A GEOMETRICALLY-EXACT FINITE ELEMENT METHOD FOR MICROPOLAR CONTINUA WITH FINITE DEFORMATIONS

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Abstract. Micropolar theory is a weakly non-local higher-order continuum theory based on the inclusion of independent (micro-)rotational degrees of freedom. Subsequent introduction of couple-stresses and an internal length scale mean the micropolar continuum is therefore capable of modelling size effects. This paper proposes a non-linear Finite Element Method based on the spatial micropolar equilibrium equations, but using the classical linear micropolar constitutive laws defined in the reference configuration. The method is verified rigorously with the Method of Manufactured Solutions, and quadratic Newton-Raphson convergence of the minimised residuals is demonstrated.

Key words: *Micropolar elasticity; Non-linear FEM; Updated Lagrangian; Method of Manufactured Solutions*

1 Introduction

The modelling of materials with microstructures has become an important research topic in the field of solid mechanics. The use of micropolar continua, first proposed by the Cosserat brothers in 1909 [1], has been proven to be an effective approach for capturing the complex behaviour of such materials, especially granular media. The theory introduces an additional kinematic field of rigid-body proper orthogonal rotations to the conventional continuum formulation, which occur in the microstructure (e.g. soil grains) independently of the macro-deformation. Micropolar continua therefore have six degrees of freedom: three for the translations and three for the rotations. A kinematic measure of the gradient of these rotations is included in the thermodynamic formulation, which is conjugate to couple-stress (torque per unit area). As a result, the theory is characterised as weakly non-local and a characteristic length is introduced, allowing the observation of size effects and the natural evolution of strain localisation.

To date, few numerical methods have been developed to model micropolar continua in the finite strain and micro-rotation (meaning occurring in the microstructure, not of the order 10^{-6}) regime. Although others have focused on both material and geometric non-linearity [2,3], the only Finite Element Method (FEM) implementations dealing with purely geometric non-linearity appear to be [4] and [5]. However, the constitutive model used in [4] is based on a small strain assumption, and [5] provides a total Lagrangian formulation. The present contribution is therefore the first fully geometrically-exact updated Lagrangian formulation which still makes rigorous use of the linear micropolar constitutive equations.

2 Micropolar theory

2.1 Kinematics

Let a micropolar continuum occupy a volume Ω in its current (deformed) configuration. The translation vector u_i emanates from the Cartesian reference position X_i of each point in the undeformed volume Ω_0 to its current position x_i in Ω, and the deformation gradient tensor $F_{i\theta} = \frac{\partial x_i}{\partial X_\theta}$ provides the fundamental link between reference and current coordinates. At every point in the micro-continuum there exists a rigid body, attached to which is a set of axes that are free to rotate independently of deformation occurring at the continuum scale. Each rotated axis k_i in the current configuration is related to its counterpart W_ψ in the reference configuration via $k_i = Q_{i\psi} W_\psi$, where $Q_{i\psi} \in \mathrm{SO}(3)$ is a proper orthogonal tensor termed the *micro-rotation* tensor. The rotation may also be parameterised as a vector φ_k of Euler angles around the reference axes; alternatively φ_k may be identified as the axis of rotation with the angle its magnitude. A skew-symmetric tensor $\Phi_{ij} = -e_{ijk}\varphi_k$ (where e_{ijk} is the third-order Levi-Civita, or *permutation*, tensor) is then used to compute the micro-rotation tensor using the (Euler-)Rodrigues formula

$$Q_{i\psi} = \delta_{i\psi} + \frac{\sin|\varphi|}{|\varphi|}\Phi_{i\psi} + \frac{1-\cos|\varphi|}{|\varphi|^2}\Phi_{ij}\Phi_{j\psi}, \tag{1}$$

where $\delta_{i\psi}$ denotes the Kronecker delta and $|\varphi|$ is the magnitude of φ_k. Two Lagrangian measures are used to quantify micropolar deformation: a strain tensor, and a measure of the rotation gradient (curvature), named the *wryness* tensor which endows the theory with its non-local property

$$E_{\gamma\pi} = Q_{i\gamma}F_{i\pi} - \delta_{\gamma\pi} \qquad \text{and} \qquad \Gamma_{\gamma\pi} = -\frac{1}{2}e_{\gamma\tau\eta}Q_{p\tau}\frac{\partial Q_{p\eta}}{\partial X_\pi}. \tag{2}$$

