ON THE DEVELOPMENT OF A MATERIAL POINT METHOD COMPATIBLE ARC-LENGTH SOLVER FOR LARGE DEFORMATION SOLID MECHANICS

Nathan D. Gavin[1]*, William M. Coombs[1], John C. Brigham[2], Charles E. Augarde[1]

[1] Department of Engineering, Durham University, Durham, UK.
nathan.d.gavin@durham.ac.uk, w.m.coombs@durham.ac.uk, charles.augarde@durham.ac.uk
[2] Swanson School of Engineering, University of Pittsburgh, PA, USA
brigham@pitt.edu

Abstract. The Material Point Method is a versatile technique, however, it may be unable to provide a valid solution if there is a snap-through or snap-back response in the equilibrium path. One approach of overcoming this issue is the use of path following techniques and one example is the arc-length method. This technique is well established in finite element analysis but not in any Material Point Method formulation. This paper details the modifications made to the standard arc-length scheme for use within the Material Point Method. A numerical example is presented to demonstrate the implementation of the Material Point-Based Arc-length scheme in tracking a snap-through response.

Key words: *Material Point Method; Arc-length method*

1 Introduction

When modelling a problem using the Material Point Method (MPM), a linear solver is most commonly the solver of choice. However, the linear solver can only be used when the equilibrium path is increasing for the incremented variable throughout the simulation. This is often not the case when modelling problems involving large deformations [1] and/or non-linear material models [2] for example, where critical points in the equilibrium path are encountered. A snap-through response is when an increase in displacement for a decrease in load is seen, and a snap-back response is when there is a reduction in displacement after the snap-through response. It is possible to model a snap-through response with a displacement controlled scheme, but a load controlled scheme is unable to model either response. A path following technique is required to trace the equilibrium paths, a common example of which is the arc-length method [3]. To the authors' knowledge, no path following techniques have been formalised for the MPM, meaning that snap-through or snap-back responses are unable to be modelled with the MPM for a load controlled scheme, limiting the number of problems the MPM can model. This paper begins with the explanation of the standard arc-length method within the context of the Finite Element Method (FEM), followed by the adaptations made to the arc-length method to incorporate it into the MPM. The MPM arc-length formulation is demonstrated with the example of a shallow arch under a point load.

2 The FEM Arc-length Method

The arc-length method is a path following technique developed by Riks [3] and Wempner [4]. An arc-length solver (or Riks solver) solves for both load and displacement based on the converged state of the previous load step and a specified arc length. The equilibrium path is formed of load factor-displacement pairs, $\{\lambda, \mathbf{u}\}$, where λ is the load factor and $\mathbf{u}$ are the nodal FEM displacements. The load factor acts as a scalar multiplier of the external force vector applied to the material domain. A key feature of the arc-length method is that the "distance" from the current converged state to the next state along the equilibrium path is limited by a specified arc length, Δl. This paper uses the cylindrical arc-length method which ensures that the Euclidean norm of the change in nodal displacement, $\Delta \mathbf{u}$, is equal to the prescribed arc length [2]. The constraint equation is given by

$$\Delta \mathbf{u}^T \Delta \mathbf{u} = \Delta l^2. \tag{1}$$

At the start of the simulation, the initial arc length is calculated as the Euclidean norm of the tangential displacement (from the external force) in the initial state of the problem [2]. If the number of Newton-Raphson (N-R) iterations needed in a load step exceeds a prescribed desired number, the arc length for the following step is decreased, reducing the “distance” away from the current converged state to the next. In each iteration of a load step, the load factor increment is found by solving a quadratic equation with coefficients dependent on the N-R displacement (from the out-of-balance forces), the tangential displacement, the nodal displacement increment so far in the load step and the arc length [2]. The nodal displacement increment is then calculated as a combination of the N-R displacement and the tangential displacement (which is scaled by the load factor increment) [2]. In the first iteration of each load step, there is no information on the nodal displacements over the iterations of the load step meaning that the initial load factor increment cannot be determined and a predictor solution is required. There are multiple options of predictor solutions [2], this work utilises the secant path predictor [5] where the direction of the initial load factor increment is based on the mesh deformation in the previous load step and the magnitude of the load factor increment is determined by the arc length for the load step and the tangential displacement [2].

3 The Material Point-Based Arc-length Method

The MPM was developed by Sulsky and co-workers [6] as an extension of the FLuid Implicit Particle (FLIP) method, itself an extension of the Particle-In-Cell (PIC) method. Combining Eulerian and Lagrangian approaches of solid mechanics, a material domain is discretised into material points (or MPs) which act as the integration points of the analysis and deform through a background mesh made up of finite elements. Each MP stores its own data depending on the problem, such as mass, stress, force and stiffness contributions, which are interpolated onto the mesh nodes influenced by the MP. The stiffness matrix and internal force vector are assembled and the unknown mesh nodal displacements are calculated. These nodal displacements are then mapped back onto the MPs and their positions are updated. The background mesh is then reset and the next load step begins. As far as the authors are aware, all of the implicit MPM formulations in published literature utilise a linear solver run under load controlled or displacement controlled schemes meaning they cannot suitably solve problems which involve snap-through or snap-back responses. Therefore, a path following technique, such as the arc-length method, should be implemented in order to increase the number of problems that the MPM can be used to solve.

