Acyclic, Star and Injective Colouring: A Complexity Picture for H-Free Graphs

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Abstract

A k-colouring c of a graph G is a mapping $V(G) \to \{1, 2, \dots\ k\}$ such that $c(u) \neq c(v)$ whenever u and v are adjacent. The corresponding decision problem is Colouring. A colouring is acyclic, star, or injective if any two colour classes induce a forest, star forest or disjoint union of vertices and edges, respectively. Hence, every injective colouring is a star colouring and every star colouring is an acyclic colouring. The corresponding decision problems are Acyclic Colouring, Star Colouring and Injective Colouring (the last problem is also known as $L(1,1)$-Labelling).

A classical complexity result on Colouring is a well-known dichotomy for H-free graphs, which was established twenty years ago (in this context, a graph is H-free if and only if it does not contain H as an *induced* subgraph). Moreover, this result has led to a large collection of results, which helped us to better understand the complexity of Colouring. In contrast, there is no systematic study into the computational complexity of Acyclic Colouring, Star Colouring and Injective Colouring despite numerous algorithmic and structural results that have appeared over the years.

We initiate such a systematic complexity study, and similar to the study of Colouring we use the class of H-free graphs as a testbed. We prove the following results:

1. We give almost complete classifications for the computational complexity of Acyclic Colouring, Star Colouring and Injective Colouring for H-free graphs.
2. If the number of colours k is fixed, that is, not part of the input, we give full complexity classifications for each of the three problems for H-free graphs.

From our study we conclude that for fixed k the three problems behave in the same way, but this is no longer true if k is part of the input. To obtain several of our results we prove stronger complexity results that in particular involve the girth of a graph and the class of line graphs.

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1 Introduction

We study the complexity of three classical colouring problems. We do this by focusing on *hereditary* graph classes, i.e., classes closed under vertex deletion, or equivalently, classes characterized by a (possibly infinite) set $\mathcal{F}$ of forbidden induced subgraphs. As evidenced by numerous complexity studies in the literature, even the case where $|\mathcal{F}| = 1$ captures a rich family of graph classes suitably interesting to develop general methodology. Hence, we usually first set $\mathcal{F} = \{H\}$ and consider the class of H*-free* graphs, i.e., graphs that do not contain H as an induced subgraph. We then investigate how the complexity of a problem restricted to H-free graphs depends on the choice of H and try to obtain a *complexity dichotomy*.

To give a well-known and relevant example, the Colouring problem is to decide, given a graph G and integer $k \geq 1$, if G has a k*-colouring*, i.e., a mapping $c : V(G) \rightarrow \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for every two adjacent vertices u and v. Král' et al. [37] proved that Colouring on H-free graphs is polynomial-time solvable if H is an induced subgraph of P_4 or $P_1 + P_3$ and NP-complete otherwise. Here, P_n denotes the n-vertex path and $G_1 + G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ the disjoint union of two vertex-disjoint graphs G_1 and G_2. If k is fixed (not part of the input), then we obtain the k-Colouring problem. No complexity dichotomy is known for k-Colouring if $k \geq 3$. In particular, the complexities of 3-Colouring for P_t-free graphs for $t \geq 8$ and k-Colouring for sP_4-free graphs for $s \geq 2$ and $k \geq 3$ are still open. Here, we write sG for the disjoint union of s copies of G. We refer to the survey of Golovach et al. [27] for further details and to [13, 36] for updated summaries.

For a colouring c of a graph G, a *colour class* consists of all vertices of G that are mapped by c to a specific colour i. We consider the following special graph colourings. A colouring of a graph G is *acyclic* if the union of any two colour classes induces a forest. The $(r+1)$-vertex *star* $K_{1,r}$ is the graph with vertices $u, v_1, \ldots, v_r$ and edges uv_i for every $i \in \{1, \ldots, r\}$. An acyclic colouring is a *star colouring* if the union of any two colour classes induces a *star forest*, that is, a forest in which each connected component is a star. A star colouring is *injective* (or an $L(1,1)$*-labelling*) if the union of any two colour classes induces an $sP_1 + tP_2$ for some integers $s \geq 0$ and $t \geq 0$. An alternative definition is to say that all the neighbours of every vertex of G are uniquely coloured. These definitions lead to the following three decision problems:

Acyclic Colouring	
Instance:	A graph G and an integer $k \geq 1$
Question:	Does G have an acyclic k-colouring?

Star Colouring	
Instance:	A graph G and an integer $k \geq 1$
Question:	Does G have a star k-colouring?

Injective Colouring	
Instance:	A graph G and an integer $k \geq 1$
Question:	Does G have an injective k-colouring?

If k is fixed, we write Acyclic k-Colouring, Star k-Colouring and Injective k-Colouring, respectively.

All three problems have been extensively studied. We note that in the literature on the Injective Colouring problem it is often assumed that two adjacent vertices may be coloured alike by an injective colouring (see, for example, [29, 30, 33]). However, in our paper, we do **not** allow this; as reflected in their definitions we only consider colourings that are proper. This enables us to compare the results for the three different kinds of colourings with each other.

So far, systematic studies mainly focused on structural characterizations, exact values, lower and upper bounds on the minimum number of colours in an acyclic colouring or star colouring (i.e., the *acyclic* and *star chromatic number*); see, e.g., [2, 9, 19, 20, 21, 34, 35, 50, 51, 53], to name just a few papers, whereas injective colourings (and the *injective chromatic number*) were mainly considered in the context of the distance constrained labelling framework (see the survey [11] and Section 6 therein). The problems have also been studied from a complexity perspective, but apart from a study on Acyclic Colouring for graphs of bounded maximum degree [45], known results are scattered and relatively sparse. We perform a *systematic* and *comparative* complexity study by focusing on the following research question both for k part of the input and for fixed k:

What are the computational complexities of Acyclic Colouring*,* Star Colouring *and* Injective Colouring *for H-free graphs?*

Before discussing our new results and techniques, we first briefly discuss some known results.

Coleman and Cai [14] proved that for every $k \geq 3$, Acyclic k-Colouring is NP-complete for bipartite graphs. Afterwards, a number of hardness results appeared for other hereditary graph classes. Alon and Zaks [3] showed that Acyclic 3-Colouring is NP-complete for line graphs of maximum degree 4. Angelini and Frati [4] showed that Acyclic 3-Colouring is NP-complete for planar graphs of maximum degree 4. Mondal et al. [45] proved that Acyclic 4-Colouring is NP-complete for graphs of maximum degree 5 and for planar graphs of maximum degree 7. Albertson et al. [1] and recently, Lei et al. [38] proved that Star 3-Colouring is NP-complete for planar bipartite graphs and line graphs, respectively. Bodlaender et al. [7], Sen and Huson [48] and Lloyd and Ramanathan [41] proved that Injective Colouring is NP-complete for split graphs, unit disk graphs and planar graphs, respectively. Mahdian [44] proved that for every $k \geq 4$, Injective k-Colouring is NP-complete for line graphs, whereas Injective 4-Colouring is known to be NP-complete for cubic graphs (see [11]); observe that Injective 3-Colouring is trivial for general graphs.

On the positive side, Lyons [43] showed that every acyclic colouring of a P_4-free graph is, in fact, a star colouring. Lyons [43] also proved that Acyclic Colouring and Star Colouring are polynomial-time solvable for P_4-free graphs; we note that Injective Colouring is trivial for P_4-free graphs, as every injective colouring must assign each vertex of a connected P_4-free graph a unique colour. The results of Lyons have been extended to P_4-tidy graphs and $(q, q-4)$-graphs [40]. Cheng et al. [12] complemented the aforementioned result of Alon and Zaks [3] by proving that Acyclic Colouring is polynomial-time solvable for claw-free graphs of maximum degree at most 3. Calamoneri [11] observed that Injective Colouring is polynomial-time solvable for comparability and co-comparability graphs. Zhou et al. [52] proved that Injective Colouring is polynomial-time solvable for graphs of bounded treewidth (which is best possible due to the W[1]-hardness result of Fiala et al. [22]).

Our Complexity Results and Methodology

The *girth* of a graph G is the length of a shortest cycle of G (if G is a forest, then its girth is ∞). To answer our research question we focus on two important graph classes, namely the classes of graphs of high girth and line graphs, which are interesting classes on their own. If a problem is NP-complete for both classes, then it is NP-complete for H-free graphs whenever H has a cycle or a claw. It then remains to analyze the case when H is a *linear forest*, i.e., a disjoint union of paths; see [8, 10, 25, 37] for examples of this approach, which we discuss in detail below.

The construction of graph families of high girth and large chromatic number is well studied in graph theory (see, e.g. [18]). To prove their complexity dichotomy for Colouring on H-free graphs, Král' et al. [37] first showed that for every integer $g \geq 3$, 3-Colouring is NP-complete for the class of graphs of girth at least g. This approach can be readily extended to any integer $k \geq 3$ [17, 42]. The basic idea is to replace edges in a graph by graphs of high girth and large chromatic number, such that the resulting graph has sufficiently high girth and is k-colourable if and only if the original graph is so (see also [28, 32]).

By a more intricate use of the above technique we are able to prove that for every $g \geq 3$, Acyclic 3-Colouring is NP-complete for the class of graphs of girth at least g. This implies that Acyclic 3-Colouring is NP-complete for H-free graphs whenever H has a cycle. We prove the same result for every $k \geq 4$ by combining known results, just as we use known results to prove that Star k-Colouring ($k \geq 3$) and Injective k-Colouring ($k \geq 4$) are NP-complete for H-free graphs if H has a cycle.

A classical result of Holyer [31] is that 3-Colouring is NP-complete for line graphs (and Leven and Galil [39] proved the same for $k \geq 4$). As line graphs are claw-free, Král' et al. [37] used Holyer's result to show that 3-Colouring is NP-complete for H-free graphs whenever H has an induced claw. For Acyclic 3-Colouring, this follows from Alon and Zaks' result [3], which we extend to work for $k \geq 4$. For Injective k-Colouring ($k \geq 4$) we can use the aforementioned result on line graphs of Mahdian [44].

The above hardness results leave us to consider the case where H is a linear forest. In Section 2 we will use a result of Atminas et al. [5] to prove a general result from which it follows that for fixed k, all three problems are polynomial-time solvable for H-free graphs if H is a linear forest. Hence, we have full complexity dichotomies for the three problems when k is fixed. However, these positive results do not extend to the case where k is part of the input: we prove NP-completeness for graphs that are P_r-free for some small value of r or have a small independence number, i.e., that are sP_1-free for some small integer s.

