QCSP on Reflexive Tournaments

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Abstract

We give a complexity dichotomy for the Quantified Constraint Satisfaction Problem QCSP(H) when H is a reflexive tournament. It is well-known that reflexive tournaments can be split into a sequence of strongly connected components $\mathrm{H}_1, \ldots, \mathrm{H}_n$ so that there exists an edge from every vertex of H_i to every vertex of H_j if and only if $i < j$. We prove that if H has both its initial and final strongly connected component (possibly equal) of size 1, then QCSP(H) is in NL and otherwise QCSP(H) is NP-hard.

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1 Introduction

The *Quantified Constraint Satisfaction Problem* QCSP(B), for a fixed *template* (structure) B, is a popular generalisation of the *Constraint Satisfaction Problem* CSP(B). In the latter, one asks if a primitive positive sentence (the existential quantification of a conjunction of atoms) φ is true on B, while in the former this sentence may also have universal quantification. Much of the theoretical research into (finite-domain[1]) CSPs has been in respect of a complexity classification project [11, 5], recently completed by [4, 22, 24], in which it is shown that all such problems are either in P or NP-complete. Various methods, including combinatorial (graph-theoretic), logical and universal-algebraic were brought to bear on this classification project, with many remarkable consequences.

Complexity classifications for QCSPs appear to be harder than for CSPs. Indeed, a classification for QCSPs will give a fortiori a classification for CSPs (if $\mathrm{B} \uplus \mathrm{K}_1$ is the disjoint union of B with an isolated element, then QCSP($\mathrm{B} \uplus \mathrm{K}_1$) and CSP(B) are polynomial-time many-one equivalent). Just as CSP(B) is always in NP, so QCSP(B) is always in Pspace. However, no polychotomy has been conjectured for the complexities of QCSP(B), though, until recently, only the complexities P, NP-complete and Pspace-complete were known. Recent work [25] has shown that this complexity landscape is considerably richer, and that dichotomies of the form P versus NP-hard (using Turing reductions) might be the sensible place to be looking for classifications.

CSP(B) may equivalently be seen as the *homomorphism* problem which takes as input a structure A and asks if there is a homomorphism from A to B. The *surjective CSP*, SCSP(B), is a cousin of CSP(B) in which one requires that this homomorphism from A to B be surjective. From the logical perspective this translates to the stipulation that all elements of B be used as witnesses to the (existential) variables of the primitive positive input φ. The surjective CSP appears in the literature under a variety of names, including *surjective homomorphism* [2], *surjective colouring* [12, 15] and *vertex compaction* [19, 20]. CSP(B) and SCSP(B) have various other cousins: see the survey [2] or, in the specific context of reflexive tournaments, [15]. The only one we will dwell on here is the *retraction* problem $\mathrm{CSP}^c(\mathrm{B})$ which can be defined in various ways but, in keeping with the present narrative, we could define logically as allowing atoms of the form $v = b$ in the input sentence φ where b is some element of B (the superscript c indicates that constants are allowed). It has only recently been shown that there exists a B so that SCSP(B) is in P while $\mathrm{CSP}^c(\mathrm{B})$ is NP-complete [23]. It is still not known whether such an example exists among the (partially reflexive) graphs.

It is well-known that the binary *cousin* relation is not transitive, so let us ask the question as to whether the surjective CSP and QCSP are themselves cousins? The algebraic operations pertaining to the CSP are *polymorphisms* and for QCSP these become *surjective* polymorphisms. On the other hand, a natural use of universal quantification in the QCSP might be to ensure some kind of surjective map (at least under some evaluation of many universally quantified variables). So it is that there may appear to be some relationship between the problems. Yet, there are known irreflexive graphs H for which QCSP(H) is in NL, while SCSP(H) is NP-complete (take the 6-cycle [18, 20]). On the other hand, one can find a 3-element B whose relations are preserved by a *semilattice-without-unit* operation such that both $\mathrm{CSP}^c(\mathrm{B})$ and SCSP(B) are in P but QCSP(B) is Pspace-complete. We are not aware of examples like this among graphs and it is perfectly possible that for (partially reflexive) graphs H, SCSP(H) being in P implies that QCSP(H) is in P.

[1] All structures considered in this article are finite.

Tournaments, both irreflexive and reflexive (and sometimes in between), have played a strong role as a testbed for conjectures and a habitat for classifications, for relatives of the CSP both complexity-theoretic [1, 10, 15] and algebraic [14, 21]. Looking at Table 1 one can see the last unresolved case is precisely QCSP on reflexive tournaments. This is the case we address in this paper. For irreflexive tournaments H, QCSP(H) is in P if and only if SCSP(H) in P, but for reflexive tournaments this is not the case. When H is a reflexive tournament, we prove that QCSP(H) is in NL if H has both initial and final strongly connected components trivial, and is NP-hard otherwise. In contrast to the proof from [10] and like the proof of [15], we will henceforth work largely combinatorially rather than algebraically. Note that we do not investigate beyond NP-hard, so our dichotomy cannot be compared directly to the trichotomy of [10] for irreflexive tournaments which distinguishes between P, NP-complete and Pspace-complete.

	QCSP	CSP	Surjective CSP	Retraction
irreflexive tournaments	trichotomy [10]	dichotomy [1]	dichotomy [1]	dichotomy [1]
reflexive tournaments	**this paper**	all trivial	dichotomy [15]	dichotomy [14]

Table 1 Our result in a wider context. The results for irreflexive tournaments were all proved in the more general setting of irreflexive semicomplete digraphs in the papers cited.

In Section 3 we prove the NP-hard cases of our dichotomy. Our proof method follows that from [15], while adapting the ideas of [8] in order to make what was developed for Surjective CSP applicable to QCSP. The QCSP is not naturally a combinatorial problem but can be seen thusly when viewed in a certain way. We indeed closely mirror [15] with [8] in the strongly connected case. For the not strongly connected case, the adaptation from the strongly connected case was straightforward for the Surjective CSP in [15]. However, the straightforward method does not work for the QCSP. Instead, we seek a direct argument that essentially sees us extending the method from [15].

In Section 4 we prove the NL cases of our dichotomy. Here, we use ideas originally developed in (the conference version of) [8] and first used in the wild in [17]. Thus, we do not introduce new proof techniques as such but rather weave our proof through the reasonably intricate synthesis of different known techniques. In Section 5 we state our dichotomy and give some directions for future work. Owing to space restrictions in the original submission, some of our proofs are omitted.

2 Preliminaries

For an integer $k \geq 1$, we write $[k] := \{1, \ldots, k\}$. A vertex $u \in V(G)$ in a digraph G is *backwards-adjacent* to another vertex $v \in V$ if $(u, v) \in E$. It is *forwards-adjacent* to another vertex $v \in V$ if $(v, u) \in E$. If a vertex u has a self-loop (u, u), then u is *reflexive*; otherwise u is *irreflexive*. A digraph G is *reflexive* or *irreflexive* if all its vertices are reflexive or irreflexive, respectively.

