some $j \in \{1,\ldots,p\} \setminus \{i\}$, and so $C \subset N_i^G \cup B^G$. Without loss of generality, we may therefore assume that $N_j^G = \emptyset$ for $j \in \{2,\ldots,p\}$ and so $D^G = N_1^G \cup B^G$. Now if $j \in \{2,\ldots,p\}$, then the vertices of A_j^G are anti-complete to D^G, so Claim 4 implies that $|A_j^G| = 1$. This implies that $L^G \setminus A_1^G$ is a clique.

By Claim 4 there is at most one vertex $x^G \in A_1^G$ that has two non-neighbours in the same non-trivial component C of $G[D^G]$ and if such a vertex exists, then it must be anti-complete to D^G. Let $A_1^{*G} = A_1^G \setminus \{x^G\}$ if such a vertex x^G exists and $A_1^{*G} = A_1^G$ otherwise. Then every vertex in A_1^{*G} has at most one non-neighbour in each component of $G[D^G]$. Note that A^{*G} is non-empty, since D^G is non-empty and G is connected.

Suppose C is a component of $G[D^G]$ on at least four vertices. Now suppose, for contradiction, that there are two vertices $y, y' \in A_1^{*G}$ with different neighbourhoods in C. Then without loss of generality there is a vertex $x \in C$ that is adjacent to y, but not to y'. Since $|C| \geq 4$ and every vertex in A_1^{*G} has at most one non-neighbour in C, there must be two vertices $z, z' \in C$ that are adjacent to both y and y'. Now $G[z, z', x, y', y]$ is a $\overline{2P_1 + P_3}$, a contradiction. We conclude that every vertex in A_1^{*G} has the same neighbourhood in C. This implies that every vertex of C is either complete or anti-complete to A_1^{*G}. If a vertex of C is anti-complete to A_1^{*G}, then it is anti-complete to A_1^G, and so it lies in B^G.

Let D^{*G} be the set of vertices in D^G that are in components of $G[D^G]$ that have at most three vertices. Then every vertex of $D^G \setminus D^{*G}$ is complete or anti-complete to A_1^{*G} and anti-complete to $A_1^G \setminus A_1^{*G}$.

Now let $G' = G[D^{*G} \cup L^G \setminus (A_1^G \setminus A_1^{*G})]$ and note that this graph is uniquely defined by G and K^G. Then $G'[D^{*G}]$ is K_4-free, so $G'[D^{*G} \cup A_1^{*G}]$ is K_5-free, so every induced K_5 in G' is entirely contained in $L^G \setminus (A_1^G \setminus A_1^{*G})$. Furthermore, since $p \geq 5$, every vertex in $L^G \setminus (A_1^G \setminus A_1^{*G})$ is contained in an induced K_5 in G'. Therefore every isomorphism q from G' to H' satisfies $q(L^G \setminus (A_1^G \setminus A_1^{*G})) = L^H \setminus (A_1^H \setminus A_1^{*H})$. Therefore a bijection $f : V(G) \to V(H)$ is an isomorphism from G to H such that $f(L^G) = L^H$ if and only if all of the following hold:

1. The restriction of f to $V(G')$ is an isomorphism from G' to H' such that $f(A_1^{*G}) = A_1^{*H}$.
2. $f(A_1^G \setminus A_1^{*G}) = A_1^H \setminus A_1^{*H}$.
3. For every component C of $G[D^G]$ with at least four vertices, $f(C)$ is a component of $H[D^H]$ on the same number of vertices and $|C \cap B^G| = |f(C) \cap B^H|$.

It is therefore sufficient to test whether there is a bijection from G to H with the above properties. Note that these properties are defined on pairwise disjoint vertex sets, and the edges in G and H between these sets are completely determined by the definition of the sets. Thus it is sufficient to independently test whether there are bijections satisfying each of these properties. If D^{*G} is empty, then G' is a complete multipartite graph, so we can easily test if Property 1 holds in this case. Otherwise, since A_j^G has no neighbours outside L^G for $j \in \{2, \dots, p\}$, every isomorphism from G' to H' satisfies $f(A_1^{*G}) = A_1^{*H}$, so it is sufficient to test if G' and H' are isomorphic, and we can do this by applying Case 1 or Case 2. The sets $A_1^G \setminus A_1^{*G}$ and $A_1^H \setminus A_1^{*H}$ consist of at most one vertex, so we can test if Property 2 can be satisfied in polynomial time. To satisfy Property 3, we only need to check whether there is a bijection q from the components of $G[D^{*G} \setminus D^G]$ to the components of $H[D^{*H} \setminus D^H]$ such that $|q(C)| = |C|$ and $|q(C) \cap B^H| = |C \cap B^G|$ for every component of $G[D^{*G} \setminus D^G]$ and this can clearly be done in polynomial time. This completes the proof of Case 3. ☐