Our complexity results for H-free graphs are summarized in the following three theorems, proven in Sections 3–5, respectively; see Table 1 for a comparison. For two graphs F and G, we write $F \subseteq_i G$ or $G \supseteq_i F$ to denote that F is an *induced* subgraph of G.

▶ **Theorem 1.** *Let H be a graph. For the class of H-free graphs it holds that:*

(i) Acyclic Colouring *is polynomial-time solvable if $H \subseteq_i P_4$ and* NP*-complete if H is not a forest or $H \supseteq_i 19P_1, 3P_3$ or $2P_5$;*

(ii) For every $k \geq 3$, Acyclic k-Colouring *is polynomial-time solvable if H is a linear forest and* NP*-complete otherwise.*

▶ **Theorem 2.** *Let H be a graph. For the class of H-free graphs it holds that:*

(i) Star Colouring *is polynomial-time solvable if $H \subseteq_i P_4$ and* NP*-complete for any $H \neq 2P_2$.*

(ii) For every $k \geq 3$, Star k-Colouring *is polynomial-time solvable if H is a linear forest and* NP*-complete otherwise.*

▶ **Theorem 3.** *Let H be a graph. For the class of H-free graphs it holds that:*

(i) Injective Colouring *is polynomial-time solvable if $H \subseteq_i P_4$ or $H \subseteq_i P_1 + P_3$ and* NP*-complete if H is not a forest or $2P_2 \subseteq_i H$ or $6P_1 \subseteq_i H$.*

	polynomial time	NP-complete
Colouring [37]	$H \subseteq_i P_4$ or $P_1 + P_3$	else
Acyclic Colouring	$H \subseteq_i P_4$	else except for at most 1991 open cases
Star Colouring	$H \subseteq_i P_4$	else except for 1 open case
Injective Colouring	$H \subseteq_i P_4$ or $P_1 + P_3$	else except for 10 open cases
k-Colouring (see [13, 27, 36])	depends on k	infinitely many open cases for all $k \geq 3$
Acyclic k-Colouring ($k \geq 3$)	H is a linear forest	else
Star k-Colouring ($k \geq 3$)	H is a linear forest	else
Injective k-Colouring ($k \geq 4$)	H is a linear forest	else

Table 1 The state-of-the-art for the three problems in this paper and the original Colouring problem; both when k is fixed and when k is part of the input.

(ii) For every $k \geq 4$, Injective k-Colouring *is polynomial-time solvable if H is a linear forest and* NP*-complete otherwise.*

In Section 6 we give a number of open problems that resulted from our systematic study; in particular we will discuss the distance constrained labelling framework in more detail.

2 A General Polynomial Result

A *biclique* or *complete bipartite graph* is a bipartite graph on vertex set $S \cup T$, such that S and T are independent sets and there is an edge between every vertex of S and every vertex of T; if $|S| = s$ and $|T| = t$, we denote this graph by $K_{s,t}$, and if $s = t$, the biclique is *balanced* and of *order* s. We say that a colouring c of a graph G satisfies the *balance biclique condition* (BB-condition) if c uses at least $k + 1$ colours to colour G, where k is the order of a largest biclique that is contained in G as a (not necessarily induced) subgraph.

Let π be some colouring property; e.g., π could mean being acyclic, star or injective. Then π *can be expressed in MSO$_2$* (monadic second-order logic with edge sets) if for every $k \geq 1$, the graph property of having a k-colouring with property π can be expressed in MSO$_2$. The general problem Colouring(π) is to decide, on a graph G and integer $k \geq 1$, if G has a k-colouring with property π. If k is fixed, we write k-Colouring(π). We now prove the following result.

▶ **Theorem 4.** *Let H be a linear forest, and let π be a colouring property that can be expressed in MSO$_2$, such that every colouring with property π satisfies the BB-condition. Then, for every integer $k \geq 1$, k-*Colouring*(π) is linear-time solvable for H-free graphs.*

Proof. Atminas, Lozin and Razgon [5] proved that that for every pair of integers ℓ and k, there exists a constant $b(\ell, k)$ such that every graph of treewidth at least $b(\ell, k)$ contains an induced P_ℓ or a (not necessarily induced) biclique $K_{k,k}$. Let G be an H-free graph, and let ℓ be the smallest integer such that $H \subseteq_i P_\ell$; observe that ℓ is a constant. Hence, we can use Bodlaender's algorithm [6] to test in linear time if G has treewidth at most $b(\ell, k) - 1$.

First suppose that the treewidth of G is at most $b(\ell, k) - 1$. As π can be expressed in MSO$_2$, the result of Courcelle [15] allows us to test in linear time whether G has a k-colouring with property π. Now suppose that the treewidth of G is at least $b(\ell, k)$. As G is H-free, G is P_ℓ-free. Then, by the result of Atminas, Lozin and Razgon [5], we find that G contains $K_{k,k}$ as a subgraph. As π satisfies the BB-condition, G has no k-colouring with property π. ◀

We now apply Theorem 4 to obtain the polynomial cases for fixed k in Theorem 1–3.

▶ **Corollary 5.** *Let H be a linear forest. For every $k \geq 1$,* Acyclic k-Colouring*,* Star k-Colouring *and* Injective k-Colouring *are polynomial-time solvable for H-free graphs.*

Proof. All three kinds of colourings use at least s colours to colour $K_{s,s}$ (as the vertices from one bipartition class of $K_{s,s}$ must receive unique colours). Hence, every acyclic, star and injective colouring of every graph satisfies the BB-condition. Moreover, it is readily seen that the colouring properties of being acyclic, star or injective can all be expressed in MSO_2. Hence, we may apply Theorem 4. ◀

3 Acyclic Colouring

In this section, we prove Theorem 1. For a graph G and a colouring c, the pair (G, c) has a *bichromatic* cycle C if C is a cycle of G with $|c(V(C)| = 2$, i.e., the vertices of C are coloured by two alternating colours (so C is even). A path P in G is an *i-j-path* if the vertices of P have alternating colours i and j. We now prove the following result.

▶ **Lemma 6.** *For every $g \geq 3$,* Acyclic 3-Colouring *is* NP*-complete for graphs of girth at least g.*

Proof. We reduce from Acyclic 3-Colouring, which is known to be NP-complete [14]. We start by taking a graph F that has a 4-colouring but no 3-colouring and that is of girth at least g. By a seminal result of Erdős [18], such a graph F exists (and its size is constant, as it only depends on g which is a fixed integer). We now repeatedly remove edges from F until we obtain a graph F' that is acyclically 3-colourable. Let xy be the last edge that we removed. As F has no 3-colouring, the edge xy exists. Moreover, by our construction, the graph $F' + xy$ is not acyclically 3-colourable. As edge deletions do not decrease the girth, $F' + xy$ and F' have girth at least g.

The basic idea (Case 1) is as follows. Let G be an instance of Acyclic 3-Colouring. We pick an edge $uv \in E(G)$. In $G - uv$ we "glue" F' by identifying u with x and y with v; see also Figure 1. We then prove that G has an acyclic 3-colouring if and only if G' has an acyclic 3-colouring. Then, by performing the same operation for each other edge of G as well, we obtain a graph G'', such that G has an acyclic 3-colouring if and only if G'' has so. As the size of G'' is polynomial in the size of G and the girth of G'' is at least g, we have proven the theorem. The challenge in this technique is that we do not know how the graph F' looks. We can only prove its existence and therefore have to consider several possibilities for the properties of the acyclic 3-colourings of F'. Hence, we distinguish between Cases 1–3, 4a, and 4b.

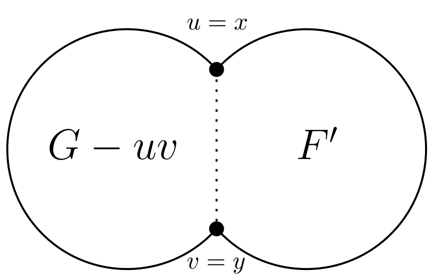

Figure 1 The graph G' from Case 1.

Case 1: *Every acyclic* 3*-colouring of* F' *assigns different colours to* x *and* y*.*
We construct the graph G' as described above and in Figure 1. We claim that G is a yes-instance of Acyclic 3-Colouring if and only if G' is a yes-instance of Acyclic 3-Colouring.

First suppose that G has an acyclic 3-colouring c. Let c^* be an acyclic 3-colouring of F'. We may assume without loss of generality that $c(u) = c^*(x)$ and $c(v) = c^*(y)$. Hence, we can define a vertex colouring c' of G' with $c'(w) = c(w)$ if $w \in V(G)$ and $c'(w) = c^*(w)$ if $w \in V(F')$. As c and c^* are 3-colourings of G and F', respectively, c' is a 3-colouring of G'. We claim that c' is acyclic. For contradiction, assume that (G', c') has a bichromatic cycle C. If all edges of C are in G or all edges of C are in F', then (G, c) or (F', c^*) has a bichromatic cycle, which is not possible as c and c^* are acyclic. Hence, at least one edge of C belongs to G and at least one edge of C belongs to F'. This means that C contains both $u = x$ and $v = y$. Recall that G contains the edge uv. Consequently, (G, c) has a bichromatic cycle, namely the cycle induced by $V(C) \cap V(G)$, a contradiction.

Now suppose that G' has an acyclic 3-colouring c'. Let c and c^* be the restrictions of c' to $V(G)$ and $V(F')$, respectively. Then c and c^* are acyclic 3-colourings of $G - uv$ and F', respectively. By our assumption and because c^* is an acyclic 3-colouring of F', we find that $c^*(x) \neq c^*(y)$, or equivalently, $c(u) \neq c(v)$. This means that c is also a 3-colouring of G and c^* is also a 3-colouring of $F' + xy$. We claim that c is acyclic on G. For contradiction, assume that (G, c) has a bichromatic cycle C. As c is an acyclic 3-colouring of $G - uv$, we deduce that C must contain the edge $uv = xy$. As $F' + xy$ has no acyclic 3-colouring by construction and c^* is a 3-colouring of $F' + xy$, we find that $(F' + xy, c^*)$ has a bichromatic cycle D. As c^* is an acyclic 3-colouring of F', this means that D contains the edge $xy = uv$. However, then (G', c') has a bichromatic cycle, namely the cycle induced by $V(C) \cup V(D)$, a contradiction.