The *directed path* on k vertices is the digraph with vertices $u_0, \ldots, u_{k-1}$ and edges (u_i, u_{i+1}) for $i = 0, \ldots, k-2$. By adding the edge (u_{k-1}, u_0), we obtain the *directed cycle* on k vertices. A digraph G is *strongly connected* if for all $u, v \in V(\mathrm{G})$ there is a directed path in $E(\mathrm{G})$ from u to v. A *double edge* in a digraph G consists in a pair of distinct vertices $u, v \in V(\mathrm{G})$, so that (u, v) and (v, u) belong to $E(\mathrm{G})$. A digraph G is *semicomplete* if for every two distinct vertices u and v, at least one of (u, v), (v, u) belongs to $E(\mathrm{G})$. A semicomplete digraph G is a *tournament* if for every two distinct vertices u and v, exactly one of (u, v), (v, u) belongs to $E(\mathrm{G})$. A reflexive tournament G is *transitive* if for every three vertices u, v, w with $(u, v), (v, w) \in E(\mathrm{G})$, also (u, w) belongs to $E(\mathrm{G})$. A digraph F is a *subgraph* of a digraph G if $V(\mathrm{F}) \subseteq V(\mathrm{G})$ and $E(\mathrm{F}) \subseteq E(\mathrm{G})$. It is *induced* if $E(\mathrm{F})$ coincides with $E(\mathrm{G})$ restricted to pairs containing only vertices of $V(\mathrm{F})$. A *subtournament* is an induced subgraph of a tournament. It is well known that a reflexive tournament H can be split into a sequence of strongly connected components $\mathrm{H}_1, \dots, \mathrm{H}_n$ for some integer $n \geq 1$ so that there exists an edge from every vertex of H_i to every vertex of H_j if and only if $i < j$. We will use the notation $\mathrm{H}_1 \Rightarrow \cdots \Rightarrow \mathrm{H}_n$ for H and we refer to H_1 and H_n as the *initial* and *final* components of H, respectively.

A *homomorphism* from a digraph G to a digraph H is a function $f : V(\mathrm{G}) \to V(\mathrm{H})$ such that for all $u, v \in V(\mathrm{G})$ with $(u, v) \in E(\mathrm{G})$ we have $(f(u), f(v)) \in E(\mathrm{H})$. We say that f is *(vertex)-surjective* if for every vertex $x \in V(\mathrm{H})$ there exists a vertex $u \in V(\mathrm{G})$ with $f(u) = x$. A digraph H′ is a *homomorphic image* of a digraph H if there is a surjective homomorphism from H to H′ that is also *edge-surjective*, that is, for all $(x', y') \in E(\mathrm{H}')$ there exists an $(x, y) \in E(\mathrm{H})$ with $x' = h(x)$ and $y' = h(y)$.

The problem H-Retraction takes as input a graph G of which H is an induced subgraph and asks whether there is a homomorphism from G to H that is the identity on H. This definition is polynomial-time many-one equivalent to the one we suggested in the introduction (see e.g. [2]). The *quantified constraint satisfaction problem* QCSP(H) takes as input a sentence $\varphi := \forall x_1 \exists y_1 \dots \forall x_n \exists y_n \; \Phi(x_1, y_1, \dots, x_n, y_n)$, where Φ is a conjunction of positive atomic (binary edge) relations. This is a yes-instance to the problem just in case $H \models \varphi$.

The *canonical query* of G (from [13]) is a primitive positive sentence φ_{G} that has the property that, for all H, G has a homomorphism to H iff $\mathrm{H} \models \varphi_{\mathrm{G}}$. It is built by mapping edges (x, y) from $E(\mathrm{G})$ to atoms $E(x, y)$ is an existentially quantified conjunctive query.

The *direct product* of two digraphs G and H, denoted $\mathrm{G} \times \mathrm{H}$, is the digraph on vertex set $V(\mathrm{G}) \times V(\mathrm{H})$ with edges $((x, y), (x', y'))$ if and only if $(x, x') \in E(\mathrm{G})$ and $(y, y') \in E(\mathrm{H})$. We denote the direct product of k copies of G by G^k. A k-ary *polymorphism* of G is a homomorphism f from G^k to G; if $k = 1$, then f is also called an *endomorphism.* A k-ary polymorphism f is *essentially unary* if there exists a unary operation g and $i \in [k]$ so that $f(x_1, \dots, x_k) = g(x_i)$ for every $(x_1, \dots, x_k) \in \mathrm{G}^k$. Let us say that a k-ary polymorphism f is *uniformly* z for some $z \in V(\mathrm{G})$ if $f(x_1, \dots, x_k) = z$ for every $(x_1, \dots, x_k) \in V(\mathrm{G}^k)$. We need the following two lemmas.

▶ **Lemma 1.** *Let H be a reflexive tournament and f be a k-ary polymorphism of* H. *If $f(x, \dots, x) = z$ for every $x \in V(\mathrm{H})$, then f is uniformly z.*

Proof. Consider some tuple $(x_1, \dots, x_k)$ which has m distinct vertices. We proceed by induction on m, where the base case $m = 1$ is given as an assumption. Suppose we have the result for m vertices and let $(x_1, \dots, x_k)$ have $m + 1$ distinct entries. For simplicity (and w.l.o.g.) we will consider this reordered and without duplicates as $(y_1, \dots, y_m, y_{m+1})$. Suppose f maps $(x_1, \dots, x_k)$ to z'. Assume $(y_m, y_{m+1}) \in E(\mathrm{H})$ (the case (y_{m+1}, y_m) is symmetric). Then consider the tuples $(y_1, \dots, y_m, y_m)$ and $(y_1, \dots, y_{m+1}, y_{m+1})$. By the inductive hypothesis, f maps each of these (when reordered and padded appropriately with duplicates) to z. Furthermore, we have co-ordinatewise edges from $(y_1, \dots, y_m, y_m)$ to $(y_1, \dots, y_m, y_{m+1})$ and from $(y_1, \dots, y_m, y_{m+1})$ to $(y_1, \dots, y_{m+1}, y_{m+1})$. Since we deduce by the definition of polymorphism that both $(z, z'), (z', z) \in E(\mathrm{H})$, it follows that $z' = z$. Thus, f maps also $(y_1, \dots, y_m, y_{m+1})$ (when reordered and padded appropriately with duplicates) to z. That is, $f(x_1, \dots, x_k) = z$. ◀

▶ **Lemma 2.** *Let* H *be the reflexive tournament* $H_1 \Rightarrow \cdots \Rightarrow H_i \Rightarrow \cdots \Rightarrow H_n$*. If* f *is a* k*-ary surjective polymorphism of* H*, then* f *preserves each of* $V(H_1), \ldots, V(H_n)$*; that is, for every* i *and every tuple of* k *vertices* $x_1, \ldots, x_k \in V(H_i)$*,* $f(x_1, \ldots, x_k) \in V(H_i)$*.*

Proof. Suppose f maps some tuple $(x_1, \ldots, x_m)$ from $V(H_i)$ to $y \in V(H_\ell)$. Let $(x'_1, \ldots, x'_m)$ be any tuple from $V(H_i)$. Since H_i is strongly connected, $f(x'_1, \ldots, x'_m)$ in $V(H_\ell)$. It follows that if $\ell \neq i$, e.g. w.l.o.g. $\ell < i$, then some component $\ell' \geq i$ can not be in the range of f. ◀

The relevance of this lemma is in its sequent corollary, which follows according to Proposition 3.15 of [3].