As the MPM allows the MPs to deform through the background mesh, it can often be the case that MPs will populate small portions of an element, causing issues with elements in the analysis having very small stiffnesses, resulting in large nodal displacements relative to the applied external forces. If an arc-length solver is implemented into the MPM in its FEM form, these large nodal displacements account for the majority of the allowed nodal displacement of the arc-length constraint which manifests in very small increments of the load factor and an inefficient tracing of the equilibrium path. For this reason, the Material Point-Based Arc-length Method (MP-BALM) reformulates the standard arc-length constraint based on the MP displacements. The change in MP displacements, $\Delta \mathbf{u}_p$, are calculated by mapping the nodal displacements to the MP locations such that

$$\Delta \mathbf{u}_p = \sum_{v=1}^{n} S_{vp}(x_p)\,\Delta \mathbf{u}_v = \bar{\mathbf{N}}_p\,\Delta \mathbf{u} \tag{2}$$

where n is the number of nodes in the background mesh, $S_{vp}(x_p)$ is the shape function of node v based on the position of the MP, x_p, $\Delta \mathbf{u}_v$ is the change in displacement of node v and $\bar{\mathbf{N}}_p$ is a diagonal matrix containing the shape functions for all of the mesh nodes for the MP in question. A diagonal matrix, $\bar{\mathbf{N}}$, is formed by summing $\bar{\mathbf{N}}_p$ for all MPs and scaling such that the values on the diagonal are between 0 and 1, these values represent the influence of the nodal displacements on the physical body being modelled. By incorporating the MP displacements into (1), the nodal displacements are scaled based on the shape function contributions of the MPs for each node, meaning that the large nodal displacements caused by partially filled elements do not take up as much of the arc-length constraint, allowing the solver to progress along the equilibrium path at more equal increments. The addition of the $\bar{\mathbf{N}}$ matrix is incorporated throughout the arc-length formulation to calculate the load factor and nodal displacement increments. As the MPs are displaced and the mesh is reset after every load step in the MPM, it is not possible to use the nodal displacement from the previous load step for the secant path predictor, therefore, a nodal displacement reconstruction is needed. This involves a least squares reconstruction of the nodal displacements using the total material point displacements and the current basis function interactions between the MPs and the nodes.

4 Numerical Results

The MP-BALM is implemented based on AMPLE [7] and is demonstrated with the 2D, plane strain analysis of a shallow arch subject to a point load, a problem which displays a snap-through response. This problem is akin to the Crisfield arch [1], however, this work presents the problem using continuum rather than beam or truss elements. Figure 1 shows the problem geometry, where the arch is formed with an inner length, L_i, and outer length, L_o, from the centre line of the arch of 9.5 m and 10 m respectively. The height of the inner arch wall, h_i, and outer arch wall, h_o, are 4.09 m and 4.45 m respectively. The arch material is modelled as linear elastic with Young's modulus of 100 kPa and a Poisson's ratio of 0.25. A point load, P, of 100 N is applied evenly between two material points at the crest of the arch, one MP either side of the centre line. The mesh is generated such that the nodes of the mesh align with the base of the arch and fixed boundary conditions are applied over the nodes of the base. The size of each element within the mesh is $0.125\,\text{m} \times 0.125\,\text{m}$ and a total of 8312 generalised interpolated material points MPs are used to create the arch. The simulation is run until the loaded MPs displace by 8.2 m. The load displacement response of the loaded MPs is shown in Figure 1. The plot shows a clear snap-through response with material softening seen after a vertical displacement of 1.5 m with further displacement seen for a reduction in amount of force. The material begins to stiffen after a vertical displacement of 5.5 m and continues to stiffen until the simulation is ended.

5 Conclusions

The standard arc-length method has been reformulated for the MPM to incorporate the material point displacements as opposed to the the more traditional approach of simply using the nodal displacements in the arc-length constraint. The MP shape functions associated with each mesh node are used to scale the nodal displacements to reduce the effect of extreme nodal displacements due to partially filled elements around the boundary of the material domain. An example of a shallow arch with an applied point

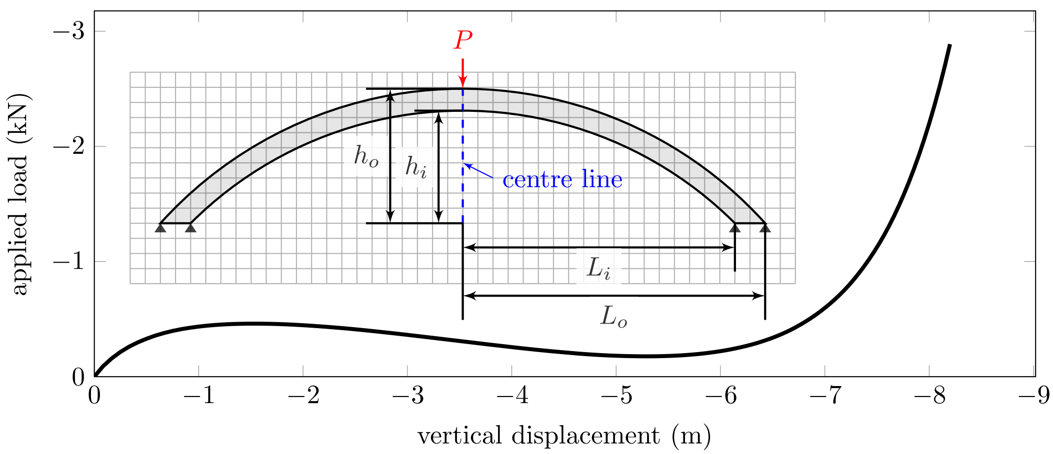

Figure 1: Shallow arch with point load applied problem geometry and load displacement response.

load was used to demonstrate the ability of the MP-BALM to track a snap-through response for a load controlled problem, a technique previously not presented in any literature. The demonstration shows the fundamental ability of the MPM to model continua at high deformation and the addition of the arc-length solver extends the scope of problems the MPM can model.

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