Let F^* be the graph obtained from F' by adding a new vertex x' and edges xx' and $x'y$. As $F' + xy$ has girth at least g, we find that F^* and $F^* - x'y$ have girth at least g. As x' has degree 1 in $F^* - x'y$ and F' has an acyclic 3-colouring, $F^* - x'y$ has an acyclic 3-colouring.

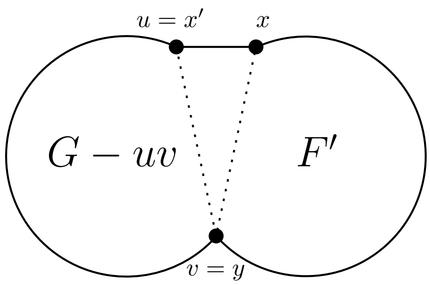

Figure 2 The graph G' from Case 2.

Case 2: *All acyclic* 3*-colourings of* F' *assign the same colour to* x *and* y *and* F^* *has no acyclic* 3*-colouring.*
In this case we let G' be the graph obtained from $G - uv$ and $F^* - x'y$ by identifying u with x' and v with y; see also Figure 2. We claim that G is a yes-instance of Acyclic 3-Colouring if and only if G' is a yes-instance of Acyclic 3-Colouring.

First suppose that G has an acyclic 3-colouring c. Let c^* be an acyclic 3-colouring of $F^* - x'y$. Then the restriction of c^* to F' is an acyclic 3-colouring of F'. By our assumption, it holds therefore that $c^*(x) = c^*(y)$ and thus $c^*(x') \neq c^*(y)$. We may assume without loss of generality that $c(u) = c^*(x')$ and $c(v) = c^*(y)$. Hence, we can define a vertex labelling c' of G' with $c'(w) = c(w)$ if $w \in V(G)$ and $c'(w) = c^*(w)$ if $w \in V(F^*)$. As c and c^* are 3-colourings of G and $F^* - x'y$, respectively, c' is a 3-colouring of G'. We claim that c' is acyclic. For contradiction, assume that (G', c') has a bichromatic cycle C. If the edges of C are all in G or all in $F^* - x'y$, then (G, c) or $(F^* - x'y, c^*)$ has a bichromatic cycle, which is not possible as c and c^* are acyclic. Hence, at least one edge of C belongs to G and at least one edge of C belongs to F'. This means that C contains both $u = x'$ and $v = y$. Recall that G contains the edge uv. Consequently, (G, c) has a bichromatic cycle, namely the cycle induced by $V(C) \cap V(G)$, a contradiction.

Now suppose that G' has an acyclic 3-colouring c'. Let c and c^* be the restrictions of c' to $V(G - uv)$ and $V(F^* - x'y)$, respectively. Then c and c^* are acyclic 3-colourings of $G - uv$ and $F^* - x'y$, respectively. Moreover, the restriction of c' to $V(F')$ is an acyclic 3-colouring of F'. By our assumption, this means that $c'(x) = c'(y)$ and thus $c^*(x') \neq c^*(y)$, or equivalently, $c(u) \neq c(v)$. Consequently, c is also a 3-colouring of G and c^* is also a 3-colouring of F^*. We claim that c is acyclic. For contradiction, assume that (G, c) has a bichromatic cycle C. As c is an acyclic 3-colouring of $G - uv$, we deduce that C must contain the edge $uv = x'y$. As F^* does not have an acyclic 3-colouring by our assumption and c^* is a 3-colouring of F^*, we find that (F^*, c^*) has a bichromatic cycle D. As c^* is an acyclic 3-colouring of $F^* - x'y$, this means that D must contain the edge $x'y = uv$. However, then (G', c') has a bichromatic cycle, namely the cycle induced by $V(C) \cup V(D)$, a contradiction.

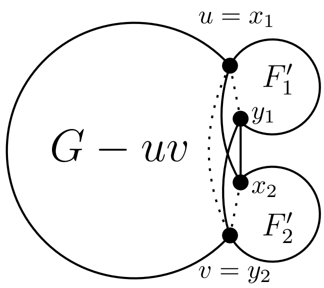

Figure 3 The graph G' with the graph F^+ from Case 3 (before we recursively repeat g times the operation of placing the graph F^+ on the y_1x_2-edge).

Case 3: *All acyclic 3-colourings of F' assign the same colour to x and y and F^* has an acyclic 3-colouring.*
We first construct a new graph F^+ as follows. We take the disjoint union of two copies F_1' and F_2' of F', where we denote the vertices x and y as x_1 and y_1 in F_1' and as x_2 and y_2 in F_2'. We add edges x_1x_2, x_2y_1, and y_1y_2 to $F_1' + F_2'$; see also Figure 3.

We claim that F^+ has an acyclic 3-colouring. First, observe that F^+ is the union of two copies of F^* sharing exactly one edge, namely y_1x_2. That is, $F_1' + x_1x_2, y_1x_2$ and $F_2' + y_1y_2, y_1x_2$ are both isomorphic to F^*. By our assumption on F^*, graphs $F_1' + x_1x_2, x_2y_1$ and $F_2' + y_1y_2, y_1x_2$ have acyclic 3-colourings c_1 and c_2, respectively. By our assumption on F', the restriction of c_1 to F_1' gives x_1, y_1 the same colour and the restriction of c_2 to F_2' gives x_2 and y_2 the same colour. We may assume without loss of generality that c_1 assigns colour 1 to x_1 and y_1 and colour 2 to x_2, and that c_2 assigns colour 2 to x_2 and y_2 and colour 1 to y_1. This yields a 3-colouring c^+ of F^+. We claim that c^+ is acyclic. For contradiction, suppose (F^+, c^+) has a bichromatic cycle C. As the restrictions of c^+ to $F_1' + x_1x_2, y_1x_2$ and $F_2' + y_1y_2, y_1x_2$ (the 3-colourings c_1 and c_2) are acyclic, C must contain the edges x_1x_2 and y_1y_2, so C has the chord y_1x_2. Hence, $(F_1' + x_1x_2, y_1x_2, c_1)$ has a bichromatic cycle on vertex set $(V(C) \setminus V(F_2)) \cup \{x_2\}$, a contradiction.

We now essentially reduce to Case 1. Set $x = x_1$, $y = y_2$ and take the graph F^+. We proved above that F^+ has an acyclic 3-colouring. As every acyclic 3-colouring c of F^+ colours x_1 and y_1 alike, c colours $x = x_1$ and $y = y_2$ differently (as y_1x_2 is an edge). Finally, the graph $F^+ + xy = F^+ + x_1y_2$ has no acyclic 3-colouring, as for every 3-colouring c of $F^+ + x_1y_2$, the 4-vertex cycle $x_1x_2y_1y_2x_1$ is bichromatic for $(F^+ + x_1y_2, c)$. The only difference with Case 1 is that the graph $F^+ + x_1y_2$ has girth 4 due to the cycle $x_1x_2y_1y_2x_1$ whereas we need the girth to be at least g just as the graph $F' + xy$ in Case 1 has girth g. Hence, before reducing to Case 1, we first recursively repeat g times the operation of placing the graph F^+ on the y_1x_2-edge; note that the size of the resulting graph G' is still polynomial in the size of G.

Case 4: *There exist acyclic 3-colourings c_1 and c_2 of F' with $c_1(x) = c_1(y)$ and $c_2(x) \neq c_2(y)$.* We first construct a new graph J. We take two disjoint copies F'_1 and F'_2 of F' and identify the two x-vertices with each other and also the two y-vertices with each other. We write $x = x_1 = x_2$ and $y = y_1 = y_2$; see also Figure 4 (left).

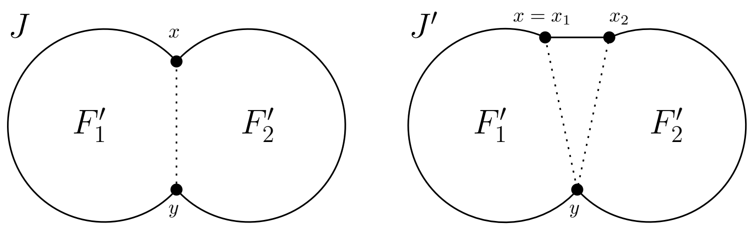

Figure 4 The graph J from Case 4 (left) and the graph J' from Case 4b (right).

We distinguish between two sub-cases.

Case 4a: *J has an acyclic 3-colouring.*
Our goal is to reduce either to Case 2 or 3 by using J instead of F'. We first observe that J and $J + xy$ have girth at least g. We also note that $J + xy$ has no acyclic 3-colouring, as otherwise $F' + xy$, being an induced subgraph of $J + xy$, has an acyclic 3-colouring. Hence, in order to reduce to Case 2 or 3 it remains to show that every acyclic 3-colouring of J assigns the same colour to x and y. For contradiction, suppose that J has an acyclic 3-colouring c such that $c(x) \neq c(y)$, say $c(x) = 1$ and $c(y) = 2$. Then in at least one of the two subgraphs F'_1 and F'_2 of J, say F'_1, there exists no 1-2 path from x to y; otherwise (J, c) has a bichromatic cycle formed by the union of the two 1-2-paths, which is not possible as c is acyclic. Let c' be the restriction of c to $V(F'_1)$. Then, as $c(x) = 1$ and $c(y) = 2$, we find that c' is a 3-colouring of $F'_1 + xy$. As there is no 1-2 path from x to y in F'_1, we find that c' is even an acyclic 3-colouring of $F'_1 + xy$, a contradiction (recall that $F' + xy$ has no acyclic 3-colouring by construction).

Case 4b: *J has no acyclic 3-colouring.*
By assumption, F' has an acyclic 3-colouring that gives x and y different colours. We first prove a claim.[1]

Claim 1. For every triple (h, i, j) with $\{h, i, j\} = \{1, 2, 3\}$, every acyclic 3-colouring c of F' with $c(x) = c(y) = h$ yields an h-i path and h-j path from x to y.

[1] Claim 1 only holds for $k = 3$ and is the reason we cannot generalize Lemma 6 to $k \geq 3$.