▶ **Corollary 3.** *Let* H *be the reflexive tournament* $H_1 \Rightarrow \cdots \Rightarrow H_i \Rightarrow \cdots \Rightarrow H_n$*. Each subset of the domain* $V(H_i)$ *is definable by a QCSP instance in one free variable.*

An endomorphism e of a digraph G is a *constant map* if there exists a vertex $v \in V(G)$ such that $e(u) = v$ for every $u \in V(G)$, and e is the *identity* if $e(u) = u$ for every $u \in G$. An *automorphism* is a bijective endomorphism whose inverse is a homomorphism. An endomorphism is *trivial* if it is either an automorphism or a constant map; otherwise it is *non-trivial*. A digraph is *endo-trivial* if all of its endomorphisms are trivial. An endomorphism e of a digraph G *fixes* a subset $S \subseteq V(G)$ if $e(S) = S$, that is, $e(x) \in S$ for every $x \in S$, and e fixes an induced subgraph F of G if it is the identity on $V(F)$. It fixes an induced subgraph F *up to automorphism* if $e(F)$ is an automorphic copy of F. An endomorphism e of G is a *retraction* of G if e is the identity on $e(V(G))$. A digraph is *retract-trivial* if all of its retractions are the identity or constant maps. Note that endo-triviality implies retract-triviality, but the reverse implication is not necessarily true (see [15]). However, on reflexive tournaments both concepts do coincide [15].

We need a series of results from [15]. The third one follows from the well-known fact that every strongly connected tournament has a directed Hamilton cycle [6].

▶ **Lemma 4** ([15]). *A reflexive tournament is endo-trivial if and only if it is retract-trivial.*

▶ **Lemma 5** ([15]). *Let* H *be an endo-trivial reflexive digraph with at least three vertices. Then every polymorphism of* H *is essentially unary.*

▶ **Lemma 6** ([15]). *If* H *is an endo-trivial reflexive tournament, then* H *contains a directed Hamilton cycle.*

▶ **Lemma 7** ([15]). *If* H *is an endo-trivial reflexive tournament, then every homomorphic image of* H *of size* $1 < n < |V(H)|$ *has a double edge.*

▶ **Corollary 8.** *If* H *is an endo-trivial reflexive digraph on at least three vertices, then* QCSP(H) *is* NP*-hard (in fact it is even* Pspace*-complete).*

Proof. This follows from Lemma 5 and [3]. ◀

3 The Proof of the NP-Hard Cases of the Dichotomy

We commence with the NP-hard cases of the dichotomy. The simpler NL cases will follow.

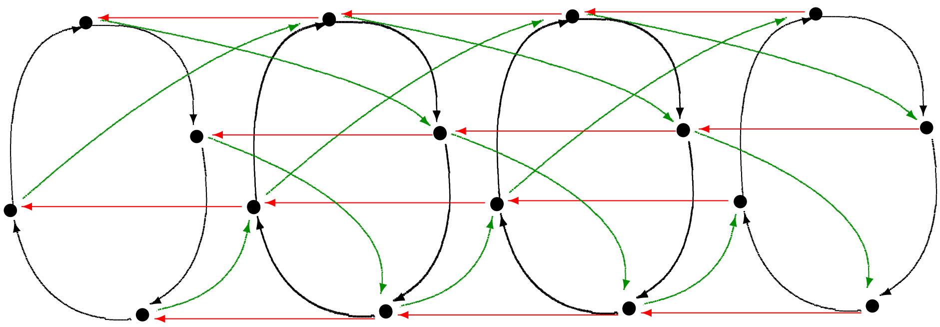

Figure 1 The gadget Cyl^*_m in the case $m := 4$ (self-loops are not drawn). We usually visualise the right-hand copy of DC^*_4 as the "bottom" copy and then we talk about vertices "above" and "below" according to the red arrows.

3.1 The NP-Hardness Gadget

We introduce the gadget Cyl^*_m from [15] drawn in Figure 1. Take m disjoint copies of the (reflexive) directed m-cycle DC^*_m arranged in a cylindrical fashion so that there is an edge from i in the jth copy to i in the $(j+1)$th copy (drawn in red), and an edge from i in the $(j+1)$th copy to $(i+1) \bmod m$ in the jth copy (drawn in green). We consider DC^*_m to have vertices $\{1, \dots, m\}$. Recall that every strongly connected (reflexive) tournament on m vertices has a Hamilton Cycle HC_m. We label the vertices of HC_m as $1, \dots, m$ in order to attach it to the gadget Cyl^*_m.[2]

The following lemma follows from induction on the copies of DC^*_m, since a reflexive tournament has no double edges.

▶ **Lemma 9** ([15]). *In any homomorphism h from Cyl^*_m, with bottom cycle DC^*_m, to a reflexive tournament, if $|h(\mathrm{DC}^*_m)| = 1$, then $|h(\mathrm{Cyl}^*_m)| = 1$.*

We will use another property, denoted (†), of Cyl^*_m, which is that the retractions from Cyl^*_m to its bottom copy of DC^*_m, once propagated through the intermediate copies, induce on the top copy precisely the set of automorphisms of DC^*_m. That is, the top copy of DC^*_m is mapped isomorphically to the bottom copy, and all such isomorphisms may be realised. The reason is that in such a retraction, the $(j+1)$th copy may either map under the identity to the jth copy, or rotate one edge of the cycle clockwise, and Cyl^*_m consists of sufficiently many (namely m) copies of DC^*_m. Now let H be a reflexive tournament that contains a subtournament H_0 on m vertices that is endo-trivial. By Lemma 6, we find that H_0 contains at least one directed Hamilton cycle HC_0. Define $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0])$ as follows. Begin with H and add a copy of the gadget Cyl^*_m, where the bottom copy of DC^*_m is identified with HC_0, to build a digraph $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$. Now ask, for some $y \in V(\mathrm{H})$ whether there is a retraction r of $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$ to H so that some vertex x (not dependent on y) in the top copy of DC^*_m in Cyl^*_m is such that $r(x) = y$. Such vertices y comprise the set $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0])$.

▶ Remark 10. If x belongs to some copy of DC^*_m that is not the top copy, we can find a vertex x' in the top copy of DC^*_m and a retraction r' from $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$ to H with $r'(x') = r(x) = y$, namely by letting r' map the vertices of higher copies of DC^*_m to the image

[2] The superscripted $*$ indicates that the corresponding graph is reflexive. This notation is inherited from [15]. It is not significant since we could safely assume every graph we work with is reflexive as the template is a reflexive tournament.

of their corresponding vertex in the copy that contains x. In particular this implies that $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0])$ contains $V(\mathrm{H}_0)$.

We note that the set $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0])$ is potentially dependent on which Hamilton cycle in H_0 is chosen. We now recall that $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$ if H retracts to H_0.

▶ **Lemma 11** ([15]). *If* H *is a reflexive tournament that retracts to a subtournament* H_0 *with Hamilton cycle* HC_0*, then* $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$.