2.2 Constitutive and balance equations

To preserve objectivity the constitutive laws are defined only in the reference frame. *Biot-like* stress $B_{\alpha\beta}$ and couple-stress $S_{\alpha\beta}$ are obtained directly from the Lagrangian strain and wryness measures as

$$B_{\alpha\beta} = \lambda\delta_{\alpha\beta}E_{\gamma\gamma} + (\mu+\nu)E_{\alpha\beta} + (\mu-\nu)E_{\beta\alpha} = D_{\alpha\beta\gamma\pi}E_{\gamma\pi} \tag{3}$$

$$S_{\alpha\beta} = \alpha\delta_{\alpha\beta}\Gamma_{\gamma\gamma} + (\beta+\gamma)\Gamma_{\alpha\beta} + (\beta-\gamma)\Gamma_{\beta\alpha} = \widehat{D}_{\alpha\beta\gamma\pi}\Gamma_{\gamma\pi} \tag{4}$$

where $D_{\alpha\beta\gamma\pi}$ and $\widehat{D}_{\alpha\beta\gamma\pi}$ are constitutive tensors which include the Lamé parameters λ and μ and micropolar constants ν, α, β and γ based on information about the material's characteristic length scale [6]. The inverse Piola transformation leads to the Cauchy stress and couple-stress

$$\sigma_{ij} = J^{-1}Q_{i\alpha}B_{\alpha\beta}F_{j\beta} \qquad \text{and} \qquad m_{ij} = J^{-1}Q_{i\alpha}S_{\alpha\beta}F_{j\beta} \tag{5}$$

respectively, where $J = \det(F)$ is the volume ratio between the original and deformed states. The spatial forms of linear and angular momentum balance in the quasi-static case read

$$\frac{\partial\sigma_{ij}}{\partial x_j} + p_i = 0 \qquad \text{and} \qquad \frac{\partial m_{ij}}{\partial x_j} - e_{ijk}\sigma_{jk} + q_i = 0, \tag{6}$$

where p_i and q_i are the body force and body couple respectively. Note that the presence of couples in the angular momentum balance equation, which is satisfied trivially in classical continua through equality of complementary shear stresses, means the stress tensor is not required to be symmetric.

3 Numerical formulation

Discretisation and application of Galerkin's method produces weak forms of (6) at a node I,

$$\int_\Omega \frac{\partial N_I}{\partial x_j}\sigma_{ij}\,\mathrm{d}\Omega = \int_\Omega N_I p_i\,\mathrm{d}\Omega \qquad \text{and} \qquad \int_\Omega \left(\frac{\partial N_I}{\partial x_j}m_{ij} + N_I e_{ijk}\sigma_{jk}\right)\mathrm{d}\Omega = \int_\Omega N_I q_i\,\mathrm{d}\Omega, \tag{7}$$

where N_I is the shape function associated with node I. Solution of the discretised boundary-value problem for a fixed external load with a Newton-Raphson scheme requires an iterative sequence of linearisation and incrementation of the internal force p^{int}_{Ii} and couple q^{int}_{Ii} expressions (the LHS of $(7)_1$ and $(7)_2$ respectively) until a convergence criterion is met. The linearisation procedure produces a tangent stiffness matrix K_{IiJj} which essentially relates an increment in deformation to an increment in force and couple, such that

$$\Delta p^{\text{int}}_{Ii} = K^{pu}_{IiJj}\Delta u_{Jj} + K^{pw}_{IiJj}\Delta w_{Jj} \qquad \text{and} \qquad \Delta q^{\text{int}}_{Ii} = K^{qu}_{IiJj}\Delta u_{Jj} + K^{qw}_{IiJj}\Delta w_{Jj} \tag{8}$$

for deformations at all nodes J, where Δw_{Jj} denotes an incremental rotation at J around axis k_j as oriented at the beginning of the current iteration. Once the deformation increments are obtained, the kinematic field is updated for the $(k+1)^{\text{th}}$ iteration from that in the k^{th} with

$$x^{k+1}_i = x^k_i + \Delta u_i \qquad \text{and} \qquad Q^{k+1}_{i\psi} = (\Delta Q_{ij})Q^k_{j\psi} \tag{9}$$

where ΔQ_{ij} is computed by substituting Δw_k for φ_k in (1). As noted in [4], the complete consistent linearisation of p^{int}_{Ii} and q^{int}_{Ii} with respect to nodal displacement and rotation increments is lengthy and arduous, and although it has been completed and used in this work, its inclusion lies beyond the scope of this contribution.