We prove Claim 1 as follows. For contradiction, suppose that F' has an acyclic 3-colouring c that colours x and y alike, say $c(x) = c(y) = 1$, such that F' contains no 1-2-path or no 1-3-path, say F' contains no 1-2-path from x to y. Then by swapping colours 2 and 3, we obtain another acyclic 3-colouring c' of F' such that F' contains no 1-3-path from x to y. In J we now colour the vertices of F'_1 by c and the vertices of F'_2 by c'. As $c(x) = c(x') = 1$ and $c(y) = c(y') = 1$, this yields a 3-colouring c_J. By assumption, c_J is not acyclic. Hence, (J, c_J) contains a bichromatic cycle C with colours 1 and i for some $i \in \{2, 3\}$. As the restrictions of c_J to F'_1 and F'_2 are acyclic, C must contain at least one vertex of $V(F'_1) \setminus \{x, y\}$ and at least one vertex of $V(F'_2) \setminus \{x, y\}$. Thus C consists of 1-i-paths from x to y in both F'_1 and F'_2. As at least one of these paths is missing in F'_1 or F'_2, this yields a contradiction.

We now construct a new graph J' as follows. We take two disjoint copies F'_1 and F'_2 of F' and still identify y_1 and y_2 as y, but instead of identifying x_1 and x_2 we add an edge between x_1 and x_2; see also Figure 4 (right).
We now prove some more claims that will enable us to reduce to Case 1.

(i) The graphs J' and $J' + x_1y$ have girth at least g.
This follows directly from the fact that respectively, F' and $F' + xy$ have girth at least g.

(ii) The graph $J' + x_1y$ has no acyclic 3-colouring.
This follows directly from the fact that $F' + xy$ is an induced subgraph of $J' + x_1y$ and has no acyclic 3-colouring by construction.

(iii) The graph J' has an acyclic 3-colouring.
This can be seen as follows. By assumption, F' has an acyclic 3-colouring c that gives x and y different colours, say $c(x) = 1$ and $c(y) = 3$. By swapping colours 1 and 2 we obtain an acyclic 3-colouring c' of F' with $c'(x) = 2$ and $c'(y) = 3$. As $c(y) = c'(y) = 3$, this yields a 3-colouring $c_{J'}$ of J'. As the restrictions of $c_{J'}$ to F'_1 and F'_2 are acyclic, any bichromatic cycle of $(J', c_{J'})$ must pass through x_1, x_2 and y. However, x_1, x_2 and y have colours 1, 2, 3, respectively. Hence, this is not possible.

(iv) Every acyclic 3-colouring of J' gives x_1 and y different colours.
For contradiction, assume J' has an acyclic 3-colouring c that colours x_1 and y alike, say $c(x_1) = c(y) = 1$ and $c(x_2) = 2$. The restriction of c to $V(F'_1)$ is an acyclic 3-colouring of F'_1 that gives x_1 and y colour 1. Hence, by Claim 1, F'_1 contains a 1-2 path from x_1 to y. The restriction of c' to $V(F'_2)$ is an acyclic 3-colouring of F'_2 that gives x_2 colour 2 and y colour 1. Then F'_2 must contain a 1-2 path from x_2 to y; otherwise we found an acyclic 3-colouring of $F'_2 + x_2y$, which is not possible by construction. The two 1-2 paths now form, with the edge x_1x_2, a bichromatic cycle of (J', c). As c is acyclic, this is not possible.

By (i)-(iv) we may take J' with x_1 and y instead of F' with x and y and reduce to Case 1. ◀

The *line graph* of a graph G has vertex set $E(G)$ and an edge between two vertices e and f if and only if e and f share an end-vertex of G. In Lemma 7 we modify the construction of [3] for line graphs from $k = 3$ to $k \geq 3$. In Lemma 8 we give a new construction for proving hardness when k is part of the input.

▶ **Lemma 7.** *For every $k \geq 3$,* Acyclic k-Colouring *is* NP*-complete for line graphs.*

Proof. For an integer $k \geq 1$, a *k-edge colouring* of a graph $G = (V, E)$ is a mapping $c : E \to \{1, \ldots, k\}$ such that $c(e) \neq c(f)$ whenever the edges e and f share an end-vertex. A *colour class* consists of all edges of G that are mapped by c to a specific colour i. The pair (G, c) has a *bichromatic* cycle C if C is a cycle of G with its edges coloured by two alternating colours. The notion of a *bichromatic* path is defined in a similar manner. We say that c is *acyclic* if (G, c) has no bichromatic cycle. For a fixed integer $k \geq 1$, the Acyclic k-Edge Colouring problem is to decide if a given graph has an acyclic k-edge colouring. Alon and Zaks proved that Acyclic 3-Edge Colouring is NP-complete for multigraphs. We note that a graph has an acyclic k-edge colouring if and only if its line graph has an acyclic k-colouring. Hence, it remains to generalize the construction of Alon and Zaks [3] from $k = 3$ to $k \geq 3$. Our main tool is the gadget graph F_k, depicted in Figure 5, about which we prove the following two claims.

(i) The edges of F_k can be coloured acyclically using k colours, with no bichromatic path between v_1 and v_{14}.

(ii) Every acyclic k-edge colouring of F_k using k colours assigns e_1 and e_2 the same colour.

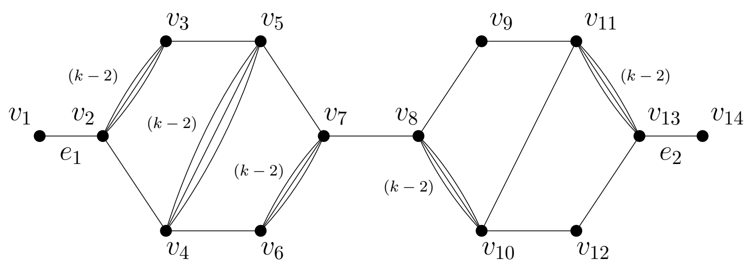

Figure 5 The gadget multigraph F_k. The labels on edges are multiplicities.

We first prove (ii). We assume, without loss of generality, that v_1v_2 is coloured by 1, v_2v_4 by 2 and the edges between v_2 and v_3 by colours $3, \ldots, k$. The edge v_3v_5 has to be coloured by 1, otherwise we have a bichromatic cycle on $v_2v_3v_5v_4$. This necessarily implies that

- the edges between v_4 and v_5 are coloured by $3, \ldots, k$,
- the edge v_5v_7 is coloured by 2,
- the edge v_4v_6 is coloured by 1,
- the edges between v_6 and v_7 are coloured by $3, \ldots, k$, and
- the edge v_7v_8 is coloured by 1.

Now assume that the edge v_8v_9 is coloured by $a \in \{2, \ldots, k\}$ and the edges between v_8 and v_{10} by colours from the set A, where $A = \{2, \ldots, k\} \setminus a$. The edge $v_{10}v_{11}$ is either coloured a or 1. However, if it is coloured 1, v_9v_{11} is assigned a colour $b \in A$ and necessarily we have either a bichromatic cycle on $v_8v_9v_{11}v_{13}v_{12}v_{10}$, coloured by b and a, or a bichromatic cycle on $v_{10}v_{11}v_{13}v_{12}$, coloured by a and 1. Thus $v_{10}v_{11}$ is coloured by a. To prevent a bichromatic cycle on $v_8v_9v_{11}v_{10}$, the edge v_9v_{11} is assigned colour 1. The rest of the colouring is now determined as $v_{10}v_{12}$ has to be coloured by 1, the edges between v_{11} and v_{13} by A, $v_{12}v_{13}$ by a, and $v_{13}v_{14}$ by 1. We then have a k-colouring with no bichromatic cycles of size at least 3 in F_k for every possible choice of a. This proves that v_1v_2 and $v_{13}v_{14}$ are coloured alike under every acyclic k-edge colouring.

We prove (i) by choosing a different from 2. Then there is no bichromatic path between v_1 and v_{14}.

We now reduce from k-Edge-Colouring to Acyclic k-Edge Colouring as follows. Given an instance G of k-edge Colouring we construct an instance G' of Acyclic k-Edge Colouring by replacing each edge uv in G by a copy of F_k where u is identified with v_1 and v is identified with v_{14}.

If G' has an acyclic k-edge colouring c' then we obtain a k-edge colouring c of G by setting $c(uv) = c'(e_1)$ where e_1 belongs to the gadget F_k corresponding to the edge uv. If

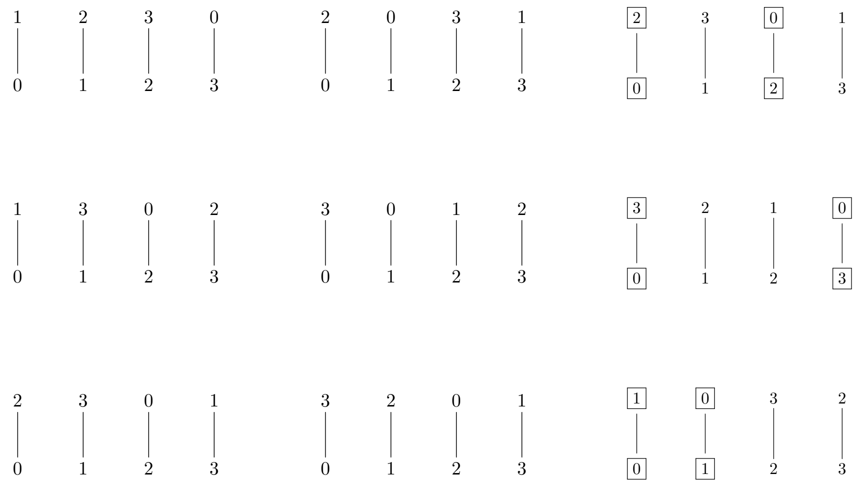

Figure 6 Acyclic colourings in the proof of Lemma 8 for a vertex representing one of the three colours (left and middle). Sample failures for an acyclic colouring from other permutations of $(0, 1, 2, 3)$ together with a failure cycle (right). Note that each row of quadruples is joined in a clique.

G has a k-edge colouring c then we obtain an acyclic k-edge colouring c' of G' by setting $c'(e_1) = c(uv)$ where e_1 belongs to the gadget corresponding to the edge uv. The remainder of each gadget F_k can then be coloured as described above. ◀

In our next result, k is part of the input.