We now review a variant of a construction from [8]. Let G be a graph containing H where $|V(\mathrm{H})|$ is of size n. Consider all possible functions $\lambda : [n] \to V(\mathrm{H})$ (let us write $\lambda \in V(\mathrm{H})^{[n]}$ of cardinality N). For some such λ, let $\mathcal{G}(\lambda)$ be the graph G enriched with constants $c_1, \ldots, c_n$ where these are interpreted over $V(\mathrm{H})$ according to λ in the natural way (acting on the subscripts). We use calligraphic notation to remind the reader the signature has changed from $\{E\}$ to $\{E, c_1, \ldots, c_n\}$ but we will still treat these structures as graphs. If we write $\mathrm{G}(\lambda)$ without calligraphic notation we mean we look at only the $\{E\}$-reduct, that is, we drop the constants. Of course, $\mathrm{G}(\lambda)$ will always be G.

Let $\mathcal{G} = \bigotimes_{\lambda \in V(\mathrm{H})^{[n]}} \mathcal{G}(\lambda)$. That is, the vertices of $\mathcal{G}$ are N-tuples over $V(\mathrm{G})$ and there is an edge between two such vertices $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N)$ if and only if $(x_1, y_1), \ldots, (x_N, y_N) \in E(\mathrm{G})$. Finally, the constants c_i are interpreted as $(x_1, \ldots, x_N)$ so that $\lambda_1(c_i) = x_1, \ldots, \lambda_N(c_i) = x_N$. An important induced substructure of $\mathcal{G}$ is $\{(x, \ldots, x) : x \in V(\mathrm{G})\}$. It is a copy of G called the *diagonal* copy and will play an important role in the sequel. To comprehend better the construction of $\mathcal{G}$ from the sundry $\mathcal{G}(\lambda)$, confer on Figure 2.

The final ingredient of our fundamental construction involves taking some structure $\mathcal{G}$ and making its canonical query with all vertices other than those corresponding to $c_1, \ldots, c_n$ becoming existentially quantified variables (as usual in this construction). We then turn the $c_1, \ldots, c_n$ to variables $y_1, \ldots, y_n$ to make $\varphi_{\mathcal{G}}(y_1, \ldots, y_n)$. Let $\mathcal{H}$ come from the given construction in which $G = H$. It is proved in [8] that $\mathrm{H}' \models \forall y_1, \ldots, y_n\ \varphi_{\mathcal{H}}(y_1, \ldots, y_n)$ if and only if $\mathrm{QCSP(H)} \subseteq \mathrm{QCSP(H')}$ (here we identify QCSP(H) with the set of sentences that form its yes-instances). By way of a side note, let us consider a k-ary relation R over H with tuples $(x^1_1, \ldots, x^1_k)$, …, $(x^r_1, \ldots, x^r_k)$. For $i \in [r]$, let λ_i map $(c_1, \ldots, c_k)$ to $(x^i_1, \ldots, x^i_k)$. Let $\mathcal{H} = \bigotimes_{\lambda \in \{\lambda_1, \ldots, \lambda_r\}} \mathcal{H}(\lambda)$. Then $\varphi_{\mathcal{H}}(y_1, \ldots, y_n)$ is the closure of R under the polymorphisms of H.

3.2 The strongly connected case: Two Base Cases

Recall that if H is a (reflexive) endo-trivial tournament, then QCSP(H) is NP-hard due to Lemma 5 combined with the results from [3] (indeed, we may even say Pspace-complete). However H may not be endo-trivial. We will now show how to deal with the case where H is not endo-trivial but retracts to an endo-trivial subtournament. For doing this we use the NP-hardness gadget, but we need to distinguish between two different cases.

▶ **Lemma 12** (Base Case I.). *Let* H *be a reflexive tournament that retracts to an endo-trivial subtournament* H_0 *with Hamilton cycle* HC_0*. Assume that* H *retracts to* H'_0 *for every isomorphic copy* $\mathrm{H}'_0 = i(\mathrm{H}_0)$ *of* H_0 *in* H *with* $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}'_0, i(\mathrm{HC}_0)]) = V(\mathrm{H})$*. Then* H_0-Retraction *can be polynomially reduced to* QCSP(H).

Proof. Let m be the size of $|V(\mathrm{H}_0)|$ and n be the size of $|V(\mathrm{H})|$. Let G be an instance of H_0-Retraction. We build an instance φ of QCSP(H) in the following fashion. First, take a copy of H together with G and build G' by identifying these on the copy of H_0 that they

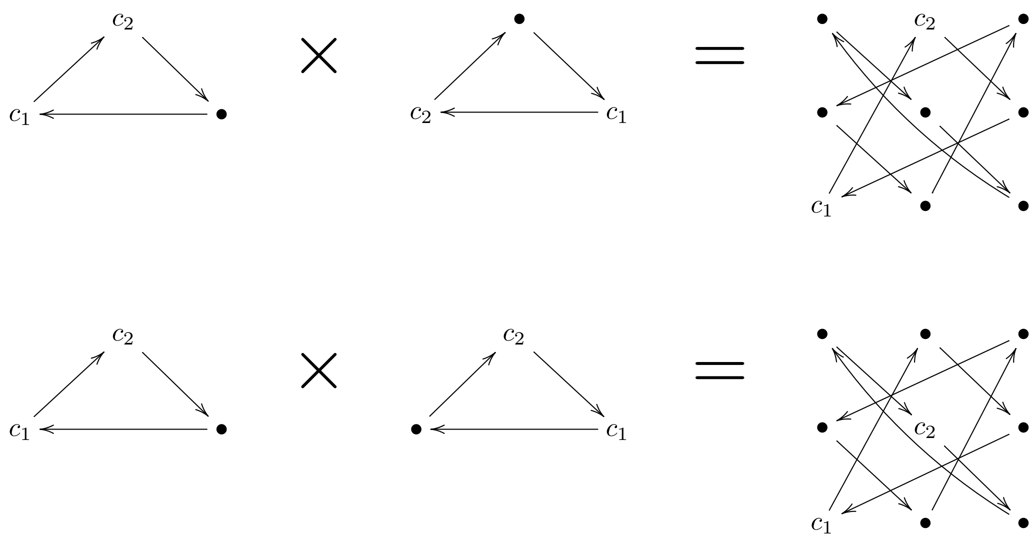

Figure 2 Illustrations of direct product with constants.

both possess as an induced subgraph. Now, consider all possible functions $\lambda : [n] \to V(\mathrm{H})$. For some such λ, let $\mathcal{G}'(\lambda)$ be the graph enriched with constants $c_1, \dots, c_n$ where these are interpreted over some subset of $V(\mathrm{H})$ according to λ in the natural way (acting on the subscripts).

Let $\mathcal{G}' = \bigotimes_{\lambda \in V(\mathrm{H})^{[n]}} \mathcal{G}'(\lambda)$. Let G'^d, H^d and H_0^d be the diagonal copies of G', H and H_0 in $\mathcal{G}'$. Let $\mathcal{H}$ be the subgraph of $\mathcal{G}'$ induced by $V(\mathrm{H}) \times \cdots \times V(\mathrm{H})$. Note that the constants $c_1, \dots, c_n$ live in $\mathcal{H}$. Now build $\mathcal{G}''$ from $\mathcal{G}'$ by augmenting a new copy of Cyl_m^* for every vertex $v \in V(\mathcal{H}) \setminus V(\mathrm{H}_0^d)$. Vertex v is to be identified with any vertex in the top copy of DC_m^* in Cyl_m^* and the bottom copy of DC_m^* is to be identified with HC_0 in H_0^d according to the identity function. (Thus, in each case, the new vertices are the middle cycles of Cyl_m^* and all but one of the vertices in the top cycle of Cyl_m^*.)