4 New GI-complete Results

We state Theorems 4, 5 and 6, which establish that Graph Isomorphism is GI-complete on $(\text{diamond}, 2P_3)$-free, $(\text{diamond}, P_6)$-free and $(\text{gem}, \overline{P_1 + 2P_2})$-free graphs, respectively. The complexity of Graph Isomorphism on $(\overline{2P_1 + P_3}, 2P_3)$-free graphs and (gem, P_6)-free graphs was previously unknown, but as these classes contain the classes of $(\text{diamond}, 2P_3)$-free graphs and $(\text{diamond}, P_6)$-free graphs, respectively, Theorems 4 and 5, respectively, imply that Graph Isomorphism is also GI-complete on these classes. In Theorems 4 and 5, GI-completeness follows from the fact that the constructions used in our proofs (which we omit) fall into the framework of so-called simple path encodings (see [25]). The construction used in the proof of Theorem 6 does not fall into this framework and we give a direct proof of GI-completeness in this case.

Theorem 4. Graph Isomorphism *is* GI-*complete on* $(\text{diamond}, 2P_3)$-*free graphs.*

Theorem 5. Graph Isomorphism *is* GI-*complete on* $(\text{diamond}, P_6)$-*free graphs.*

Theorem 6. Graph Isomorphism *is* GI-*complete on* $(\text{gem}, P_1 + 2P_2)$-*free graphs. Furthermore,* $(\text{gem}, P_1 + 2P_2)$-*free graphs have unbounded clique-width.*

Proof Sketch. Let G be a graph. Let $v_1^G, \dots, v_n^G$ be the vertices of G and let $e_1^G, \dots, e_m^G$ be the edges of G. We construct a graph $q(G)$ from G as follows:

1. Create a complete multipartite graph with partition $(A_1^G, \dots, A_n^G)$, where $|A_i^G| = d_G(v_i^G)$ for $i \in \{1, \dots, n\}$ and let $A^G = \bigcup A_i^G$.
2. Create a complete multipartite graph with partition $(B_1^G, \dots, B_m^G)$, where $|B_i^G| = 2$ for $i \in \{1, \dots, m\}$ and let $B^G = \bigcup B_i^G$.
3. Take the disjoint union of the two graphs above, then for each edge $e_i^G = v_{i_1}^G v_{i_2}^G$ in G in turn, add an edge from one vertex of B_i^G to a vertex of $A_{i_1}^G$ and an edge from the other vertex of B_i^G to a vertex of $A_{i_2}^G$. Do this in such a way that the edges added between A^G and B^G form a perfect matching.

It can be checked that $q(G)$ is (gem, P_1+2P_2)-free for every graph G. Let G and H be graphs. Let G^* and H^* be the graphs obtained from G and H, respectively, by adding four pairwise adjacent vertices that are adjacent to every vertex of G and H, respectively. Note that every vertex of G^* and H^* has degree at least 3. We claim that G is isomorphic to H if and only if $q(G^*)$ is isomorphic to $q(H^*)$. As the latter two graphs are $(\text{gem}, P_1 + 2P_2)$-free, this proves the first result.

Let H_n be the $n \times n$ grid. We use Lemma 5 with $m = 1$ combined with Fact 2 to prove that the set of graphs $\{q(H_n) \mid n \in \mathbb{N}\}$, which are $(\text{gem}, P_1 + 2P_2)$-free as stated above, has unbounded clique-width. □

5 Classifying the Complexity of Graph Isomorphism for (H_1, H_2)-free Graphs

Given four graphs H_1, H_2, H_3, H_4, the classes of (H_1, H_2)-free graphs and (H_3, H_4)-free graphs are *equivalent* if the unordered pair H_3, H_4 can be obtained from the unordered pair H_1, H_2 by some combination of the operations:

(i) complementing both graphs in the pair, and
(ii) if one of the graphs in the pair is K_3, replacing it with the paw or vice versa.

Note that two graphs G and H are isomorphic if and only if their complements $\overline{G}$ and $\overline{H}$ are isomorphic. Therefore, for every pair of graphs H_1, H_2, the Graph Isomorphism problem is polynomial-time solvable or GI-complete for (H_1, H_2)-free graphs if and only if the same is true for $(\overline{H_1}, \overline{H_2})$-free graphs. Since Graph Isomorphism can be solved component-wise, and it can easily be solved on complete multipartite graphs in polynomial time, Lemma 2 implies that for every graph H_1, the Graph Isomorphism problem is polynomial-time solvable or GI-complete for (H_1, K_3)-free graphs if and only if the same is true for (H_1, paw)-free graphs. Thus if two classes are equivalent, then the complexity of Graph Isomorphism is the same on both of them. Here is the summary of known results for the complexity of Graph Isomorphism on (H_1, H_2)-free graphs (see Section 2 for notation; we omit the proof).