▶ **Lemma 8.** Acyclic Colouring *is* NP*-complete for* $(19P_1, 3P_3, 2P_5)$*-free graphs.*

Proof. We reduce from 3-Colouring with maximum degree 4 which is known to be NP-complete [26]. Let G be an instance of 3-Colouring with $|V(G)| = n$ vertices and maximum degree 4. We will construct an instance G' of Acyclic Colouring where $k = 4n$. Our vertex gadget is built from two k-cliques, J_0 and J_1, with a matching between them. We number the vertices of each of the cliques 0 to $k-1$. The matching we insert into the graph is $(0, 0), \ldots, (k-1, k-1)$. In addition, we place an edge from i in J_0 to j in J_1 if and only if $\lfloor i/4 \rfloor < \lfloor j/4 \rfloor$. Suppose that some assignment of colours is given to J_0. By recolouring, we assume it is the identity colouring of i to i on J_0. Then the possible acyclic k-colourings of vertices $(\lfloor i/4 \rfloor + 0, \lfloor i/4 \rfloor + 1, \lfloor i/4 \rfloor + 2, \lfloor i/4 \rfloor + 3)$ in J_1 are

$$\begin{array}{l}
(\lfloor i/4 \rfloor + 1, \lfloor i/4 \rfloor + 2, \lfloor i/4 \rfloor + 3, \lfloor i/4 \rfloor + 0),\\
(\lfloor i/4 \rfloor + 1, \lfloor i/4 \rfloor + 3, \lfloor i/4 \rfloor + 0, \lfloor i/4 \rfloor + 2),\\
(\lfloor i/4 \rfloor + 2, \lfloor i/4 \rfloor + 3, \lfloor i/4 \rfloor + 1, \lfloor i/4 \rfloor + 0),\\
(\lfloor i/4 \rfloor + 2, \lfloor i/4 \rfloor + 0, \lfloor i/4 \rfloor + 3, \lfloor i/4 \rfloor + 1),\\
(\lfloor i/4 \rfloor + 3, \lfloor i/4 \rfloor + 0, \lfloor i/4 \rfloor + 1, \lfloor i/4 \rfloor + 2),\\
(\lfloor i/4 \rfloor + 3, \lfloor i/4 \rfloor + 2, \lfloor i/4 \rfloor + 0, \lfloor i/4 \rfloor + 1).
\end{array}$$

They are built from the permutations of $(0, 1, 2, 3)$ that do not contain a transposition. We draw all of them, to demonstrate it is not an acyclic colouring, in Figure 6 (keep in mind that vertices in a row are joined in a clique).

In our reduction, the first two acyclic k-colourings will represent colour 1, the second two colour 2 and the third two colour 3 of the sought 3-colouring of G. To force similarly coloured copies of J_0 we add a new k-clique J_2 with edges from i in J_0 to j in J_2 if and only if $i < j$. To prevent the existence of bichromatic cycles in our later construction, we add a k-clique J_3 with edges from i in J_2 to j in J_3 if and only if $i < j$. This enforces that in any acyclic k-colouring of G', the i-th vertices (where $i \in \{0, \ldots, k-1\}$) in cliques J_0, J_2, J_3 would have the same colour. Therefore, by the way we placed the edges between different cliques from $\{J_0, J_2, J_3\}$, there is no bichromatic path with vertices from more than one clique in $\{J_0, J_2, J_3\}$.

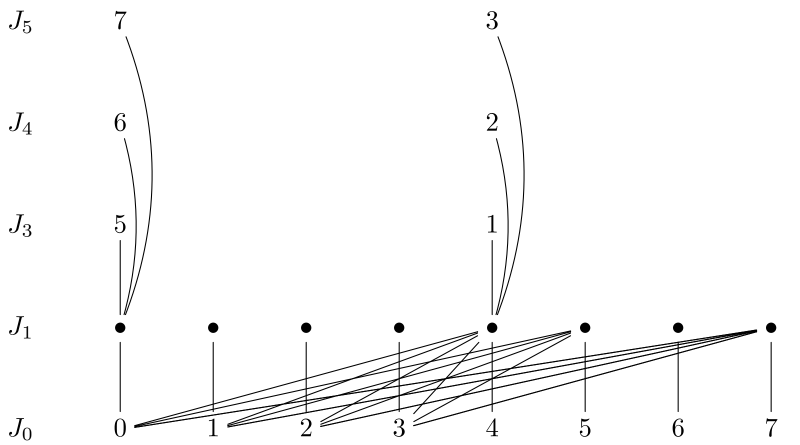

Figure 7 Edge construction in the proof of Lemma 8 between vertices 0 and 1 of G. Everything in a row is joined in a clique. Edges are omitted between J_0 and J_3, J_4, J_5, though they enforce the colouring.

We now construct edge gadgets. We take another two k-cliques to join J_2, say J_4 and J_5. We will want them coloured exactly like J_0, so for i in J_2 and j in J_4 or J_5, where $i < j$, we will add an edge ij. Suppose we have an edge in G between p and q for some $p, q \in \{0, \ldots, n-1\}$. Then we place an edge from the vertex $4p$ in J_1 to $4q+1$ in J_3 and from $4q$ in J_1 to $4p+1$ in J_3 (recall that $p, q \in \{0, \ldots, n-1\}$ and cliques J_1 and J_3 are of size $4n$, so these edges are well defined). See Figure 7. Now we place an edge from $4p$ in J_1 to $4q+2$ in J_4 and of $4q$ in J_1 to $4p+2$ in J_4. Finally, we place an edge from $4p$ in J_1 to $4q+3$ in J_5 and from $4q$ in J_1 to $4p+3$ in J_5. This concludes the construction for the edge pq in $E(G)$.

Suppose we have an edge $rs \in E(G)$ so that $\{p, q\} \cap \{r, s\} = \emptyset$. Then we build a gadget for rs using the same additional three cliques that we used for the edge pq. However, if we have edges with a common endpoint, e.g. $pq, ps \in E(G)$, then by adding the edges from $4p$ in J_1 to $4q+1$ in J_3, from $4q$ in J_1 to $4p+1$ in J_3, from $4p$ in J_1 to $4s+1$ in J_3, and from $4s$ in J_1 to $4p+1$ in J_3 we introduce new 4-cycles, one of them induced by the vertices $4q$ and $4p$ in J_1 and $4p+1$ and $4s+1$ in J_3. To avoid this, we add three additional k-cliques to build the gadget for ps. By Vizing's Theorem [49], we obtain in polynomial time a 5-edge colouring of G (as G has maximum degree 4). Using this 5-edge colouring, we build gadgets for all the edges with at most $5 \times 3 = 15$ additional k-cliques (we use 3 additional cliques for each colour class).

The clique structure of G' is drawn in Figure 8. As G' consists of at most 18 cliques, G' is $19P_1$-free. Furthermore, any induced linear forest where each connected component has size at least 3 contains vertices in at most five cliques. Hence G' is $(3P_3, 2P_5)$-free. It remains to prove that G has a 3-colouring if and only if G' has an acyclic k-colouring.

First, suppose that G' has an acyclic k-colouring c'. Then each k-clique of G' has to use each colour exactly once. We can permute colours so that vertex i in J_0 (where $0 \leq i \leq 4n-1$) has colour i. It follows from the connections between cliques that the i-th vertices in cliques $J_2, \ldots, J_{17}$ also have colour i and the vertices $4j, 4j+1, 4j+2, 4j+3$, $(0 \leq j \leq n-1)$ in J_1 have colours from the set $\{4j, 4j+1, 4j+2, 4j+3\}$. For each vertex i in G, set $c(i) = 1$ if the colours of $(4i, 4i+1, 4i+2, 4i+3)$ in J_1 under c' correspond to one of the first two possible colourings (listed above); set $c(i) = 2$ if it corresponds to one of the second two possible colourings; set $c(i) = 3$ if it corresponds to one of the last two colourings. We claim that c is a 3-colouring of G. Suppose that pq is an edge in G with edge gadget using cliques J_3, J_4, J_5. Since c' is acyclic and c' is identity on J_3, we have $c'(4p) \neq 4p+1$ in J_1 or $c'(4q) \neq 4q+1$ in J_1. Both $4p$ and $4q$ are the first vertices of the respective quadruples, so p and q are not both coloured 1. Similarly, the edges going between cliques J_1 and J_4 ensure that they are not both coloured 2 and the edges going between cliques J_1 and J_5 ensure that they are not both coloured 3. Hence, $c(p) \neq c(q)$ and c is a 3-colouring of G.

Now suppose G has a 3-colouring c. We construct a labelling c' of G' where we colour each quadruple in J_1 corresponding to a vertex of G by the first of each pair of colourings listed in the table for each of the three colours, respectively. The labelling c' in other cliques of G' is the identity. By the construction of G' and particularly by the properties of edge gadgets in G', we find that c' is a k-colouring of G'.

Finally, we need to verify that c' is acyclic. We will begin with bichromatic cycles between two cliques. No bichromatic cycle can appear in J_0 and J_1 forming the vertex gadget. This is due to the edges from the former to the latter always pointing to a higher number (or the same but here we chose a 3-colouring to avoid such situation). A similar explanation works for all the clique pairs $(0,2), (2,3), \ldots, (2,17)$ in Figure 8. The last possibility is a bichromatic cycle formed through J_1 from one of the cliques J_3 to J_{17}. However, such a cycle would have to pass through an actual edge gadget (where it is forbidden by the 3-colouring) or through vertices in different edge gadgets, where it must form a cycle with four colours. Now we need to consider bichromatic cycles passing through three or more cliques, but they would have to involve a bichromatic path through J_0, J_2, J_3 which is not possible. This completes the proof. ◀

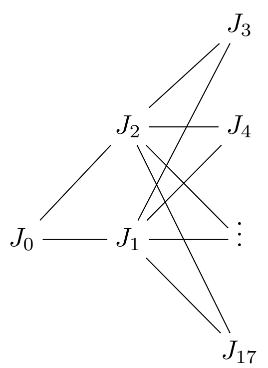

Figure 8 Connections between cliques in the construction from the proof of Lemma 8.

We combine the above results with results of Coleman and Cai [14] and Lyons [43] to prove Theorem 1.

Figure 9 The gadget replacing edges uv (on the left) and its natural star 3-colouring (on the right) in the proof of Lemma 9.