Finally, build φ from the canonical query of $\mathcal{G}''$ where we additionally turn the constants $c_1, \dots, c_n$ to outermost universal variables. The size of φ is doubly exponential in n (the size of H) but this is constant, so still polynomial in the size of G.

We claim that G retracts to H_0 if and only if $\varphi \in \mathrm{QCSP}(\mathrm{H})$.

First suppose that G retracts to H_0. Let λ be some assignment of the universal variables of φ to H. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H})$ it suffices to prove that there is a homomorphism from $\mathcal{G}''$ to H that extends λ. Then for this it suffices to prove that there is a homomorphism h from $\mathcal{G}'$ that extends λ. Let us explain why. Because H retracts to H_0, we have $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$ due to Lemma 11. Hence, if $h(x) = y$ for two vertices $x \in V(\mathcal{H}) \setminus V(\mathrm{H}_0^d)$ and $y \in V(\mathrm{H})$, we can always find a retraction of the graph $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$ to H that maps x to y, and we mimic this retraction on the corresponding subgraph in $\mathcal{G}''$. The crucial observation is that this can be done independently for each vertex in $V(\mathcal{H}) \setminus V(\mathrm{H}_0^d)$, as two vertices of different copies of Cyl_m^* are only adjacent if they both belong to $\mathcal{H}$.

Henceforth let us consider the homomorphic image of $\mathcal{G}'$ that is $\mathcal{G}'(\lambda)$. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H})$ it suffices to prove that there is a homomorphism from $\mathrm{G}'(\lambda)$ to H that extends λ.

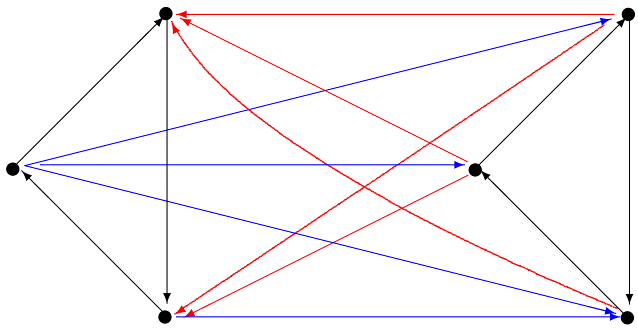

Figure 3 An interesting tournament H on six vertices (self-loops are not drawn). This tournament does not retract to the DC_3^* on the left-hand side, yet $\mathrm{Spill}_3(\mathrm{H}[\mathrm{DC}_3^*, \mathrm{DC}_3]) = V(\mathrm{H})$.

Note that it will be sufficient to prove that G' retracts to H. Let h be the natural retraction from G' to H that extends the known retraction from G to H_0. We are done.

Suppose now $\varphi \in \mathrm{QCSP}(\mathrm{H})$. Choose some surjection for λ, the assignment of the universal variables of φ to H. Recall $N = |V(\mathrm{H})^{[n]}|$. The evaluation of the existential variables that witness $\varphi \in \mathrm{QCSP}(\mathrm{H})$ induces a surjective homomorphism s from $\mathcal{G}''$ to H which contains within it a surjective homomorphism s' from $\mathcal{H} = \mathrm{H}^N$ to H. Consider the diagonal copy of $\mathrm{H}_0^d \subset \mathrm{H}^d \subset \mathrm{G}'^d$ in $\mathcal{G}'$. By abuse of notation we will also consider each of s and s' acting just on the diagonal. If $|s'(\mathrm{H}_0^d)| = 1$, by construction of $\mathcal{G}''$, we have $|s'(\mathrm{H}^d)| = 1$. Indeed, this was the property we noted in Lemma 9. By Lemma 1, this would mean s' is uniformly mapping $\mathcal{H}$ to one vertex, which is impossible as s' is surjective. Now we will work exclusively in the diagonal copy G'^d. As $1 < |s'(\mathrm{H}_0^d)| < m$ is not possible either due to Lemma 7, we find that $|s'(\mathrm{H}_0^d)| = m$, and indeed s' maps H_0^d to a copy of itself in H which we will call $\mathrm{H}_0' = i(\mathrm{H}_0^d)$ for some isomorphism i.

We claim that $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0^d)]) = V(\mathrm{H})$. In order to see this, consider a vertex $y \in V(\mathrm{H})$. As s' is surjective, there exists a vertex $x \in V(\mathcal{H})$ with $s'(x) = y$. By construction, x belongs to some top copy of DC_m^* in Cyl_m^* in $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$. We can extend i^{-1} to an isomorphism from the copy of Cyl_m^* (which has $i(\mathrm{HC}_0^d)$ as its bottom cycle) in the graph $\mathrm{F}(\mathrm{H}_0', i(\mathrm{HC}_0^d))$ to the copy of Cyl_m^* (which has HC_0^d as its bottom cycle) in the graph $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$. We define a mapping r^* from $\mathrm{F}(\mathrm{H}_0', i(\mathrm{HC}_0^d))$ to H by $r^*(u) = s' \circ i^{-1}(u)$ if u is on the copy of Cyl_m^* in $\mathrm{F}(\mathrm{H}_0', i(\mathrm{HC}_0^d))$ and $r^*(u) = u$ otherwise. We observe that $r^*(u) = u$ if $u \in V(\mathrm{H}_0')$ as s' coincides with i on H_0. As H_0^d separates the other vertices of the copy of Cyl_m^* from $V(\mathrm{H}^d) \setminus V(\mathrm{H}_0^d)$, in the sense that removing H_0^d would disconnect them, this means that r^* is a retraction from $\mathrm{F}(\mathrm{H}_0', i(\mathrm{HC}_0^d))$ to H. We find that r^* maps $i(x)$ to $s' \circ i^{-1}(i(x)) = s'(x) = y$. Moreover, as x is in the top copy of DC_m^* in $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$, we conclude that y always belongs to $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0^d)])$.

As $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0^d)]) = V(\mathrm{H})$, we find, by assumption of the lemma, that there exists a retraction r from H to H_0'. Now, recalling that we can view s' acting just on the diagonal copy H^d of H, $i^{-1} \circ r \circ s'$ is the desired retraction of G to H_0. ◀

We now need to deal with the situation in which we have an isomorphic copy $\mathrm{H}_0' = i(\mathrm{H}_0)$ of H_0 in H with $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0)]) = V(\mathrm{H})$, such that H does not retract to H_0' (see Figure 3 for an example). We cannot deal with this case in a direct manner and first show another base case. For this we need the following lemma and an extension of endo-triviality that we discuss afterwards.

▶ **Lemma 13** ([15]). *Let* H *be a reflexive tournament, containing a subtournament* H_0 *so that* *any endomorphism of* H *that fixes* H_0 *as a graph is an automorphism. Then any endomorphism of* H *that maps* H_0 *to an isomorphic copy* $\mathrm{H}_0' = i(\mathrm{H}_0)$ *of itself is an automorphism of* H.

Let H_0 be an induced subgraph of a digraph H. We say that the pair $(\mathrm{H}, \mathrm{H}_0)$ is *endo-trivial* if all endomorphisms of H that fix H_0 are automorphisms.