4 Verification

The formulation's accuracy and convergence properties are assessed by means of the Method of Manufactured Solutions (MMS). In the MMS, a synthetic solution field is designed and the corresponding body force/couple and boundary conditions generated via the governing equations. Numerical accuracy is then observed by comparing the numerically-approximated solution of the problem with the pre-determined analytical solution. To that end, the arbitrary displacement-rotation field

$$u_1 = \cdots = \varphi_3 = a\sin(2\pi X_1)\sin(2\pi X_2)\sin(2\pi X_3), \tag{10}$$

was chosen, where $X_i \in [0,1]$ defines a unit cube domain and $a = \frac{1}{100}$ ensures displacement is sufficiently small to ensure numerical stability. This particular trigonometric function was selected as it is continuously and infinitely differentiable and cannot be captured exactly by Lagrange interpolation. In this study, the problem is simulated using a three-dimensional FE implementation of the formulation, with a discretisation of tri-linear hexahedral elements. The Euclidean norms of the displacement error and Cauchy stress error (normalised by the analytical norm) are then integrated over the domain; the resulting convergence graphs for elements of decreasing dimension h are shown on log-log scales in Figures 1(a) and (b). As linear elements are used, quadratic displacement and (approaching) linear stress error decay is observed. Asymptotically quadratic Newton-Raphson convergence is demonstrated in Figure 1(c) for a separate problem involving very large displacements and rotations. The force residual is computed by taking its Euclidean norm normalised by the external load, and the energy residual is the scalar product of the force residual and the incremental deformations.

5 Conclusions

A geometrically-exact FEM for modelling the behaviour of micropolar continua under finite strains and micro-rotations has been presented. The governing equations are solved in the spatial frame, making the

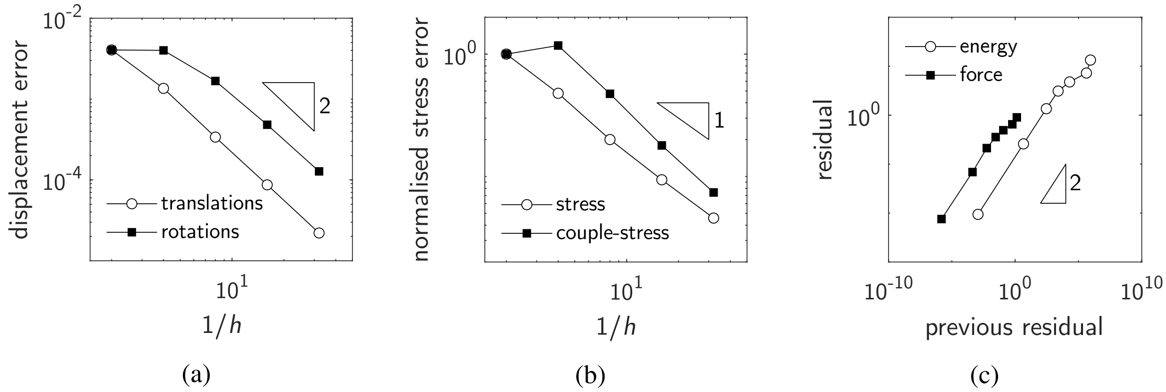

Figure 1: Convergence with mesh refinement of (a) translations/rotations and (b) Cauchy stress/couple-stress. Newton-Raphson convergence through a single loadstep is depicted in (c).

method updated Lagrangian, however the constitutive parameters used are defined in the reference frame and therefore all have physical meaning. Additionally, the method is computationally efficient (in terms of its convergence rate) and can be used to model complex problems accurately.

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