Theorem 1 (restated). *Let H be a graph. For the class of H-free graphs it holds that:*

(i) Acyclic Colouring *is polynomial-time solvable if $H \subseteq_i P_4$ and* NP*-complete if H is not a forest or $H \supseteq_i 19P_1, 3P_3, 2P_5$ or P_{11};*

(ii) For every $k \geq 3$, Acyclic k-Colouring *is polynomial-time solvable if H is a linear forest and* NP*-complete otherwise.*

Proof. We first prove (ii). First suppose that H contains an induced cycle C_p. If $p = 3$, then we use the result of Coleman and Cai [14], who proved that for every $k \geq 3$, Acyclic k-Colouring is NP-complete for bipartite graphs. Suppose that $p \geq 3$. If $k = 3$, then we let $g = p + 1$ and use Lemma 6. If $k \geq 4$, we reduce from Acyclic 3-Colouring for graphs of girth $p + 1$ by adding a dominating clique of size $k - 3$. Now assume H has no cycle so H is a forest. If H has a vertex of degree at least 3, then H has an induced $K_{1,3}$. As every line graph is $K_{1,3}$-free, we can use Lemma 7. Otherwise H is a linear forest and we use Corollary 5.

We now prove (i). Due to (ii), we may assume that H is a linear forest. If $H \subseteq_i P_4$, then we use the result of Lyons [43] that states that Acyclic Colouring is polynomial-time solvable for P_4-free graphs. If $H \supseteq_i 19P_1, 3P_3, 2P_5$ or P_{11}, then we use Lemma 8. ◀

4 Star Colouring

In this section we prove Theorem 2. We first prove the following lemma.

▶ **Lemma 9.** *Let H be a graph with an* even *cycle. Then, for every $k \geq 3$,* Star k-Colouring *is* NP*-complete for H-free graphs.*

Proof. We reduce from 3-Colouring for graphs of girth at least $p + 1$. Given an instance G of this problem, we construct an instance G' of Star 3-Colouring as follows. Take three vertex disjoint copies of P_3 and form a triangle using one endpoint of each; see Figure 9. Replace each edge uv in G by this gadget with u and v identified with the non-adjacent endpoints of two paths. Then G' is C_p-free since, aside from triangles, the construction cannot introduce any cycle shorter than those present in G.

We first show that any star 3-colouring of G' colours u and v differently. Assume not, their neighbours must be coloured differently since otherwise any 3-colouring of the remainder of the gadget will result in a bichromatic P_4. Without loss of generality, assume that u and v are coloured 1, the neighbour u' of u is coloured 2 and the neighbour v' of v is coloured 3. Let x be the neighbour of u' in the triangle and y the neighbour of v' in the triangle. Neither x or y can be coloured 1 since this will result in a bichromatic P_4. Therefore x is coloured 3, y is coloured 2 and the third vertex z of the triangle is coloured 1. This is a contradiction since we have a bichromatic P_4 on the vertices u', x, y, v'. Therefore, we obtain a 3-colouring c of G by setting $c(v) = c'(v)$ for some star 3-colouring c' of G'.

We extend a given 3-colouring of G to a star 3-colouring of G', by locally star 3-colouring as in the right hand side of Figure 9 (or automorphically). Hence, G is 3-colourable if and only if G' is star 3-colourable.

We obtain NP-completeness for $k \geq 4$ by a reduction from Star 3-Colouring for C_p-free graphs by adding a dominating clique of size $k - 3$. ◀

In Lemma 10 we extend the recent result of Lei et al. [38] from $k = 3$ to $k \geq 3$. In Lemma 11 we show a result where k is part of the input. A graph is *co-bipartite* if it is the complement of a bipartite graph.

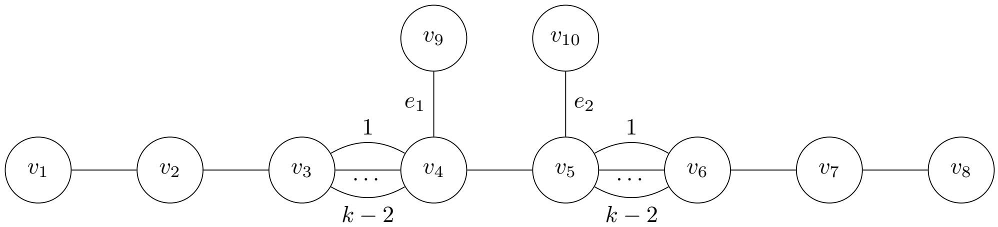

Figure 10 The gadget F_k in the proof of Lemma 10.

▶ **Lemma 10.** *For every $k \geq 3$, Star k-Colouring is NP-complete for line graphs.*

Proof. Recall that for an integer $k \geq 1$, a *k-edge colouring* of a graph $G = (V, E)$ is a mapping $c : E \to \{1, \ldots, k\}$ such that $c(e) \neq c(f)$ whenever the edges e and f share an end-vertex. Recall also that the notions of a colour class and bichromatic subgraph for colourings has its natural analogue for edge colourings. An edge k-colouring c is a *star* k-edge colouring if the union of any two colour classes induces a star forest. For a fixed integer $k \geq 1$, the Star k-Edge Colouring problem is to decide if a given graph has an star k-edge colouring. Lei et al. [38] proved that Star 3-Edge Colouring is NP-complete. Dvořák et al. [16] observed that a graph has a star k-edge colouring if and only if its line graph has a star k-colouring. Hence, it suffices to follow the proof of Lei et al.[38] in order to generalize the case $k = 3$ to $k \geq 3$. As such, we give a reduction from k-Edge Colouring to Star k-Edge Colouring which makes use of the gadget F_k in Figure 10. First we consider separately the case where the edges $e_1 = v_4v_9$ and $e_2 = v_5v_{10}$ are coloured alike and the case where they are coloured differently to show that in any star k-edge colouring of the gadget F_k shown in Figure 10, v_1v_2 and v_7v_8 are assigned the same colour.

Assume $c(e_1) = c(e_2) = 1$. We may then assume that the edge v_4v_5 is assigned colour 2 and the remaining $k - 2$ colours are used for the multiple edges v_3v_4 and v_5v_6. The edge v_2v_3, and similarly v_6v_7, must then be assigned colour 1 to avoid a bichromatic P_5 on the vertices $\{v_2, v_3, v_4, v_5, v_6\}$ using any two of the multiple edges in a single colour. The edge v_1v_2, and similarly v_7v_8 must then be assigned colour 2 to avoid a bichromatic P_5 on the vertices $\{v_1, v_2, v_3, v_4, v_9\}$.

Next assume e_1 and e_2 are coloured differently. Without loss of generality, let $c(e_1) = 1$, $c(e_2) = 2$ and $c(v_4v_5) = 3$. The multiple edges v_3v_4 must then be assigned colours 2 and $4 \ldots k$ and v_5v_6 colour 1 and colours $4 \ldots k$. To avoid a bichromatic P_5 on the vertices $\{v_2, v_3, v_4, v_5, v_6\}$, v_2v_3 must be coloured 1. Similarly, v_6v_7 must be assigned colour 2. Finally, to avoid a bichromatic P_5 on the vertices $\{v_1, v_2, v_3, v_4, v_9\}$, v_1v_2 must be coloured 3. By a similar argument, v_7v_8 must also be coloured 3, hence v_1v_2 and v_7v_8 must be coloured alike.

We can then replace every edge e in some instance G of k-Edge-Colouring by a copy of F_k, identifying its endpoints with v_1 and v_8, to obtain an instance G' of Star k-Edge-Colouring. If G is k-edge-colourable we can star k-edge-colour G' by setting $c'(v_1v_2) = c'(v_7v_8) = c(e)$. If G' is star k-edge-colourable, we obtain a k-edge-colouring of G by setting $c(e) = c'(v_1v_2)$. ◀

We now let k be part of the input. The *complement* of a graph G is the graph $\overline{G}$ with vertex set $V(G)$ and an edge between two vertices u and v if and only if $uv \notin E(G)$. A k-colouring of G can be seen as a partition of $V(G)$ into k independent sets. Hence, a k-colouring of G corresponds to a *clique-covering* of $\overline{G}$, which is a partition of $V(\overline{G}) = V(G)$ into k cliques. A graph is *co-bipartite* if it is the complement of a bipartite graph.

▶ **Lemma 11.** Star Colouring *is* NP*-complete for co-bipartite graphs.*

Proof. We show that finding an optimal star colouring of a co-bipartite graph G is equivalent to finding a maximum balanced biclique in its complement $\overline{G}$. An optimal star colouring of G corresponds to an optimal clique-covering of $\overline{G}$ such that the graph induced by the vertices of any two cliques in the covering partition is $\overline{P_4} = P_4$-free and $\overline{C_4} = 2P_2$-free. Since $\overline{G}$ is triangle-free, the clique-covering number of $\overline{G}$ is $n - M$ where n is the number of vertices of G and M is the number of edges in a maximum matching such that no two edges induce either $2P_2$ or P_4. Since $\overline{G}$ is bipartite, a maximum matching of this form is a maximum balanced biclique. It is NP-complete to find the maximum size of a balanced biclique in a bipartite graph [26]. Therefore Star Colouring is NP-complete for co-bipartite graphs. ◀

We combine the above results with results of Albertson et al. [1] and Lyons [43] to prove Theorem 2.

Theorem 2 (restated). *Let H be a graph. For the class of H-free graphs it holds that:*

(i) Star Colouring *is polynomial-time solvable if $H \subseteq_i P_4$ and* NP*-complete for any $H \neq 2P_2$.*

(ii) For every $k \geq 3$, Star k-Colouring *is polynomial-time solvable if H is a linear forest and* NP*-complete otherwise.*

Proof. We first prove (ii). First suppose that H contains an induced odd cycle. Then the class of bipartite graphs is contained in the class of H-free graphs. Lemma 7.1 in Albertson et al. [1] implies, together with the fact that for every $k \geq 3$, k-Colouring is NP-complete, that for every $k \geq 3$, Star k-Colouring is NP-complete for bipartite graphs. If H contains an induced even cycle, then we use Lemma 9. Now assume H has no cycle, so H is a forest. If H contains a vertex of degree at least 3, then H contains an induced $K_{1,3}$. As every line graph is $K_{1,3}$-free, we can use Lemma 10. Otherwise H is a linear forest, in which case we use Corollary 5.