▶ **Lemma 14** (Base Case II)**.** *Let* H *be a reflexive tournament with a subtournament* H_0 *with Hamilton cycle* HC_0 *so that* $(\mathrm{H}, \mathrm{H}_0)$ *and* H_0 *are endo-trivial and* $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$. *Then* H-Retraction *can be polynomially reduced to* QCSP(H).

Proof. Let G be an instance of H-Retraction. Let m be the size of $|V(\mathrm{H}_0)|$ and n be the size of $|V(\mathrm{H})|$. We build an instance φ of QCSP(H) in the following fashion. Consider all possible functions $\lambda : [n] \to V(\mathrm{H})$. For some such λ, let $\mathcal{G}(\lambda)$ be the graph enriched with constants $c_1, \ldots, c_n$ where these are interpreted over some subset of $V(\mathrm{H})$ according to λ in the natural way (acting on the subscripts).

Let $\mathcal{G} = \bigotimes_{\lambda \in V(\mathrm{H})^{[n]}} \mathcal{G}(\lambda)$. Let G^d, H^d and H_0^d be the diagonal copies of G, H and H_0 in $\mathcal{G}$. Let $\mathcal{H}$ be the subgraph of $\mathcal{G}$ induced by $V(\mathrm{H}) \times \cdots \times V(\mathrm{H})$. Note that the constants $c_1, \ldots, c_n$ live in $\mathcal{H}$. Now build $\mathcal{G}'$ from $\mathcal{G}$ by augmenting a new copy of Cyl_m^* for every vertex $v \in V(\mathcal{H}) \setminus V(\mathrm{H}_0^d)$. Vertex v is to be identified with any vertex in the top copy of DC_m^* in Cyl_m^* and the bottom copy of DC_m^* is to be identified with HC_0 in H_0^d according to the identity function.

Finally, build φ from the canonical query of $\mathcal{G}'$ where we additionally turn the constants $c_1, \ldots, c_n$ to outermost universal variables.

First suppose that G retracts to H by r. Let λ be some assignment of the universal variables of φ to H. To prove $\varphi \in$ QCSP(H) it suffices to prove that there is a homomorphism from $\mathcal{G}'$ to H that extends λ and for this it suffices to prove that there is a homomorphism from $\mathcal{G}$ that extends λ. This is always possible since we have $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$ by assumption.

Henceforth let us consider the homomorphic image of $\mathcal{G}$ that is $\mathcal{G}(\lambda)$. To prove $\varphi \in$ QCSP(H) it suffices to prove that there is a homomorphism from $\mathrm{G}(\lambda)$ to H that extends λ. Note that it will be sufficient to prove that G retracts to H. Well this was our original assumption so we are done.

Suppose now $\varphi \in$ QCSP(H). Choose some surjection for λ, the assignment of the universal variables of φ to H. Recall $N = |V(\mathrm{H})^{[n]}|$. The evaluation of the existential variables that witness $\varphi \in$ QCSP(H) induces a surjective homomorphism s from $\mathcal{G}'$ to H which contains within it a surjective homomorphism s' from $\mathcal{H} = \mathrm{H}^N$ to H. Consider the diagonal copy of $\mathrm{H}_0^d \subset \mathrm{H}^d \subset \mathrm{G}^d$ in $(\mathrm{G})^N$. By abuse of notation we will also consider each of s and s' acting just on the diagonal. If $|s'(\mathrm{H}_0^d)| = 1$, by construction of $\mathcal{G}'$, we have $|s'(\mathrm{H}^d)| = 1$. By Lemma 1, this would mean s' is uniformly mapping $\mathcal{H}$ to one vertex, which is impossible as s' is surjective. Now we will work exclusively on the diagonal copy G^d. As $1 < |s'(\mathrm{H}_0^d)| < m$ is not possible either due to Lemma 7, we find that $|s'(\mathrm{H}_0^d)| = m$, and indeed s' maps H_0^d to a copy of itself in H which we will call $\mathrm{H}_0' = i(\mathrm{H}_0^d)$ for some isomorphism i.

As $(\mathrm{H}, \mathrm{H}_0)$ is endo-trivial, Lemma 13 tells us that the restriction of s' to H^d is an automorphism of H^d, which we call α. The required retraction from G to H is now given by $\alpha^{-1} \circ s'$. ◀

3.3 The strongly connected case: Generalising the Base Cases

We now generalise the two base cases to more general cases via some recursive procedure. Afterwards we will show how to combine these two cases to complete our proof. We will first need a slightly generalised version of Lemma 13, which nonetheless has virtually the same proof.

▶ **Lemma 15** ([15]). *Let* $\mathrm{H}_2 \supset \mathrm{H}_1 \supset H_0$ *be a sequence of strongly connected reflexive tournaments, each one a subtournament of the one before. Suppose that any endomorphism of* H_1 *that fixes* H_0 *is an automorphism. Then any endomorphism* h *of* H_2 *that maps* H_0 *to an isomorphic copy* $\mathrm{H}_0' = i(\mathrm{H}_0)$ *of itself also gives an isomorphic copy of* H_1 *in* $h(\mathrm{H}_1)$*.*

The following two lemmas generalise Lemmas 12 and 14. The proof of the second is omitted.

▶ **Lemma 16** (General Case I). *Let* $\mathrm{H}_0, \mathrm{H}_1, \ldots, \mathrm{H}_k, \mathrm{H}_{k+1}$ *be reflexive tournaments, the first* k *of which have Hamilton cycles* $\mathrm{HC}_0, \mathrm{HC}_1, \ldots, \mathrm{HC}_k$*, respectively, so that* $\mathrm{H}_0 \subseteq H_1 \subseteq \cdots \subseteq \mathrm{H}_k \subseteq \mathrm{H}_{k+1}$*. Assume that* H_0*,* $(\mathrm{H}_1, \mathrm{H}_0)$*, ...,* $(\mathrm{H}_k, \mathrm{H}_{k-1})$ *are endo-trivial and that*

$$\begin{array}{lcl} \mathrm{Spill}_{a_0}(\mathrm{H}_1[\mathrm{H}_0, \mathrm{HC}_0]) & = & V(\mathrm{H}_1) \\ \mathrm{Spill}_{a_1}(\mathrm{H}_2[\mathrm{H}_1, \mathrm{HC}_1]) & = & V(\mathrm{H}_2) \\ \vdots & \vdots & \vdots \\ \mathrm{Spill}_{a_{k-1}}(\mathrm{H}_k[\mathrm{H}_{k-1}, \mathrm{HC}_{k-1}]) & = & V(\mathrm{H}_k). \end{array}$$

Moreover, assume that H_{k+1} *retracts to* H_k *and also to every isomorphic copy* $\mathrm{H}_k' = i(\mathrm{H}_k)$ *of* H_k *in* H_{k+1} *with* $\mathrm{Spill}_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}_k', i(\mathrm{HC}_k)]) = V(\mathrm{H}_{k+1})$*. Then* H_k*-*Retraction *can be polynomially reduced to* $\mathrm{QCSP}(\mathrm{H}_{k+1})$*.*

Proof. Let $a_{k+1}, \ldots, a_0$ be the cardinalities of $|V(\mathrm{H}_{k+1})|, \ldots, |V(\mathrm{H}_0|)$, respectively. Let $n = a_{k+1}$. Let G be an instance of H_k-Retraction. We will build an instance φ of $\mathrm{QCSP}(\mathrm{H}_{k+1})$ in the following fashion. First, take a copy of H_{k+1} together with G and build G' by identifying these on the copy of H_k that they both possess as an induced subgraph.

