Sparse grid approximation spaces for space-time boundary integral formulations of the heat equation

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Discretization by piecewise polynomials is a well-established and well-understood approach for the numerical solution of partial differential equations. For time-dependent problems, independent piecewise polynomial approximations can be used in space and time. Given stability of the joint space-time approximation, the accuracy of the method can be expressed in terms of the discretization parameters. It is clear, however, that the space and time discretizations must be balanced for an efficient numerical simulation, since the underrefined discretization space will dictate the accuracy, whereas the overrefined space will determine the overall computational cost.

In this note we address optimal balancing of several piecewise polynomial discretization spaces for the first kind space-time boundary integral formulations for the homogeneous heat equation with Dirichlet boundary conditions. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain with a smooth boundary $\Gamma := \partial\Omega$ and $\mathcal{I} := [0, T]$ be the time interval of interest. After reduction to the mantle of the space-time cyllinder $\Sigma := \Gamma \times \mathcal{I}$, cf. [6, 3], the problem is rephrased as the boundary integral equation

$$V\psi(x,t) := \int_0^T \int_\Gamma G(x-y, t-s)\, dyds = f(x,t), \quad x \in \Gamma, t \in \mathcal{I}, \tag{1}$$

where ψ is the unknown flux, f is the known data (depending on the Dirichlet data) and G is the fundamental solution of the heat equation

$$G(x,t) = \begin{cases} (4\pi t)^{-d/2} e^{-|x|^2/4t}, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{2}$$

We write $H^{r,s}(\Sigma) := L^2(\mathcal{I}, H^r(\Gamma)) \cap H^s(\mathcal{I}, L^2(\Gamma))$ for $r, s \geq 0$, equipped with the graph norm, and $H^{-r,-s}(\Sigma) := H^{r,s}(\Sigma)'$ for its dual. The single layer operator $V : H^{-1/2,-1/4}(\Sigma) \to H^{1/2,1/4}(\Sigma)$ is an isomorphism and satisfies the following coercivity estimate [1]

$$\exists c_V > 0 : \quad \langle Vq, q\rangle \geq c_V \|q\|^2_{H^{-1/2,-1/4}(\Sigma)}, \quad \forall q \in H^{-1/2,-1/4}(\Sigma). \tag{3}$$

This remarkable property being typical for *elliptic* operators immediately implies that any conforming discretization $\mathcal{X}_L \subset H^{-1/2,-1/4}(\Sigma)$ of (1) is stable and that the discrete solution $\psi_L \in \mathcal{X}_L$ is quasi-optimal, i.e.

$$\|\psi - \psi_L\|_{H^{-1/2,-1/4}(\Sigma)} \leq \frac{||V||}{c_V} \inf_{\eta_L \in \mathcal{X}_L} \|\psi - \eta_L\|_{H^{-1/2,-1/4}(\Sigma)}. \tag{4}$$

This allows to construct a number of conforming discretization spaces $\mathcal{X}_L$ for the numerical solution of (1). In particular, let the polynomial degrees in the space and time domain p_x and p_t be fixed and consider the following nested sequence of discretizations in space and time on meshes defined by bisection

$$\mathcal{X}_0^x \subset \mathcal{X}_1^x \subset \cdots \subset \mathcal{X}_i^x \subset \cdots \subset H^{-\frac{1}{2}}(\Gamma), \quad \mathcal{X}_0^t \subset \mathcal{X}_1^t \subset \cdots \subset \mathcal{X}_j^t \subset \cdots \subset H^{-\frac{1}{4}}(\mathcal{I})$$

The individual subspaces admit L^2-orthogonal representations

$$\mathcal{X}_i^x = W_0^x \oplus W_1^x \oplus \cdots \oplus W_i^x, \qquad \mathcal{X}_j^t = W_0^t \oplus W_1^t \oplus \cdots \oplus W_j^t,$$

so that for any $\psi \in L^2(\Sigma)$ holds

$$\psi = \sum_{(\ell_x,\ell_t)\in\mathbb{N}_0^2} w_{(\ell_x,\ell_t)}, \qquad w_{(\ell_x,\ell_t)} \in W_{\ell_x}^x \otimes W_{\ell_t}^t. \tag{5}$$

Conforming discretizations can now be derived from (5) by restricting the non-negative quadrant to finite, possibly anisotropic index sets $I_L^\sigma, \hat{J}_L^\sigma \subset \mathbb{N}_0^2$, where σ indicates the anisotropy and thereby the optimal balance between space and time discretizations.