We now prove (i). Due to (ii), we may assume that H is a linear forest. If $H \subseteq_i P_4$, then we use the result of Lyons [43] that states that Star Colouring is polynomial-time solvable for P_4-free graphs. If $3P_1 \subseteq_i H$, then we use Lemma 11 after observing that co-bipartite graphs are $3P_1$-free. Otherwise $H = 2P_2$, but this case was excluded from the statement of the theorem. ◀

5 Injective Colouring

In this section we prove Theorem 3. We first show three lemmas.

▶ **Lemma 12.** *For every $k \geq 4$,* Injective k-Colouring *is* NP*-complete for C_3-free graphs.*

Proof. We reduce from Injective k-Colouring. Given an instance G of Injective k-Colouring, construct an instance G' of Injective k-Colouring for triangle-free graphs as follows. For each edge uv of G, remove the edge uv and add two vertices u'_v adjacent to u and v'_u adjacent to v. Next, place an independent set of $k-2$ vertices adjacent to both u'_v and v'_u. Then G' is triangle-free since the edge gadget described is triangle-free, any two vertices of G are now at distance at least 4 and no vertex not belonging to an edge gadget has two adjacent neighbours belonging to edge gadgets. We claim that G' has an injective k-colouring if and only if G has an injective k-colouring.

First assume that G has an injective k-colouring c. Colour the vertices of G' corresponding to vertices of G as they are coloured by c. We can extend this to an injective k-colouring c' of G' by considering the gadget corresponding to each edge uv of G. Set $c'(u'_v) = c'(v)$ and $c'(v'_u) = c'(u)$. We can now assign the remaining $k-2$ colours to the vertices of the independent sets. Clearly c' creates no bichromatic P_3 involving vertices in at most one edge gadget. Assume there exists a bichromatic P_3 involving vertices in more than one edge gadget, then this path must consist of a vertex u of G together with two gadget vertices u'_v and u'_w which are coloured alike. This is a contradiction since it implies the existence of a bichromatic path v, u, w in G.

Now assume that G' has an injective k-colouring c'. Let c be the restriction of c' to those vertices of G' which correspond to vertices of G. To see that c is an injective colouring of G, note that we must have $c'(u'_v) = c'(v)$ and $c'(v'_u) = c'(u)$ for any edge uv. Therefore, if c induces a bichromatic P_3 on u, v, w, then c' induces a bichromatic P_3 on v'_u, v, v'_w. We conclude that c is injective. ◀

In our next two results, k is part of the input.

▶ **Lemma 13.** Injective Colouring *is polynomial-time solvable for P_4-free graphs and $(P_1 + P_3)$-free graphs.*

Proof. Since connected P_4-free graphs have diameter at most 2, no two vertices can be coloured alike in an injective colouring. Hence the injective chromatic number of a P_4-free graph is equal to the number of its vertices.

We now consider $(P_1 + P_3)$-free graphs. First, note that an injective colouring of G is equivalent to a clique-covering of its complement $\overline{G}$ such that the graph induced by the vertices of the union of any two clique classes is $(P_1 + P_2)$-free (as $\overline{P_3} = P_1 + P_2$). Since G is $(P_1 + P_3)$-free, $\overline{G}$ is $\overline{P_1 + P_3}$-free. By a result of Olariu [46], each connected component of $\overline{G}$ is either triangle-free or complete multi-partite. Let $D_1, \ldots, D_p$ be the vertex sets of the connected components of $\overline{G}$ for some $p \geq 1$. Then in G, every D_i is complete to every D_j. Hence, the injective chromatic number of G is the sum of the injective chromatic numbers of the subgraphs G_i induced by D_i ($i \in \{1, \ldots, p\}$). As such, it remains to determine the injective chromatic number of each G_i, which we do below.

Let $1 \leq i \leq p$. If $\overline{G_i}$ is complete multi-partite, then G_i is a disjoint union of cliques and its injective chromatic number is equal to the size of its largest connected component. In the other case, $\overline{G_i}$ is triangle-free. Then each clique class in a clique-covering has size at most 2, and any clique class of size 2 must dominate the remaining vertices of $\overline{G_i}$ to avoid a bichromatic $P_1 + P_2$. Thus, the clique-covering is a matching, each edge of which dominates

$\overline{G_i}$, together with the remaining vertices which each form clique classes of size 1. Therefore, we find an optimal $(P_1 + P_2)$-free clique-covering of $\overline{G}$ by finding a maximum matching in the graph consisting of dominating edges of $\overline{G_i}$. The injective chromatic number of G_i is then the number of vertices of G_i minus the number of edges in such a matching. ◀

▶ **Lemma 14.** Injective Colouring *is* NP*-complete for* $6P_1$*-free graphs.*

Proof. We first show that Colouring remains NP-complete given a partition of the instance G into four cliques. The Clique Covering problem is NP-complete for planar graphs [37]. A 4-colouring of a planar graph G can be found in quadratic time [47] and gives a partition of $\overline{G}$ into four cliques. Hence, given a planar instance G of clique-covering, we construct an instance $(\overline{G}, c)$ of Colouring where c is a 4-colouring of G such that the chromatic number of $\overline{G}$ is equal to the clique-covering number of G.

We now give a reduction from this problem to Injective Colouring for $6P_1$-free graphs. Given a graph G and a partition c into four cliques $C^1 \dots C^4$, let G' be the graph obtained from G by deleting those vertices with no neighbours outside of their own clique C^i. Then G can be coloured with k colours if and only if G' can be coloured with k colours and the maximum size of a clique in the partition c of G is at most k. To see this, note that the vertices of $G \setminus G'$ then have degree at most $k - 1$, hence we can greedily colour these vertices given a k-colouring of G'.

This instance (G', c) of Colouring given a partition of G' into four cliques can then be transformed in polynomial time to an instance G'' of Injective Colouring as follows. Add a fifth clique C^0 with one vertex v_e for each edge $e = xy$ in G' which has endpoints in two different cliques of c. For each such edge, replace e by two edges xv_e and yv_e. G' has a colouring with k colours if and only if G'' has an injective colouring with $k + m$ colours where m is the number of edges in G with endpoints in different cliques. To see this, note that in any injective colouring of G'', the set of colours used in C^0 is disjoint from the set of those used in the cliques $C^1 \dots C^4$. Therefore if G'' can be injective coloured with $m + k$ colours then G' can be coloured with k colours. On the other hand, colour the vertices of $C^1 \dots C^4$ as they are coloured in some k colouring of G' and C^0 with m further colours. This is an injective colouring of G'' since any induced P_3 contains either two vertices of C^1 or one vertex of C^0 and two vertices adjacent in G'. In either case the path must be coloured with three distinct colours. This implies that G'' has an injective colouring with $k + m$ colours if and only if G' has a colouring with k colours. ◀

We combine the above results with results of Bodlaender et al. [7] and Mahdian [44] to prove Theorem 3.

Theorem 3 (restated). *Let* H *be a graph. For the class of* H*-free graphs it holds that:*

(i) Injective Colouring *is polynomial-time solvable if* $H \subseteq_i P_4$ *or* $H \subseteq_i P_1 + P_3$ *and* NP*-complete if* H *is not a forest or* $2P_2 \subseteq_i H$ *or* $6P_1 \subseteq_i H$.

(ii) For every $k \geq 4$, Injective k-Colouring *is polynomial-time solvable if* H *is a linear forest and* NP*-complete otherwise.*

Proof. We first prove (ii). If $C_3 \subseteq_i H$, then we use Lemma 12. Now suppose $C_p \subseteq_i H$ for some $p \geq 4$. Mahdian [44] proved that for every $g \geq 4$ and $k \geq 4$, Injective k-Colouring is NP-complete for line graphs of bipartite graphs of girth at least g. These graphs may not be C_3-free but for $g \geq p + 1$ they are C_p-free. Now assume H has no cycle, so H is a forest. If H contains a vertex of degree at least 3, then H contains an induced $K_{1,3}$. As every line graph is $K_{1,3}$-free, we can use the aforementioned result of Mahdian [44] again. Otherwise H is a linear forest, in which case we use Corollary 5.

We now prove (i). Due to (ii), we may assume that H is a linear forest. If $H \subseteq_i P_4$ or $H \subseteq_i P_1 + P_3$, then we use Lemma 13. Now suppose that $2P_2 \subseteq_i H$. Then the class of $(2P_2, C_4, C_5)$-free graphs (split graphs) are contained in the class of H-free graphs. Recall that Bodlaender et al. [7] proved that Injective Colouring is NP-complete for split graphs. If $6P_1 \subseteq_i H$, then we use Lemma 14. ◀

6 Conclusions

Our complexity study led to three complete and three almost complete complexity classifications (Theorems 1–3). Due to our systematic approach we were able to identify some interesting open questions for future research, which we collect below.

▷ Open Problem 1. For $k \geq 4$ and $g \geq 4$, determine the complexity of Acyclic k-Colouring for graphs of girth at least g.

For solving Open Problem 1 it would be helpful to have a better understanding of the structure of the critical graphs used in the proof of Lemma 6. We also aim to prove analogous results for the other two problems.

▷ Open Problem 2. For every $g \geq 4$, determine the complexities of Star Colouring and Injective Colouring for graphs of girth at least g.

Naturally we also aim to settle the remaining open cases for our three problems in Table 1. In particular, there is one case left for Star Colouring.

▷ Open Problem 3. Determine the complexity of Star Colouring for $2P_2$-free graphs.

Recall that the other two problems and also Colouring are all NP-complete for $2P_2$-free graphs. However, none of the hardness constructions carry over to Star Colouring. In this context, the next open problem for split graphs ($(2P_2, C_4, C_5)$-free graphs) is also interesting.

▷ Open Problem 4. Determine the complexity of Star Colouring for split graphs.

We proved that Injective Colouring is NP-complete for triangle-free graphs, but the following problem is still open.

▷ Open Problem 5. Determine the complexity of Injective Colouring for bipartite graphs.

Jin et al. [33] proved that the variant of Injective Colouring where adjacent vertices may be coloured alike is NP-complete for bipartite graphs. However, their hardness construction does not carry over to Injective Colouring.