Consider all possible functions $\lambda : [n] \to V(\mathrm{H}_{k+1})$. For some such λ, let $\mathcal{G}'(\lambda)$ be the graph enriched with constants $c_1, \ldots, c_n$ where these are interpreted over some subset of $V(\mathrm{H}_{k+1})$ according to λ in the natural way (acting on the subscripts).

Let $\mathcal{G}' = \bigotimes_{\lambda \in V(\mathrm{H}_{k+1})^{[n]}} \mathcal{G}'(\lambda)$. Let G'^d, H^d_{k+1} and H^d_k etc. be the diagonal copies of G'^d, H_{k+1} and H_k in $\mathcal{G}'$. Let $\mathcal{H}_{k+1}$ be the subgraph of $\mathcal{G}'$ induced by $V(\mathrm{H}_{k+1}) \times \cdots \times V(\mathrm{H}_{k+1})$. Note that the constants $c_1, \ldots, c_n$ live in $\mathcal{H}_{k+1}$. Now build $\mathcal{G}''$ from $\mathcal{G}'$ by augmenting a new copy of $\mathrm{Cyl}^*_{a_k}$ for every vertex $v \in V(\mathcal{H}_{k+1}) \setminus V(\mathrm{H}^d_k)$. Vertex v is to be identified with any vertex in the top copy of DC_{a_k} in $\mathrm{Cyl}^*_{a_k}$ and the bottom copy of DC_{a_k} is to be identified with HC_k in H^d_k according to the identity function.

Then, for each $i \in [k]$, and $v \in V(\mathrm{H}^d_i) \setminus V(\mathrm{H}^d_{i-1})$, add a copy of $\mathrm{Cyl}^*_{a_{i-1}}$, where v is identified with any vertex in the top copy of $\mathrm{DC}^*_{a_{i-1}}$ in $\mathrm{Cyl}^*_{a_{i-1}}$ and the bottom copy of DC^*_{i-1} is to be identified with H_{i-1} according to the identity map of $\mathrm{DC}^*_{a_{i-1}}$ to HC_{i-1}.

Finally, build φ from the canonical query of $\mathcal{G}''$ where we additionally turn the constants $c_1, \ldots, c_n$ to outermost universal variables.

First suppose that G retracts to H_k. Let λ be some assignment of the universal variables of φ to H_{k+1}. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$ it suffices to prove that there is a homomorphism from $\mathcal{G}''$ to H_{k+1} that extends λ and for this it suffices to prove that there is a homomorphism from $\mathcal{G}'$ that extends λ. Let us explain why. We map the various copies of $\mathrm{Cyl}^*_{a_{i-1}}$ in G'' in any suitable fashion, which will always exist due to our assumptions and the fact that $\mathrm{Spill}_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}_k, \mathrm{HC}_k]) = V(\mathrm{H}_{k+1})$, which follows from our assumption that H_{k+1} retracts to H_k and Lemma 11.

Henceforth let us consider the homomorphic image of $\mathcal{G}'$ that is $\mathcal{G}'(\lambda)$. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$ it suffices to prove that there is a homomorphism from $\mathrm{G}'(\lambda)$ to H_{k+1} that extends λ. Note that it will be sufficient to prove that G' retracts to H_{k+1}. Let h be the natural retraction from G' to H_{k+1} that extends the known retraction from G to H_k. We are done.

Suppose now $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$. Choose some surjection for λ, the assignment of the universal variables of φ to H_{k+1}. Let $N = |V(\mathrm{H}_{k+1})^{[n]}|$. The evaluation of the existential variables that witness $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$ induces a surjective homomorphism s from $\mathcal{G}'$ to H_{k+1} which contains within it a surjective homomorphism s' from $\mathcal{H} = \mathrm{H}_{k+1}^N$ to H_{k+1}. Consider the diagonal copy of $\mathrm{H}_0^d \subset \cdots \subset \mathrm{H}_k^d \subset \mathrm{H}_{k+1}^d \subset G'^d$ in $\mathcal{G}'$. By abuse of notation we will also consider each of s and s' acting just on the diagonal. If $|s'(\mathrm{H}_0^d)| = 1$, by construction of $\mathcal{G}''$, we could follow the chain of spills to deduce that $|s'(\mathrm{H}_{k+1}^d)| = 1$, which is not possible by Lemma 1. Moreover, $1 < |s'(H_0^d)| < |V(H_0^d)|$ is impossible due to Lemma 7. Now we will work exclusively on the diagonal copy G'^d.

Thus, $|s'(\mathrm{H}_0^d)| = |V(\mathrm{H}_0^d)|$ and indeed s' maps H_0^d to an isomorphic copy of itself in H_{k+1} which we will call $\mathrm{H}_0' = i(\mathrm{H}_0^d)$. We now apply Lemma 15 as well as our assumed endo-trivialities to derive that s' in fact maps H_k^d by the isomorphism i to a copy of itself in H_{k+1} which we will call H_k'. Since s' is surjective, we can deduce that $\mathrm{Spill}_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}_k', i(\mathrm{HC}_k^d)]) = V(\mathrm{H}_{k+1})$ in the same way as in the proof of Lemma 12. and so there exists a retraction r from H_{k+1} to H_k'. Now $i^{-1} \circ r \circ s'$ gives the desired retraction of G to H_k. ◀

▶ **Lemma 17** (General Case II). *Let* $\mathrm{H}_0, \mathrm{H}_1, \ldots, \mathrm{H}_k, \mathrm{H}_{k+1}$ *be reflexive tournaments, the first* $k+1$ *of which have Hamilton cycles* $\mathrm{HC}_0, \mathrm{HC}_1, \ldots, \mathrm{HC}_k$*, respectively, so that* $\mathrm{H}_0 \subseteq H_1 \subseteq \cdots \subseteq \mathrm{H}_k \subseteq \mathrm{H}_{k+1}$. *Suppose that* H_0, $(\mathrm{H}_1, \mathrm{H}_0)$, $\ldots$, $(\mathrm{H}_k, \mathrm{H}_{k-1}), (\mathrm{H}_{k+1}, \mathrm{H}_k)$ *are endo-trivial and that*

$$\begin{array}{lcl} \mathrm{Spill}_{a_0}(\mathrm{H}_1[\mathrm{H}_0, \mathrm{HC}_0]) & = & V(\mathrm{H}_1) \\ \mathrm{Spill}_{a_1}(\mathrm{H}_2[\mathrm{H}_1, \mathrm{HC}_1]) & = & V(\mathrm{H}_2) \\ \vdots & \vdots & \vdots \\ \mathrm{Spill}_{a_{k-1}}(\mathrm{H}_k[\mathrm{H}_{k-1}, \mathrm{HC}_{k-1}]) & = & V(\mathrm{H}_k) \\ \mathrm{Spill}_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}_k, \mathrm{HC}_k]) & = & V(\mathrm{H}_{k+1}) \end{array}$$

Then H_{k+1}-Retraction *can be polynomially reduced to* $\mathrm{QCSP}(\mathrm{H}_{k+1})$.