Theorem 7. *For a class $\mathcal{G}$ of graphs defined by two forbidden induced subgraphs, the following holds:*

1. Graph Isomorphism *is solvable in polynomial time on $\mathcal{G}$ if $\mathcal{G}$ is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:*
   (i) H_1 *or* $H_2 \subseteq_i P_4$
   (ii) $\overline{H_1}$ *and* $H_2 \subseteq_i K_{1,t} + P_1$ *for some* $t \geq 1$
   (iii) $\overline{H_1}$ *and* $H_2 \subseteq_i tP_1 + P_3$ *for some* $t \geq 1$
   (iv) $H_1 \subseteq_i K_t$ *and* $H_2 \subseteq_i 2K_{1,t}, K_{1,t}^{+}$ *or* P_5 *for some* $t \geq 1$
   (v) $H_1 \subseteq_i$ paw *and* $H_2 \subseteq_i P_2 + P_4, P_6, S_{1,2,2}$ *or* $K_{1,t}^{++} + P_1$ *for some* $t \geq 1$
   (vi) $H_1 \subseteq_i$ diamond *and* $H_2 \subseteq_i P_1 + 2P_2$
   (vii) $H_1 \subseteq_i$ gem *and* $H_2 \subseteq_i P_1 + P_4$ *or* P_5

(viii) $H_1 \subseteq_i \overline{2P_1 + P_3}$ *and* $H_2 \subseteq_i P_2 + P_3$.

2. Graph Isomorphism *is* GI*-complete on* $\mathcal{G}$ *if* $\mathcal{G}$ *is equivalent to a class of* (H_1, H_2)*-free graphs such that one of the following holds:*
   (i) neither H_1 *nor* H_2 *is a path star forest*
   (ii) neither $\overline{H_1}$ *nor* $\overline{H_2}$ *is a path star forest*
   (iii) $H_1 \supseteq_i K_3$ *and* $H_2 \supseteq_i 2P_1 + 2P_2, P_1 + 2P_3, 2P_1 + P_4$ *or* $3P_2$
   (iv) $H_1 \supseteq_i K_4$ *and* $H_2 \supseteq_i K_{1,4}^{++}, P_1 + 2P_2$ *or* $P_1 + P_4$
   (v) $H_1 \supseteq_i K_5$ *and* $H_2 \supseteq_i K_{1,3}^{++}$
   (vi) $H_1 \supseteq_i C_4$ *and* $H_2 \supseteq_i K_{1,3}, 3P_1 + P_2$ *or* $2P_2$
   (vii) $H_1 \supseteq_i$ diamond *and* $H_2 \supseteq_i K_{1,3}, P_2 + P_4, 2P_3$ *or* P_6
   (viii) $H_1 \supseteq_i$ gem *and* $H_2 \supseteq_i P_1 + 2P_2$.

Open Problem 1. *What is the complexity of* Graph Isomorphism *on* (H_1, H_2)*-free graphs in the following cases?*

(i) $H_1 = K_3$ *and* $H_2 \in \{P_7, S_{1,2,3}\}$
(ii) $H_1 = K_4$ *and* $H_2 = S_{1,1,3}$
(iii) $H_1 =$ diamond *and* $H_2 \in \{P_1 + P_2 + P_3, P_1 + P_5\}$
(iv) $H_1 =$ gem *and* $H_2 = P_2 + P_3$
(v) $H_1 = \overline{2P_1 + P_3}$ *and* $H_2 = P_5$

Note that all of the classes of (H_1, H_2)-free graphs in Open Problem 1 are incomparable. We omit the proof of the next theorem.

Theorem 8. *Let* $\mathcal{G}$ *be a class of graphs defined by two forbidden induced subgraphs. Then* $\mathcal{G}$ *is not equivalent to any of the classes listed in Theorem 7 if and only if it is equivalent to one of the seven cases listed in Open Problem 1.*

6 Conclusions

By combining known and new results we determined the complexity of Graph Isomorphism in terms of polynomial-time solvability and GI-completeness for (H_1, H_2)-free graphs for all but seven pairs (H_1, H_2). This also led to a new class of (H_1, H_2)-free graphs whose clique-width is unbounded. In particular, we developed a technique for showing polynomial-time solvability for $(\overline{2P_1 + P_3}, H)$-free graphs, which we illustrated for the case $H = P_2 + P_3$. For future work we have some preliminary results for the case where $H = P_5$.