Finally, we recall that Injective Colouring is also known as $L(1,1)$-labelling. The general distance constrained labelling problem $L(a_1,\ldots,a_p)$-Labelling is to decide if a graph G has a labelling $c : V(G) \to \{1,\ldots,k\}$ for some integer $k \geq 1$ such that for every $i \in \{1,\ldots,p\}$, $|c(u)-c(v)| \geq a_i$ whenever u and v are two vertices of distance i in G. If k is fixed, we write $L(a_1,\ldots,a_p)$-k-Labelling instead. By applying Theorem 4 we obtain the following result.

▶ **Theorem 15.** *For all $k \geq 1, a_1 \geq 1, \ldots, a_k \geq 1$, the $L(a_1,\ldots,a_p)$-k-Labelling problem is polynomial-time solvable for H-free graphs if H is a linear forest.*

We leave a more detailed and systematic complexity study of problems in this framework for future work (see, for example, [11, 23, 24] for some complexity results for both general graphs and special graph classes).

References

1 Michael O. Albertson, Glenn G. Chappell, Henry A. Kierstead, André Kündgen, and Radhika Ramamurthi. Coloring with no 2-colored P_4's. *Electronic Journal of Combinatorics*, 11, 2004.

2 Noga Alon, Colin McDiarmid, and Bruce A. Reed. Acyclic coloring of graphs. *Random Structures and Algorithms*, 2:277–288, 1991.

3 Noga Alon and Ayal Zaks. Algorithmic aspects of acyclic edge colorings. *Algorithmica*, 32:611–614, 2002.

4 Patrizio Angelini and Fabrizio Frati. Acyclically 3-colorable planar graphs. *Journal of Combinatorial Optimization*, 24:116–130, 2012.

5 Aistis Atminas, Vadim V. Lozin, and Igor Razgon. Linear time algorithm for computing a small biclique in graphs without long induced paths. *Proceedings of SWAT 2012, LNCS*, 7357:142–152, 2012.

6 Hans L. Bodlaender. A linear-time algorithm for finding tree-decompositions of small treewidth. *SIAM Journal on Computing*, 25:1305–1317, 1996.

7 Hans L. Bodlaender, Ton Kloks, Richard B. Tan, and Jan van Leeuwen. Approximations for lambda-colorings of graphs. *Computer Journal*, 47:193–204, 2004.

8 Marthe Bonamy, Konrad K. Dabrowski, Carl Feghali, Matthew Johnson, and Daniël Paulusma. Independent feedback vertex set for P_5-free graphs. *Algorithmica*, 81:1342–1369, 2019.

9 Oleg V. Borodin. On acyclic colorings of planar graphs. *Discrete Mathematics*, 25:211–236, 1979.

10 Hajo Broersma, Petr A. Golovach, Daniël Paulusma, and Jian Song. Updating the complexity status of coloring graphs without a fixed induced linear forest. *Theoretical Computer Science*, 414:9–19, 2012.

11 Tiziana Calamoneri. The $L(h,k)$-labelling problem: An updated survey and annotated bibliography. *Computer Journal*, 54:1344–1371, 2011.

12 Christine T. Cheng, Eric McDermid, and Ichiro Suzuki. Planarization and acyclic colorings of subcubic claw-free graphs. *Proc. of WG 2011, LNCS*, 6986:107–118, 2011.

13 Maria Chudnovsky, Shenwei Huang, Sophie Spirkl, and Mingxian Zhong. List-three-coloring graphs with no induced $P_6 + rP_3$. *CoRR*, abs/1806.11196, 2018.

14 Thomas F. Coleman and Jin-Yi Cai. The cyclic coloring problem and estimation of sparse Hessian matrices. *SIAM Journal on Algebraic Discrete Methods*, 7:221–235, 1986.

15 Bruno Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Information and Computation*, 85:12–75, 1990.

16 Zdeněk Dvořák, Bojan Mohar, and Robert Šámal. Star chromatic index. *Journal of Graph Theory*, 72(3):313–326, 2013.

17 Thomas Emden-Weinert, Stefan Hougardy, and Bernd Kreuter. Uniquely colourable graphs and the hardness of colouring graphs of large girth. *Combinatorics, Probability and Computing*, 7:375–386, 1998.

18 Paul Erdős. Graph theory and probability. *Canadian Journal of Mathematics*, 11:34–38, 1959.

19 Guillaume Fertin, Emmanuel Godard, and André Raspaud. Minimum feedback vertex set and acyclic coloring. *Information Processing Letters*, 84:131–139, 2002.

20 Guillaume Fertin and André Raspaud. Acyclic coloring of graphs of maximum degree five: Nine colors are enough. *Information Processing Letters*, 105:65–72, 2008.

21 Guillaume Fertin, André Raspaud, and Bruce A. Reed. Star coloring of graphs. *Journal of Graph Theory*, 47(3):163–182, 2004.

22 Jiří Fiala, Petr A. Golovach, and Jan Kratochvíl. Parameterized complexity of coloring problems: Treewidth versus vertex cover. *Theoretical Computer Science*, 412:2513–2523, 2011.

23 Jiří Fiala, Petr A. Golovach, Jan Kratochvíl, Bernard Lidický, and Daniël Paulusma. Distance three labelings of trees. *Discrete Applied Mathematics*, 160:764–779, 2012.

24 Jiří Fiala, Ton Kloks, and Jan Kratochvíl. Fixed-parameter complexity of lambda-labelings. *Discrete Applied Mathematics*, 113:59–72, 2001.

25 Esther Galby, Paloma T. Lima, Daniël Paulusma, and Bernard Ries. Classifying k-edge colouring for H-free graphs. *Information Processing Letters*, 146:39–43, 2019.

26 Michael R. Garey and David S. Johnson. *Computers and Intractability; A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., USA, 1990.

27 Petr A. Golovach, Matthew Johnson, Daniël Paulusma, and Jian Song. A survey on the computational complexity of colouring graphs with forbidden subgraphs. *Journal of Graph Theory*, 84:331–363, 2017.

28 Petr A. Golovach, Daniël Paulusma, and Jian Song. Coloring graphs without short cycles and long induced paths. *Discrete Applied Mathematics*, 167:107–120, 2014.

29 Geňa Hahn, Jan Kratochvíl, Jozef Širáň, and Dominique Sotteau. On the injective chromatic number of graphs. *Discrete Mathematics*, 256:179–192, 2002.

30 Pavol Hell, André Raspaud, and Juraj Stacho. On injective colourings of chordal graphs. *Proc. LATIN 2008, LNCS*, 4957:520–530, 2008.

31 Ian Holyer. The NP-completeness of edge-coloring. *SIAM Journal on Computing*, 10:718–720, 1981.

32 Shenwei Huang, Matthew Johnson, and Daniël Paulusma. Narrowing the complexity gap for colouring (C_s, P_t)-free graphs. *Computer Journal*, 58:3074–3088, 2015.

33 Jing Jin, Baogang Xu, and Xiaoyan Zhang. On the complexity of injective colorings and its generalizations. *Theoretical Computer Science*, 491:119–126, 2013.

34 Ross J. Kang and Tobias Müller. Frugal, acyclic and star colourings of graphs. *Discret. Appl. Math.*, 159:1806–1814, 2011.

35 T. Karthick. Star coloring of certain graph classes. *Graphs and Combinatorics*, 34:109–128, 2018.

36 Tereza Klimošová, Josef Malík, Tomáš Masařík, Jana Novotná, Daniël Paulusma, and Veronika Slívová. Colouring $(P_r + P_s)$-free graphs. *Proc. ISAAC 2018, LIPIcs*, 123:5:1–5:13, 2018.

37 Daniel Král', Jan Kratochvíl, Zsolt Tuza, and Gerhard J. Woeginger. Complexity of coloring graphs without forbidden induced subgraphs. *Proc. WG 2001, LNCS*, 2204:254–262, 2001.

38 Hui Lei, Yongtang Shi, and Zi-Xia Song. Star chromatic index of subcubic multigraphs. *Journal of Graph Theory*, 88:566–576, 2018.

39 Daniel Leven and Zvi Galil. NP-completeness of finding the chromatic index of regular graphs. *Journal of Algorithms*, 4:35–44, 1983.

40 Cláudia Linhares-Sales, Ana Karolinna Maia, Nícolas A. Martins, and Rudini Menezes Sampaio. Restricted coloring problems on graphs with few P_4's. *Annals of Operations Research*, 217:385–397, 2014.

41 Errol L. Lloyd and Subramanian Ramanathan. On the complexity of distance-2 coloring. *Proc. ICCI 1992*, pages 71–74, 1992.

42 Vadim V. Lozin and Marcin Kamiński. Coloring edges and vertices of graphs without short or long cycles. *Contributions to Discrete Mathematics*, 2(1), 2007.

43 Andrew Lyons. Acyclic and star colorings of cographs. *Discrete Applied Mathematics*, 159:1842–1850, 2011.

44 Mohammad Mahdian. On the computational complexity of strong edge coloring. *Discrete Applied Mathematics*, 118:239–248, 2002.

45 Debajyoti Mondal, Rahnuma Islam Nishat, Md. Saidur Rahman, and Sue Whitesides. Acyclic coloring with few division vertices. *Journal of Discrete Algorithms*, 23:42–53, 2013.

46 Stephan Olariu. Paw-free graphs. *Information Processing Letters*, 28:53–54, 1988.

47 Neil Robertson, Daniel P. Sanders, Paul D. Seymour, and Robin Thomas. The four-colour theorem. *Journal of Combinatorial Theory, Series B*, 70:2–44, 1997.

48 Arunabha Sen and Mark L. Huson. A new model for scheduling packet radio networks. *Wireless Networks*, 3:71–82, 1997.

49 Vadim Georgievich Vizing. On an estimate of the chromatic class of a p-graph. *Diskret Analiz*, 3:25–30, 1964.

50 David R. Wood. Acyclic, star and oriented colourings of graph subdivisions. *Discrete Mathematics and Theoretical Computer Science*, 7:37–50, 2005.

51 Xiao-Dong Zhang and Stanislaw Bylka. Disjoint triangles of a cubic line graph. *Graphs and Combinatorics*, 20:275–280, 2004.

52 Xiao Zhou, Yasuaki Kanari, and Takao Nishizeki. Generalized vertex-coloring of partial k-trees. *IEICE Transactions on Fundamentals of Electronics, Communication and Computer Sciences*, E83-A:671–678, 2000.

53 Enqiang Zhu, Zepeng Li, Zehui Shao, and Jin Xu. Acyclically 4-colorable triangulations. *Information Processing Letters*, 116:401–408, 2016.