▶ **Corollary 18.** *Let* H *be a non-trivial strongly connected reflexive tournament. Then* QCSP(H) *is* NP*-hard.*

Proof. As H is a strongly connected reflexive tournament, which has more than one vertex by our assumption, H is not transitive. Note that H-Retraction is NP-complete (see Section 4.5 in [15], using results from [14, 5, 16]). Thus, if H is endo-trivial, the result follows from Lemma 12 (note that we could also have used Corollary 8).

Suppose H is not endo-trivial. Then, by Lemma 4, H is not retract-trivial either. This means that H has a non-trivial retraction to some subtournament H_0. We may assume that H_0 is endo-trivial, as otherwise we will repeat the argument until we find a retraction from H to an endo-trivial (and consequently strongly connected) subtournament.

Suppose that H retracts to all isomorphic copies $\mathrm{H}_0' = i(\mathrm{H}_0)$ of H_0 within it, except possibly those for which $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0)]) \neq V(\mathrm{H})$. Then the result follows from Lemma 12. So there is a copy $\mathrm{H}_0' = i(\mathrm{H}_0)$ to which H does not retract for which $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0)]) = V(\mathrm{H})$. If $(\mathrm{H}, \mathrm{H}_0')$ is endo-trivial, the result follows from Lemma 14. Thus we assume $(\mathrm{H}, \mathrm{H}_0')$ is not endo-trivial and we deduce the existence of $\mathrm{H}_0' \subset \mathrm{H}_1 \subset \mathrm{H}$ (H_1 is strictly between H and H_0') so that $(\mathrm{H}_1, \mathrm{H}_0')$ and H_0' are endo-trivial and H retracts to H_1. Now we are ready to break out. Either H retracts to all isomorphic copies of $\mathrm{H}_1' = i(\mathrm{H}_1)$ in H, except possibly for those so that $\text{Spill}_m(\text{H}[\text{H}'_1, i(\text{HC}_1)]) \neq V(\text{H})$, and we apply Lemma 16, or there exists a copy H'_1, with $\text{Spill}_m(\text{H}[\text{H}'_1, i(\text{HC}_1)]) = V(\text{H})$, to which it does not retract. If (H, H'_1) is endo-trivial, the result follows from Lemma 17. Otherwise we iterate the method, which will terminate because our structures are getting strictly larger. ◀

3.4 An initial strongly connected component that is non-trivial

This section follows a similar methodology to the previous two sections. However, the proofs are a little more involved and are omitted from this version of the paper.

▶ **Corollary 19.** *Let* H *be a reflexive tournament with an initial strongly connected component that is non-trivial. Then* QCSP(H) *is* NP*-hard.*

4 The Proof of the NL Cases of the Dichotomy

A particular role in the tractable part of our dichotomy will be played by TT_2^*, the reflexive transitive 2-tournament, which has vertex set $\{0, 1\}$ and edge set $\{(0, 0), (0, 1), (1, 1)\}$.

▶ **Lemma 20.** *Let* $\text{H} = \text{H}_1 \Rightarrow \cdots \Rightarrow \text{H}_n$ *be a reflexive tournament on* $m + 2$ *vertices with* $V(\text{H}_1) = \{s\}$ *and* $V(\text{H}_n) = \{t\}$*. Then there exists a surjective homomorphism from* $(\text{TT}_2^*)^m$ *to* H*.*

Proof. Build a surjective homomorphism f from $(\text{TT}_2^*)^m$ to H in the following fashion. Let $\overline{x}_i$ be the m-tuple which has 1 in the ith position and 0 in all other positions. For $i \in [m]$, let f map $\overline{x}_i$ to i. Let f map $(0, \ldots, 0)$ to s and everything remaining to t.

By construction, f is surjective. To see that f is a homomorphism, let $((y_1, \ldots, y_m), (z_1, \ldots, z_m)) \in E((\text{TT}_2^*)^m)$, which is the case exactly when $y_i \leq z_i$ for all $i \in [m]$. Let $f(y_1, \ldots, y_m) = u$ and $f(z_1, \ldots, z_m) = v$. First suppose that $y_1, \ldots, y_m$ are all 0. Then $u = s$. As s has an out-edge to every vertex of H, we find that $(u, v) \in E(\text{H})$. Now suppose that $y_1, \ldots, y_m$ contains a single 1. If $(y_1, \ldots, y_m) = (z_1, \ldots, z_m)$, then $u = v$. As H is reflexive, we find that $(u, v) \in \text{H}$. If $(y_1, \ldots, y_m) \neq (z_1, \ldots, z_m)$, then $v = t$. As t has an in-edge from every vertex of H, we find that $(u, v) \in E(\text{H})$. Finally suppose that $y_1, \ldots, y_m$ contains more than one 1. Then $u = v = t$. As H is reflexive, we find that $(u, v) \in E(\text{H})$. ◀

We also need the following lemma, which follows from combining some known results.

▶ **Lemma 21.** *If* H *is a transitive reflexive tournament then* QCSP(H) *is in* NL*.*

Proof. It is noted in [15] that H has the ternary median operation as a polymorphism. It follows from well-known results (e.g. in [7, 9]) that QCSP(H) is in NL. ◀

The other tractable cases are more interesting.

We are now ready to prove the main result of this section.

▶ **Theorem 22.** *Let* $\text{H} = \text{H}_1 \Rightarrow \cdots \Rightarrow \text{H}_n$ *be a reflexive tournament. If* $|V(H_1)| = |V(H_n)| = 1$*, then* QCSP(H) *is in* NL*.*

Proof. Let $|V(\text{H})| = m + 2$ for some $m \geq 0$. By Lemma 20, there exists a surjective homomorphism from $(\text{TT}_2^*)^m$ to H. There exists also a surjective homomorphism from H to TT_2^*; we map s to 0 and all other vertices of H to 1. It follows from [8] that $\text{QCSP(H)} = \text{QCSP}(\text{TT}_2^*)$ meaning we may consider the latter problem. We note that TT_2^* is a transitive reflexive tournament. Hence, we may appply Lemma 21. ◀

5 Final result and remarks

We are now in a position to prove our main dichotomy theorem.

▶ **Theorem 23.** *Let* $\mathrm{H} = \mathrm{H}_1 \Rightarrow \cdots \Rightarrow \mathrm{H}_n$ *be a reflexive tournament. If* $|V(\mathrm{H}_1)| = |V(\mathrm{H}_n)| = 1$*, then* QCSP(H) *is in* NL*; otherwise it is* NP*-hard.*

Proof. The NL case follow from Theorem 22. The NP-hard cases follow from Corollary 18 and Corollary 19, bearing in mind the case with a non-trivial final strongly connected component is dual to the case with a non-trivial initial strongly connected component (map edges (x, y) to (y, x)). ◀

Theorem 23 resolved the open case in Table 1. Recall that the results for the irreflexive tournaments in this table were all proven in a more general setting, namely for irreflexive semicomplete graphs. A natural direction for future research is to determine a complexity dichotomy for QCSP and SCSP for reflexive semicomplete graphs. We leave this as an interesting open direction.

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