In view of (4) it is natural to take the $H^{-1/2,-1/4}(\Sigma)$-norm as the error measure. As a comparison criterion we take the asymptotic convergence rate γ of the error in this norm with respect to the dimension of the discretization space N_L for smooth solutions ψ:

$$\gamma := \sup\left\{\tilde{\gamma} : \|\psi - \psi_L\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \le c N_L^{-\tilde{\gamma}}, \text{ where } N_L \to \infty\right\}. \tag{6}$$

In the context of particular discretizations considered below, the smoothness requirement may be replaced by $\psi \in H^{\mu,\lambda}(\Sigma)$ or $H^{\mu,\lambda}_{\text{mix}}(\Sigma)$ with $0 \le \mu < p_x + 1$ and $0 \le \lambda < p_t + 1$, cf. [2, Remark 1]. This allows the exclusion of the borderline case $\mu = p_x + 1$, $\lambda = p_t + 1$, where the convergence estimates are usually corrupted by logarithmic terms, and thereby simplifing the argument. Here the space $H^{\mu,\lambda}_{\text{mix}}(\Sigma)$ stands for the hilbertian tensor product $H^r(\Gamma) \otimes H^s(\mathcal{I})$.

1. *Anisotropic full-tensor product* discretizations are a natural choice:

$$I_L^\sigma = \{(\ell_x,\ell_t) \ : \ \ell_x \le L/\sigma, \ell_t \le \sigma L\} \tag{7}$$

   Notice that the error measure is given by the anisotropic norm (4), thus nontrivial values $\sigma \neq 1$ are expected in this case.

2. *Anisotropic sparse-tensor product* discretizations are defined as in [4]

$$\hat{J}_L^\sigma = \{(\ell_x,\ell_t) \ : \ \ell_x\sigma + \ell_t/\sigma \le L\} \tag{8}$$

   This choice is potentially more efficient for smooth solutions, since it excludes the largest orthogonal subspace combinations (implying $\hat{J}_L^\sigma \subset I_L^\sigma$) without compromising the accuracy.

The outcomes of the error analysis are summarized in the table below, cf. [2] for the details. The argument is based on appropriate norm equivalences / bounds [2, (17)–(19)] that can be found in [8, Proposition 3], [5, Proposition 1] and derived along the lines of [7, Proposition 3].

The numerical results in [2, 9] validate our theoretical findings. The interested reader will find there also extensions to adaptive sparse grids and numerical solution by combination technique.

We finally remark that for some values (p_x, p_t) algorithmic accelerations are possible (e.g. when the matrix of the algebraic system is block triangular [9], etc.). Such effects are not considered here.

Full tensor product, $d = 2$

(p_x, p_t)	conv. rate γ	scaling σ^2
$(0,0)$	$\frac{15}{22} \approx 0.68$	$\frac{6}{5}$
$(1,0)$	$\frac{5}{6} \approx 0.83$	2
$(1,1)$	$\frac{45}{38} \approx 1.18$	$\frac{10}{9}$
$(3,1)$	$\frac{3}{2} = 1.50$	2

Sparse grids, $d = 2$

(p_x, p_t)	conv. rate γ	scaling σ^2
$(0,0)$	$\frac{7}{6} \approx 1.17$	1
$(1,0)$	$\frac{5}{4} = 1.25$	1
$(1,1)$	$\frac{13}{6} \approx 2.17$	1
$(3,1)$	$\frac{9}{4} = 2.25$	1

Full tensor product, $d = 3$

(p_x, p_t)	conv. rate γ	scaling σ^2
$(0,0)$	$\frac{15}{32} \approx 0.47$	$\frac{6}{5}$
$(1,0)$	$\frac{5}{8} \approx 0.63$	2
$(1,1)$	$\frac{45}{56} \approx 0.80$	$\frac{10}{9}$
$(3,1)$	$\frac{9}{8} \approx 1.13$	2

Sparse grids, $d = 3$

(p_x, p_t)	conv. rate γ	scaling σ^2
$(0,0)$	$\frac{3}{4} = 0.75$	2
$(1,0)$	$\frac{9}{8} \approx 1.13$	2
$(1,1)$	$\frac{5}{4} = 1.25$	2
$(3,1)$	$\frac{17}{8} \approx 2.13$	2

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