References

1. Babai, L.: Graph isomorphism in quasipolynomial time [extended abstract]. Proc. STOC 2016 pp. 684–697 (2016)
2. Belmonte, R., Otachi, Y., Schweitzer, P.: Induced minor free graphs: Isomorphism and clique-width. Algorithmica **80**(1), 29–47 (2018)
3. Blanché, A., Dabrowski, K.K., Johnson, M., Lozin, V.V., Paulusma, D., Zamaraev, V.: Clique-width for graph classes closed under complementation. Proc. MFCS 2017, LIPIcs **83**, 73:1–73:14 (2017)

4. Booth, K.S., Colbourn, C.J.: Problems polynomially equivalent to graph isomorphism. Tech. Rep. CS-77-04, Department of Computer Science, University of Waterloo (1979)
5. Brandstädt, A., Dabrowski, K.K., Huang, S., Paulusma, D.: Bounding the clique-width of H-free split graphs. Discrete Applied Mathematics **211**, 30–39 (2016)
6. Brandstädt, A., Dabrowski, K.K., Huang, S., Paulusma, D.: Bounding the clique-width of H-free chordal graphs. Journal of Graph Theory **86**(1), 42–77 (2017)
7. Colbourn, C.J.: On testing isomorphism of permutation graphs. Networks **11**(1), 13–21 (1981)
8. Dabrowski, K.K., Dross, F., Paulusma, D.: Colouring diamond-free graphs. Journal of Computer and System Sciences **89**, 410–431 (2017)
9. Dabrowski, K.K., Huang, S., Paulusma, D.: Bounding clique-width via perfect graphs. Journal of Computer and System Sciences (in press)
10. Dabrowski, K.K., Johnson, M., Paulusma, D.: Clique-width for hereditary graph classes. Proc. BCC 2019, London Mathematical Society Lecture Note Series **456** (2019), (to appear)
11. Dabrowski, K.K., Lozin, V.V., Paulusma, D.: Clique-width and well-quasi ordering of triangle-free graph classes. Proc. WG 2017, LNCS **10520**, 220–233 (2017)
12. Dabrowski, K.K., Paulusma, D.: Clique-width of graph classes defined by two forbidden induced subgraphs. The Computer Journal **59**(5), 650–666 (2016)
13. Grohe, M., Neuen, D., Schweitzer, P., Wiebking, D.: An improved isomorphism test for bounded-tree-width graphs. Proc. ICALP 2018, LIPIcs **107**, 67:1–67:14 (2018)
14. Grohe, M., Schweitzer, P.: Isomorphism testing for graphs of bounded rank width. Proc. FOCS 2015 pp. 1010–1029 (2015)
15. Kamiński, M., Lozin, V.V., Milanič, M.: Recent developments on graphs of bounded clique-width. Discrete Applied Mathematics **157**(12), 2747–2761 (2009)
16. Kratsch, S., Schweitzer, P.: Graph isomorphism for graph classes characterized by two forbidden induced subgraphs. Discrete Applied Mathematics **216, Part 1**, 240–253 (2017)
17. Lokshtanov, D., Pilipczuk, M., Pilipczuk, M., Saurabh, S.: Fixed-parameter tractable canonization and isomorphism test for graphs of bounded treewidth. SIAM Journal on Computing **46**(1), 161–189 (2017)
18. Lozin, V.V.: Bipartite graphs without a skew star. Discrete Mathematics **257**(1), 83–100 (2002)
19. Lozin, V.V., Rautenbach, D.: On the band-, tree-, and clique-width of graphs with bounded vertex degree. SIAM Journal on Discrete Mathematics **18**(1), 195–206 (2004)
20. Neuen, D.: Graph isomorphism for unit square graphs. Proc. ESA 2016, LIPIcs **57**, 70:1–70:17 (2016)
21. Olariu, S.: Paw-free graphs. Information Processing Letters **28**(1), 53–54 (1988)
22. Ponomarenko, I.N.: Isomorphism problem for classes of graphs closed under contractions. Zapiski Nauchnykh Seminarov (LOMI) **174**, 147–177 (1988), (in Russian, English translation in Journal of Soviet Mathematics, **55**(2):1621–1643, (1991))
23. de Ridder, H.N., et al.: Information System on Graph Classes and their Inclusions (2001–2019), `http://www.graphclasses.org`
24. Schöning, U.: Graph isomorphism is in the low hierarchy. Journal of Computer and System Sciences **37**(3), 312–323 (1988)
25. Schweitzer, P.: Towards an isomorphism dichotomy for hereditary graph classes. Theory of Computing Systems **61**(4), 1084–1127 (2017)